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RESEARCH ARTICLE

DYNAMIC BUCKLING OF A LIGHTLY DAMPED IMPERFECT DISCRETIZED SPHERICAL CAP STRESSED BY AN AXIAL IMPULSE

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ABSTRACT

The effects of damping has been a major concern of researchers in dynamical systems. In this

study, perturbation and asymptotic expansions were employed in the solution of the equations of motion of a viscously and lightly damped discretized imperfect spherical cap subjected to impulse loading. The dynamic buckling behaviour of the structure revealed that the dynamic buckling load I_D increases with light viscous damping $\bar{\xi}$. This shows that damping enhances the dynamic stability of structures. Consequently, we opine that damping should be incorporated in the construction of dynamical systems.

Keywords and Phrases: perturbation, asymptotic expansion, viscous, axisymmetric, damping, elastic.

1. INTRODUCTION

The dynamic stability of elastic structures under various loading histories which are time – dependent, aroused a wide range of inquests into the subject area. Some of the researchers include Svalbonas and Kalnins [4], Wang and Tian [5 - 6], Ette and Osuji [7], Aksogan and Sofiyev [8], Ette et al. [9], Ette [10] among others. The dynamic stability of elastic structures under the stress of dynamic loads is a primary suitability criterion for the choice of such structures for practical purposes. However, some researchers have come up with the inclusion of damping in the dynamical system so as to ameliorate the devastating effects of dynamic loads. These include Ette and Osuji [11 - 12], Osuji et al. [13] and others. In this investigation we shall use the perturbation technique with asymptotic expansions of the nonlinear equations of motion of the system. The original investigation from where this study is an extension, was made by Danielson [14] who used the concept of Mathieu-type of instability in a singular perturbation analysis of the problem. Danielson discretized the normal displacement W(x, y, $\bar{t}$) on a point on the spherical cap in the form

$$W(x, y,\bar{t}) = \xi_0(\bar{t})W_0(x, y) + \xi_1(\bar{t})W_1(x, y) + \xi_2(\bar{t})W_2(x, y) \quad (1.1)$$

where W_0, W_1 and W_2 are the symmetric pre-buckling mode, axisymmetric buckling mode and a non-axisymmetric buckling mode all of which are functions of the space variable (x; y) and $\xi_0(\bar{t})$, $\xi_1(\bar{t})$ and $\xi_2(\bar{t})$ are the respective time dependent amplitudes. He equally discretized the imperfection function $\overline{W}$ (x, y) in the shape of the buckling modes namely:

$$\overline{W}(x, y) = \bar{\xi}_1 W_1 + \bar{\xi}_2 W_2 \quad (1.2)$$

where $\bar{\xi}_1$ and $\bar{\xi}_2$ are the amplitudes of the axisymmetric and the non-axisymmetric imperfections. We shall let 0 < $\bar{\xi}_1$ < 1, 0 < $\bar{\xi}_2$ < 1 and assume that they are non-related mathematically. By substituting (1.1) and (1.2) into the relevant compatibility dynamic equilibrium equations of the simple quadratic elastic model structure and simplifying, Danielson obtained the following dynamic equilibrium equations for a discretized

imperfect spherical cap under a step load.

$$\frac{1}{\omega_o^2}\frac{d^2\xi_0}{d\bar{t}^2} + \xi_0 = \lambda f(\bar{t}) \quad (1.3)$$

$$\frac{1}{\omega_1^2}\frac{d^2\xi_1}{d\bar{t}^2} + \xi_1(1-\xi_0) - K_1\xi_1^2 + K_2\xi_2^2 = \bar{\xi}_1\xi_0 \quad (1.4)$$

$$\frac{1}{\omega_2^2}\frac{d^2\xi_2}{d\bar{t}^2} + \xi_2(1-\xi_0) + \xi_1\xi_2 = \bar{\xi}_2\xi_0 \quad (1.5)$$

$$\xi_\alpha(0) = \frac{d\xi_\alpha}{d\bar{t}}(0) = 0, \quad \alpha = 0,1,2 \quad (1.6)$$

Where for a step load, $f(\bar{t}) = 1$. Here, ω_i are the circular frequencies of the associated modes ξ_i, $i = 0,1,2$, while λ is a non-dimensional load parameter such that 0< λ < 1. Danielson solved the above coupled nonlinear differential equations by using the following assumptions:

(a) Quantities of the order of shell thickness divided by the radius can be neglected compared to unity.

(b) Tangential and boundary effects are negligible.

(c) $\bar{\xi}_1$ can be set equal to zero assuming that non-axisymmetric imperfections are the main cause of the reduction in the elastic strength of the structure.

(d) The effects of the quadratic term $K_1\bar{\xi}_1^2$ may be neglected compared to the effects of coupling between the buckling modes for initial buckling behaviour.

(e) The ratio of the subsequent frequencies namely $\frac{\omega_i}{\omega_{i-1}}$ is taken as $(1-\nu)$ where ν is the Poisson's ratio.

In [15] , Ette extended Danielson's earlier study in [14] to the case of an axial impulse and unlike that of Danielson, obtained the following striking results by incorporating all nonlinear terms as well as all the imperfection terms –

(i) By neglecting any imperfection, we automatically neglect the coupling effect of the buckling mode that is in the shape of the mode neglected, with other buckling modes.

(ii) The effects of the nonlinearity of any mode that is in the shape of the neglected mode is also neglected.

(iii) The only condition in which the coupling effects of any mode (be it pre-buckling or buckling mode) is felt is if the imperfection in the shape of the mode coupling is not neglected. Ette [15] showed that his findings and observation also hold for step loading case.

The present study is an extension of Ette's [15] findings to the case where (a) the discretized imperfect spherical cap is viscously and lightly damped. (b) The damping parameter $\bar{\xi}$ is independent of the imperfection parameters $\bar{\xi}_1$ and $\bar{\xi}_2$ i.e. $\bar{\xi}$ is not related to $\bar{\xi}_1$ or $\bar{\xi}_2$. It is also worthy of note that this study is a direct extension of Osuji *et al.* [13] wherein the simple quadratic elastic model structure was trapped by an impulse; see Figure 1. Relatively recent investigations on the subject matter include Ette *et al.* [9], Ette and Osuji [11 - 12] and Osuji *et al.* [13].

2. MATERIALS AND METHOD

FORMULATION OF THE PROBLEM

By substituting $I\delta(\bar{t})$ for $\lambda f(\bar{t})$ in (1.3) as well as the damping terms $C_1\frac{d\xi_0}{d\bar{t}}$, $C_1\frac{d\xi_1}{d\bar{t}}$ and $C_1\frac{d\xi_2}{d\bar{t}}$ into (1.3), (1.4) and (1.5) respectively, we have;

$$\frac{1}{\omega_0^2}\frac{d^2\xi_0}{d\bar{t}^2}+C_1\frac{d\xi_0}{d\bar{t}}+\xi_0=I\delta(\bar{t}) \quad (2.1.1)$$

$$\frac{1}{\omega_1^{\;2}}\frac{d^2\xi_1}{d\bar{t}^2}+C_1\frac{d\xi_1}{d\bar{t}}+\xi_1(1-\xi_0)-K_1\xi_1^{\;2}+K_2\xi_2^{\;2}=\bar{\xi}_1\xi_0 \qquad (2.1.2)$$

$$\frac{1}{\omega_2^{\;2}}\frac{d^2\xi_2}{d\bar{t}^2}+C_1\frac{d\xi_2}{d\bar{t}}+\xi_2(1-\xi_0)+\xi_1\xi_2=\bar{\xi}_2\xi_0 \qquad (2.1.3)$$

$$\xi_\alpha(0^-)=\frac{d\xi_0}{d\bar{t}}(0^-)=0,\ \alpha=0,1,2 \qquad (2.1.4)$$

where $I=$ impulse amplitude and $I\delta(\bar{t})$ is the Dirac-delta function of time $\bar{t}$.

2.2 ASYMPTOTIC SOLUTION

Let $t=\omega_0\bar{t}$. Therefore, we have $\frac{d\bar{\xi}_\alpha}{d\bar{t}}=\omega_0\frac{d\xi_\alpha}{dt},\frac{d^2\xi_\alpha}{d\bar{t}^2}=\omega_0{}^2\frac{d^2\xi_\alpha}{dt^2},\alpha=0,1,2$ (2.2.1)

We now substitute (2.2.1) into (2.1.1) – (2.1.3) and rewrite the resulting equations with the damping constant $\bar{\xi}$, where $2\bar{\xi}=C_1\omega_0, 0<\bar{\xi}<<1$(2.2.2) Thus integrating the resulting equation from the substitution into (2.1.1) above, from (0^-)

to (0^+), we have

$$\frac{d^2\xi_0}{dt^2}+2\bar{\xi}\frac{d\xi_0}{d\xi_0}+\xi_0=0,\quad t>0^+ \qquad (2.2.3a)$$

$$\xi_0(0^+)=0,\ \frac{d\xi_0}{dt}(0^+)=I,\quad \frac{d\xi_r}{dt}(0^+)=\quad r=1,\ 2 \qquad (2.2.3b)$$

Solving the differential equation (2.2.3a,b), we have

$$\xi_0(t)=\frac{I}{\varphi}e^{-\bar{\xi}t}\sin\varphi t,\ \ \varphi=\left(1-\bar{\xi}^2\right)^{\frac{1}{2}} \qquad (2.2.4)$$

Substituting $\xi_0(t)$ from (2.2.4) into the resulting equation from (2.1.2) and (2.1.3)

we have respectively;

$$\frac{d^2\xi_1}{dt^2}+2\bar{\xi}Q^2\frac{d\xi_1}{dt}+\xi_1Q^2-\xi_1\varepsilon\frac{e^{-\bar{\xi}t}\sin\varphi t}{\varphi}-Q^2K_1\xi_1^{\;2}+K_2Q^2\xi_2^{\;2}=\bar{\xi}_1\varepsilon\frac{e^{-\bar{\xi}t}\sin\varphi t}{\varphi} \qquad (2.2.5)$$

$$\frac{d^2\xi_2}{dt^2}+2\bar{\xi}R^2\frac{d\xi_2}{dt}+R^2\xi_2-\xi_2S\varepsilon\frac{e^{-\bar{\xi}t}\sin\varphi t}{\varphi}+R^2\xi_1\xi_2=\bar{\xi}_2S\varepsilon\frac{e^{-\bar{\xi}t}\sin\varphi t}{\varphi} \qquad (2.2.6)$$

where $0<\varepsilon<<1$ and

$$\varepsilon=I\left(\frac{\omega_1}{\omega_0}\right)^2,\ \ Q=\frac{\omega_1}{\omega_0},\ \ R=\frac{\omega_2}{\omega_0},\quad S=\left(\frac{\omega_2}{\omega_1}\right)^2 \qquad (2.2.7)$$

Since $0<\bar{\xi}<<1$, and

$$\varphi = \left(1-\bar{\xi}^2\right)^{\frac{1}{2}} = \left(1-\frac{\bar{\xi}^2}{2}-\ldots\right)\text{, then } \sin\varphi t \cong \sin t \tag{2.2.8}$$

Assuming (2.2.8) in (2.2.5) and (2.2.6), we have respectively

$$\frac{d^2\xi_1}{dt^2} + 2\bar{\xi}Q^2\frac{d\xi_1}{dt} + \xi_1 Q^2 - \xi_1\varepsilon\left(1+\frac{\bar{\xi}_1^{\,2}}{2}+\ldots\right)\sin t - Q^2K_1\xi_1^{\,2} + K_2Q^2\xi_2^{\,2} = \bar{Z}_1\varepsilon\left(1+\frac{\bar{\xi}_1^{\,2}}{2}+\ldots\right)\sin t \tag{2.2.9}$$

$$\frac{d^2\xi_2}{dt^2} + 2\bar{\xi}R^2\frac{d\xi_2}{dt} + \xi_2 R^2 - \xi_2 S\varepsilon\left(1+\frac{\bar{\xi}_1^{\,2}}{2}+\ldots\right)\sin t + R^2\xi_1\xi_2 = \bar{Z}_2 S\varepsilon\left(1+\frac{\bar{\xi}_1^{\,2}}{2}+\ldots\right)\sin t \tag{2.2.10}$$

where for convenience, we set $\bar{\xi}_1 = \bar{Z}_1$ and $\bar{\xi}_2 = \bar{Z}_2$. Let

$$\xi_1(t) = \eta\left(t,\tau;\varepsilon,\bar{\xi}\right) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\eta_{ij}\left(t,\tau;\varepsilon,\bar{\xi}\right)\varepsilon^i\bar{\xi}^j \tag{2.2.11a}$$

$$\xi_2(t) = \varsigma\left(t,\tau;\varepsilon,\bar{\xi}\right) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\varsigma_{ij}\left(t,\tau;\varepsilon,\bar{\xi}\right)\varepsilon^i\bar{\xi}^j \tag{2.2.11b}$$

where

$$\tau = \bar{\xi}t,\ \frac{d\xi_k}{dt} = \xi_{k,t} + \bar{\xi}\xi_{k,\tau}, \quad k = 1,\ 2. \tag{2.2.11c}$$

Using (2.2.11c) and substituting (2.2.11a) into (2.2.9) and equating coefficients of $\varepsilon^i\bar{\xi}^j$ in the resulting equation for i = 1,2 and j = 0, 1 we have

$$(\varepsilon):\ \eta_{10,tt} + Q^2\eta_{10} = \bar{Z}_1\sin t \tag{2.2.12a}$$

$$(\varepsilon\bar{\xi}):\ \eta_{11,tt} + Q^2\eta_{11} = -2\eta_{10,t\tau} - 2Q^2\eta_{10,t} \tag{2.2.12b}$$

$$(\varepsilon\bar{\xi}^2):\ \eta_{12,tt} + Q^2\eta_{12} = -2Q^2\eta_{11,t} - 2\eta_{11,t\tau} - \eta_{10,\tau\tau} - 2Q^2\eta_{10,\tau} + \frac{\bar{Z}_1\sin t}{2} \tag{2.2.12c}$$

$$(\varepsilon^2):\ \eta_{20,tt} + Q^2\eta_{20} = \eta_{10}\sin t + K_1Q^2\eta_{10}^{\,2} - K_2Q^2\varsigma_{10}^{\,2} \tag{2.2.12d}$$

$$(\varepsilon^2\bar{\xi}):\ \eta_{21,tt} + Q^2\eta_{21} = -2Q^2\eta_{20,t} - 2\eta_{20,t\tau} + \eta_{11}\sin t + 2Q^2[K_1\eta_{10}\eta_{11} - K_2\varsigma_{10}\varsigma_{11}] \tag{2.2.12e}$$

$$\eta_{ij}(0,0) = 0,\ i = 1,2,\ldots,\quad j = 0,1,2,\ldots,\ \eta_{10,t}(0,0) = 0 \tag{2.2.13a}$$

$$\eta_{1k,t}(0,0) + \eta_{1p,\tau}(0,0) = 0,\ p = k-1,\ k = 1,2,\ldots \tag{2.2.13b}$$

$$\eta_{20,t}(0,0)=0 \tag{2.2.13c}$$

$$\eta_{2k,t}(0,0)+\eta_{2p,\tau}(0,0)=0,\ \ p=k-1,\ \ k=1,2,... \tag{2.2.13d}$$

Similarly, from (2.2.10), we obtain the following

$$(\varepsilon):\ \varsigma_{10,tt}+R^2\varsigma_{10}=\bar{Z}_2\sin t \tag{2.2.14a}$$

$$(\varepsilon\bar{\xi}):\ \varsigma_{11,tt}+R^2\varsigma_{11}=-2R^2\varsigma_{10,t}-2\varsigma_{10,t\tau} \tag{2.2.14b}$$

$$(\varepsilon\bar{\xi}^2):\ \varsigma_{12,tt}+R^2\varsigma_{12}=-2R^2\varsigma_{11,t}-2\varsigma_{11,t\tau}-\varsigma_{10,\tau\tau}-2R^2\varsigma_{10,\tau}+\frac{\bar{Z}_2 S\sin t}{2} \tag{2.2.14c}$$

$$(\varepsilon^2):\ \varsigma_{20,tt}+R^2\varsigma_{20}=S\varsigma_{10}\sin t-R^2\eta_{10}\varsigma_{10} \tag{2.2.14d}$$

$$(\varepsilon^2\bar{\xi}):\ \varsigma_{21,tt}+R^2\varsigma_{21}=-2\varsigma_{20,t\tau}-2R^2\varsigma_{20,t}+S\varsigma_{11}\sin t-R^2[\eta_{11}\varsigma_{10}+\eta_{10}\varsigma_{11}] \tag{2.2.14e}$$

$$\varsigma_{ij}(0,0)=0,\ \ i=1,2,...,\quad j=0,1,2,...,\ \varsigma_{10,t}(0,0)=0 \tag{2.2.15a}$$

$$\varsigma_{1k,t}(0,0)+\varsigma_{1p,\tau}(0,0)=0,\ \ p=k-1,\ \ k=1,2,... \tag{2.2.15b}$$

$$\varsigma_{20,t}(0,0)=0 \tag{2.2.15c}$$

$$\varsigma_{2k,t}(0,0)+\varsigma_{2p,\tau}(0,0)=0,\ \ p=k-1,\ \ k=1,2,... \tag{2.2.15d}$$

Now solving the differential equation (2.2.12a) using (2.2.13a) for i = 1, j = 0,

we have

$$\eta_{10}(t,\tau)=\alpha_{10}(\tau)\cos Qt+\beta_{10}(\tau)\sin Qt+\bar{Z}_1 h_1\sin t \tag{2.2.16a}$$

$$\alpha_{10}(0)=0,\ \ \beta_{10}(0)=\frac{-\bar{Z}_1 h_1}{Q} \tag{2.2.16b}$$

where

$$h_1=\frac{1}{Q^2-1},\quad Q\neq 1 \tag{2.2.16c}$$

We now substitute (2.2.16a,b) into (2.2.12b) and to ensure a uniformly valid solution in the time scale t, equate to zero the coefficients of $\cos Qt$ and $\sin Qt$ to get respectively;

$$\beta_{10}'+Q^2\beta_{10}=0 \ \text{ and }\ \alpha_{10}'+Q^2\alpha_{10}=0 \tag{2.2.17a}$$

where $(...)'=\dfrac{d(...)}{d\tau}$

On solving (2.2.17a) using the initial conditions in (2.2.16b), we have;

$$\alpha_{10}(\tau)=0,\ \ \beta_{10}(\tau)=-\frac{\bar{Z}_1 h_1 e^{-Q^2\tau}}{Q} \tag{2.2.17b}$$

Thus we have from (2.2.16a) and (2.2.17b)

$$\eta_{10}(t,\tau)=\beta_{10}(\tau)\sin Qt+\bar{Z}_1h_1\sin t=\frac{-\bar{Z}_1h_1e^{-Q^2\tau}}{Q}+\bar{Z}_1h_1\sin t \qquad (2.2.17c)$$

We now solve the remaining non-homogenous equation in the substitution into (2.2.12b) using (2.2.13a,b) for i = 1, j = 1, k = 1, to get

$$\eta_{11}(t,\tau)=\alpha_{11}(\tau)\cos Qt+\beta_{11}(\tau)\sin Qt-2Q^2\bar{Z}_1h_1^{\,2}\cos t \qquad (2.2.18a)$$

$$\alpha_{11}(\tau)=2Q^2\bar{Z}_1h_1^{\,2},\ \ \beta_{11}(\tau)=0 \qquad (2.2.18b)$$

We now substitute for η_{11} and η_{10} into (2.2.12c) from (2.2.18a) and (2.2.17c) respectively and to ensure a uniformly valid solution in the time scale t, equate to zero the coefficients of $\cos Qt$ and $\sin Qt$ to get respectively;

$$\beta_{11}'+Q^2\beta_{11}=0 \quad \text{and} \quad \alpha_{11}'+Q^2\alpha_{11}=\frac{\beta_{10}''}{2Q}+Q\beta_{10}' \qquad (2.2.19a)$$

On solving (2.2.19a) using the initial conditions in (2.2.18b), we have

$$\alpha_{11}(\tau)=\frac{e^{-Q^2\tau}}{2Q}\left[\int_0^{\tau}e^{Q^2s}\left(\beta_{10}''(s)+2Q^2\beta_{10}'(s)\right)ds+2Q\alpha_{11}(0)\right],\ \beta_{11}(\tau)=0 \qquad (2.2.19b)$$

Therefore, we have from (2.2.18a) and (2.2.19b) that

$$\eta_{11}=\alpha_{11}(\tau)\cos Qt-2Q^2\bar{Z}_1h_1^{\,2}\cos t \qquad (2.2.19c)$$

We now solve the remaining differential equation in the substitution into (2.2.12c) using the first of (2.2.13a,b) for i = 1, j = 2 and k = 2 to get

$$\eta_{12}(t,\tau)=\alpha_{12}(\tau)\cos Qt+\beta_{12}(\tau)\sin Qt+\left[\frac{\bar{Z}_1h_1}{2}-4Q^4\bar{Z}_1h_1^{\,3}\right]\sin t \qquad (2.2.20a)$$

$$\alpha_{12}(\tau)=0,\ \ \beta_{12}(0)=\bar{Z}_1h_1h_2 \qquad (2.2.20b)$$

where

$$h_2=4Q^3h_1^{\,2}-\frac{1}{2Q}+\frac{Q}{2}-Q^2+2Q^4h_1 \qquad (2.2.20c)$$

To solve (2.2.12d), we first solve (2.2.14a,b) using (2.2.15a) for i = 1, j = 0 and get

$$\varsigma_{10}=\gamma_{10}(\tau)\cos Rt+\theta_{10}\sin Rt+\bar{Z}_2Sf_1\sin t \qquad (2.2.21a)$$

$$\gamma_{10}(0)=0,\ \theta_{10}(0)=-\frac{\bar{Z}_2Sf_1}{R},\quad f_1=\frac{1}{R^2-1},\quad R\neq 1 \qquad (2.2.21b)$$

We now substitute (2.2.21a) into (2.2.14b) and to ensure a uniformly valid solution in the time scale t, we equate to zero the coefficients of $\cos Rt$ and $\sin Rt$ respectively to get

$$\theta_{10}' + R^2\theta_{10} = 0 \quad \text{and} \quad \gamma_{10}' + R^2\gamma_{10} = 0 \tag{2.2.22a}$$

On solving (2.2.22a) using the initial conditions in (2.2.21b), we have

$$\theta_{10}(\tau) = -\frac{\bar{Z}_2 Sf_1 e^{-R^2\tau}}{R}, \quad \gamma_{10}(0) = 0 \tag{2.2.22b}$$

Therefore, we have from (2.2.21a,b) and (2.2.22b) that

$$\varsigma_{10} = \theta_{10}(\tau)\sin Rt + \bar{Z}_2 Sf_1 \sin t \tag{2.2.22c}$$

We now solve the remaining equation in the substitution into (2.2.14b) using (2.2.15a,b) for i = 1, j = 1 and k = 1, to get

$$\varsigma_{11} = \gamma_{11}(\tau)\cos Rt + \theta_{11}(\tau)\sin Rt - 2R^2\bar{Z}_2 Sf_1^2 \cos t \tag{2.2.23a}$$

$$\gamma_{11}(0) = 2R^2\bar{Z}_2 Sf_1^2, \qquad \theta_{11}(0) = 0 \tag{2.2.23b}$$

We next substitute η_{10} and ς_{10} from (2.2.17c) and (2.2.22c) respectively into (2.2.12d) and the resultant equation is similarly solved to get

$$\eta_{20}(t,\tau) = \alpha_{20}(\tau)\cos Qt + \beta_{20}(\tau)\sin Qt + \frac{r_0}{Q^2} + \frac{r_1\cos 2t}{Q^2-4} + \frac{r_2\cos(1-Q)t}{2Q-1}$$
$$-\frac{r_3\cos(1+Q)t}{2Q+1} - r_4\left\{\frac{\cos(1-R)t}{Q^2-(1-R)^2} - \frac{\cos(1+R)t}{Q^2-(1+R)^2}\right\} - \frac{q_1\cos 2Qt}{3Q^2} + \frac{q_2\cos 2Rt}{Q^2-4R^2} \tag{2.2.24a}$$

$$\alpha_{20}(0) = K_1\left(Q\bar{Z}_1 h_1\right)^2 g_5 + K_2\left(Q\bar{Z}_2 Sf_1\right)^2 g_6 + \bar{Z}_1 h_1 g_7, \quad \beta_{20}(0) = 0 \tag{2.2.24b}$$

where

$$g_5 = \frac{g_3}{2Q+1} - \frac{g_3^{\,2}}{6} - \frac{g_1 g_3^{\,2}}{2} + \frac{1}{2(Q^2-4)} + \frac{g_3}{2Q-1} \tag{2.2.24c}$$

$$g_6 = \frac{g_4}{Q^2-(1-R)^2} - \frac{g_4}{Q^2-(1+R)^2} + \frac{g_2 g_3^{\,2}}{2} - \frac{1}{2(Q^2-4)} + \frac{g_3}{2Q-1} \tag{2.2.24d}$$

$$g_7 = \frac{g_3}{2(2Q+1)} - \frac{g_3^{\,2}}{2} + \frac{1}{2(Q^2-4)} + \frac{g_3}{2(2Q-1)} \tag{2.2.24e}$$

$$g_1 = 1 + \frac{1}{Q^2}, \quad g_2 = 1 + \frac{1}{R^2}, \quad g_3 = \frac{1}{Q}, \quad g_4 = \frac{1}{R} \tag{2.2.24f}$$

$$Q \neq \frac{1}{2}, 2 \;; \quad Q \neq (1-R),\ (1+R)$$

We now substitute for η_{20}, η_{11}, η_{10}, ς_{10} and ς_{11} obtained above into (2.2.12e) and by solving for uniformly valid solution in the time scale t, with respect to $\cos Qt$ and $\sin Qt$

obtain the following;

$$\beta_{20}(\tau)=0 \text{ and } \alpha_{20}(\tau)=\left[K_1\left(Q\bar{Z}_1h_1\right)^2 g_5+K_2\left(Q\bar{Z}_2Sf_1\right)^2 g_6+\bar{Z}_1h_1g_7\right]e^{-Q^2\tau} \tag{2.2.25a}$$

Hence,

$$\begin{aligned}\eta_{20}(t,\tau)=\alpha_{20}(\tau)\cos Qt+\frac{r_0}{Q^2}+\frac{r_1\cos 2t}{Q^2-4}+\frac{r_2\cos(1-Q)t}{2Q-1}-\frac{r_3\cos(1+Q)t}{2Q+1}\\ -r_4\left\{\frac{\cos(1-R)t}{Q^2-(1-R)^2}-\frac{\cos(1+R)t}{Q^2-(1+R)^2}\right\}-\frac{q_1\cos 2Qt}{3Q^2}+\frac{q_2\cos 2Rt}{Q^2-4R^2}\end{aligned} \tag{2.2.25b}$$

Solving the remaining part of the equation in the substitution in (2.2.12e), we obtain

$$\begin{aligned}\eta_{21}(t,\tau)=\alpha_{21}(\tau)\cos Qt+\beta_{21}(\tau)\sin Qt+\frac{r_5\sin 2t}{Q^2-4}+\frac{r_6\sin 2Qt}{3Q^2}+\frac{r_7\sin 2Rt}{Q^2-4R^2}\\ +\frac{r_8\sin(1-Q)t}{2Q-1}+\frac{r_9\sin(1+Q)t}{2Q+1}+\frac{r_{10}\sin(1+R)t}{Q^2-(1+R)^2}+\frac{r_{11}\sin(1-R)t}{Q^2-(1-R)^2}\end{aligned} \tag{2.2.26a}$$

$$\alpha_{21}(0)=0,\ \beta_{21}(0)=K_1\left(Q\bar{Z}_1h_1\right)^2 g_{21}+K_2\left(Q\bar{Z}_2Sf_1\right)^2 g_{22}+\bar{Z}_1h_1g_{23} \tag{2.2.26b}$$

where

r_0 is a function of τ and r_i, i = 1, 2, …, 11 are functions of τ which are coefficients of $\cos Qt$, $\sin Qt$, $\cos Rt$, $\sin Rt$, $\cos(1-Q)t$, $\sin(1-Q)t$, $\cos(1+Q)t$ $\sin(1+Q)t$, $\cos(1+R)t$, $\cos(1-R)t$, $\sin(1-R)t$ and $\sin(1+R)t$ respectively. Also $q_1(\tau)=\frac{-K_1Q^2\beta_{10}{}^2}{2}$, $q_2(\tau)=\frac{K_2Q^2\theta_{10}{}^2}{2}$, while g_j, j = 8, 9, …, 23 are constant terms independent of τ.

Similarly, we solve equations (2.2.14c – e) to get the following results;

$$\varsigma_{12}(t,\tau)=\gamma_{12}(\tau)\cos Rt+\theta_{12}(\tau)\sin Rt+\left(\frac{\bar{Z}_2S}{2}-4R^4\bar{Z}_2Sf_1{}^2\right)\sin t \tag{2.2.27a}$$

$$\gamma_{12}(0)=0,\quad \theta_{12}(0)=-\left(\frac{\bar{Z}_2S}{2}-4R^4\bar{Z}_2Sf_1{}^2\right)\frac{f_1}{R}-\frac{\gamma_{11}'(0)}{R} \tag{2.2.27b}$$

$$\begin{aligned}\varsigma_{20}(t,\tau)=\gamma_{20}(\tau)\cos Rt+\frac{r_{12}\cos 2t}{R^2-4}+\frac{r_{13}\cos(1-R)t}{2R-1}-\frac{r_{14}\cos(1+R)t}{2R+1}\\ -\frac{r_{15}}{Q}\left\{\frac{\cos(Q-R)t}{2R-Q}+\frac{\cos(Q+R)t}{2R+Q}\right\}-r_{16}\left\{\frac{\cos(1-Q)t}{R^2-(1-Q)^2}+\frac{\cos(1+Q)t}{R^2-(1+Q)^2}\right\}+\frac{r_{17}}{R^2}\end{aligned} \tag{2.2.27c}$$

$$\gamma_{20}(0)=\bar{Z}_1\bar{Z}_2SL_{20}{}^*+\bar{Z}_2Sl_{20}{}^*,\ \theta_{20}(0)=0 \tag{2.2.27d}$$

where

$$L_{20}{}^{*}=\frac{g_3L_{15}}{2R-Q}+\frac{g_3L_{15}}{2R+Q}+\frac{L_{16}}{R^2-(1-Q)^2}-\frac{L_{16}}{R^2-(1+Q)^2}$$
$$\frac{L_{12}}{R^2-4}-\frac{L_{13}}{2R-1}+\frac{L_{14}}{2R+1}+g_4{}^2L_{17}$$

$$l_{20}{}^{*}=\frac{l_{12}}{R^2-4}-\frac{l_{13}}{2R-1}+\frac{l_{14}}{2R+1}-g_4{}^2l_{17}$$

$$l_{12}=\frac{-Sf_1}{2},\ L_{12}=\frac{R^2h_1}{2},\ l_{13}=\frac{-Sf_1}{2R},\ L_{13}=\frac{Rh_1f_1}{2}$$

$$l_{14}=\frac{Sf_1}{2R},\ L_{14}=\frac{-Rh_1f_1}{2},\ L_{15}=\frac{h_1f_1R}{2Q},\ L_{16}=\frac{-R^2h_1f_1}{2Q},\ l_{17}=\frac{Sf_1}{2},\ L_{17}=\frac{-R^2h_1}{2}$$

$$\varsigma_{21}(t,\tau)=\gamma_{21}(\tau)\cos Rt+\theta_{21}(\tau)\sin Rt+\frac{r_{18}\sin 2t}{R^2-4}+\frac{r_{19}\sin(1-R)t}{2R-1}-\frac{r_{20}\sin(1+R)t}{2R+1}$$
$$+\frac{r_{21}\sin(Q-R)t}{2QR-Q^2}-\frac{r_{22}\sin(Q+R)t}{2QR+Q^2}+\frac{r_{23}\sin(1-Q)t}{R^2-(1-Q)^2}+\frac{r_{24}\sin(1+Q)t}{R^2-(1+Q)^2} \quad (2.2.28a)$$

$$\gamma_{21}(0)=0,\quad \theta_{21}(0)=\bar{Z}_2Sl_{21}{}^{*}+\bar{Z}_1\bar{Z}_2SL_{21}{}^{*} \quad (2.2.28b)$$

where r_i , i = 18, 19, …,24 are functions of τ which are coefficients of $\sin 2t$, $\sin(1-R)t,\ \sin(1+R)t,\ \sin(Q-R)t,\ \sin(Q+R)t,\ \sin(1-Q)t$ and $\sin(1+Q)t$ respectively.

$$l_{21}{}^{*}=\frac{g_4(1+R)l_{20}}{2R+1}-\frac{2g_4l_{18}}{R^2-4}-\frac{g_4(1-R)l_{19}}{2R-1}+g_4g_9{}^2l_{20}{}^{*}-\frac{g_4l'_{13}}{2R-1}-\frac{g_4l'_{14}}{2R+1}$$

$$L_{21}{}^{*}=\frac{g_4(1+R)L_{20}}{2R+1}-\frac{g_4(Q+R)L_{22}}{2QR+Q^2}-\frac{2g_4L_{18}}{R^2-4}-\frac{g_4(1-R)L_{19}}{2R-1}-\frac{g_4(Q-R)L_{21}}{2QR-Q^2}$$
$$-\frac{g_4(1-Q)L_{23}}{R^2-(1-Q)^2}-\frac{g_4(1+Q)L_{24}}{R^2-(1+Q)^2}-\frac{g_4L'_{13}}{2R-1}-\frac{g_4L'_{14}}{2R+1}+g_4g_9{}^2L_{20}{}^{*}$$
$$+g_4L'_{15}\left\{\frac{1}{2QR-Q^2}+\frac{1}{2QR+Q^2}\right\}+g_4L'_{16}\left\{\frac{1}{R^2-(1-Q)^2}+\frac{1}{R^2-(1+Q)^2}\right\}$$

$$l'_{13}=\frac{RSf_1}{2},\ L'_{13}=\frac{R^3h_1f_1}{2},\ l'_{14}=\frac{-RSf_1}{2},\ L'_{14}=\frac{R^2h_1f_1}{2},\ L'_{15}=\frac{R^3Qh_1f_1}{2},$$
$$L'_{16}=\frac{R^2Qh_1f_1}{2},\ l_{18}=\frac{2R^2Sf_1}{R^2-4}-R^2Sf_1^2,\ L_{18}=\frac{2R^4h_1}{R^2-4}+R^2Q^2h_1{}^2f_1+R^4h_1f_1{}^2$$
$$l_{19}=\frac{2R(1-R)Sf_1}{2R-1}+SR^2f_1{}^2,\ L_{19}=RQ^2h_1{}^2f_1-R^4h_1f_1{}^2$$

$$l_{20} = \frac{2RSf_1(1+R)}{2R+1} + R^2 S f_1^2,\ L_{20} = \frac{-2R^3 h_1 f_1(1+R)}{2R+1} - RQ^2 h_1^2 f_1 - R^4 h_1 f_1^2$$

$$L_{21} = \frac{2R^4 h_1 f_1^2}{Q} - \frac{R^3 h_1 f_1(Q-R)}{2Q^2 R - Q^3} - \frac{R^3 h_1 f_1(Q-R)}{2QR - Q^2} - RQ^2 h_1^2 f_1$$

$$L_{22} = \frac{R^3 h_1 f_1(Q+R)}{2Q^2 R + Q^3} - \frac{R^3 h_1 f_1(Q+R)}{2QR + Q^2} + RQ^2 h_1^2 f_1 + \frac{2R^4 h_1 f_1^2}{Q}$$

$$L_{23} = \frac{R^4 h_1 f_1(1-Q)}{Q\left[R^2 - (1-Q)^2\right]} - \frac{R^2 Q h_1 f_1(1-Q)}{R^2 - (1-Q)^2} - R^2 Q^2 h_1^2 f_1 + \frac{R^4 h_1 f_1^2}{Q}$$

$$L_{24} = \frac{R^4 h_1 f_1(1+Q)}{Q\left[R^2 - (1+Q)^2\right]} + \frac{R^2 Q h_1 f_1(1+Q)}{R^2 - (1+Q)^2} - R^2 Q^2 h_1^2 f_1 - \frac{R^4 h_1 f_1^2}{Q}$$

So far the total (net) displacement $\xi(t,\tau;\varepsilon,\bar{\xi})$ is the sum of the two displacements $\eta(t,\tau;\varepsilon,\bar{\xi})$ and $\varsigma(t,\tau;\varepsilon,\bar{\xi})$ where, from (2.2.11a,b), we have

$$\eta(t,\tau;\varepsilon,\bar{\xi}) = \varepsilon\left[\eta_{10} + \bar{\xi}\eta_{11} + \ldots\right] + \varepsilon^2\left[\eta_{20} + \bar{\xi}\eta_{21} + \ldots\right] + \ldots \tag{2.2.29a}$$

$$\varsigma(t,\tau;\varepsilon,\bar{\xi}) = \varepsilon\left[\varsigma_{10} + \bar{\xi}\varsigma_{11} + \ldots\right] + \varepsilon^2\left[\varsigma_{20} + \bar{\xi}\varsigma_{21} + \ldots\right] + \ldots \tag{2.2.29b}$$

2.3 MAXIMUM DISPLACEMENT

Let the maximum displacement of $\eta(t,\tau;\varepsilon,\bar{\xi})$ be η_a attained at $t = t_a,\ \tau = \tau_a$ i.e. $\eta_a = \eta(t_a,\tau_a;\varepsilon,\bar{\xi})$ and the maximum displacement of $\varsigma(t,\tau;\varepsilon,\bar{\xi})$ be ς_c attained at $t = t_c,\ \tau = \tau_c$, i.e. $\varsigma_c = \varsigma(t_c,\tau_c;\varepsilon,\bar{\xi})$. The condition for η_a is

$$\eta_{,t}(t_a,\tau_a) + \bar{\xi}\eta_{,\tau}(t_a,\tau_a) = 0 \tag{2.3.1a}$$

Let

$$t_a = t_0 + \bar{\xi}t_{01} + \ldots + \varepsilon\left[t_{10} + \bar{\xi}t_{11} + \ldots\right] + \varepsilon^2\left[t_{20} + \xi t_{21} + \ldots\right] + \ldots \tag{2.3.1b}$$

Therefore from (2.2.11c), we have

$$\tau_a = \bar{\xi}t_a = \bar{\xi}\left\{t_0 + \bar{\xi}t_{01} + \ldots + \varepsilon\left[t_{10} + \bar{\xi}t_{11} + \ldots\right] + \varepsilon^2\left[t_{20} + \xi t_{21} + \ldots\right] + \ldots\right\} \tag{2.3.1c}$$

We next expand every function of t_a in a Taylor series about $t_a = t_0$ and every function of τ_a about $\tau_a = 0$. Using (2.2.29a), we therefore have

$$\begin{aligned}\eta_a = \eta(t_a,\tau_a) = \varepsilon\left[\eta_{10} + \bar{\xi}\left\{t_{01}\eta_{10,t} + t_0\eta_{10,\tau} + \eta_{11}\right\}\right]_{(t_0,0)} + \varepsilon^2[t_{10}\eta_{10,t} + \eta_{20}\\ + \bar{\xi}\{t_{11}\eta_{10,t} + t_{10}\eta_{10,\tau} + t_{01}t_{10}\eta_{10,tt} + t_0 t_{10}\eta_{10,t\tau} + t_{10}\eta_{11,t} + t_{01}\eta_{20,t}\\ + t_0\eta_{20,\tau} + \eta_{21}\}]_{(t_0,0)} + O(\varepsilon\bar{\xi}^2) + O(\varepsilon^2\bar{\xi}^2)\end{aligned} \tag{2.3.2}$$

To evaluate the time parameters as expressed in (2.3.2), we expand (2.3.1a) in Taylor series using (2.3.1b, c) and equate to zero relevant coefficients of $\varepsilon^i\bar{\xi}^j$, i = 1, 2, …,j = 0,1,… and obtain the following:

$$(\varepsilon): \eta_{10,t}\Big|_{(t_0,0)} = 0, \quad (\varepsilon\bar{\xi}): \left[t_{01}\eta_{10,tt} + t_0\eta_{10,t\tau} + \eta_{11,t} + \eta_{10,\tau}\right]_{(t_0,0)} = 0 \tag{2.3.3a}$$

$$(\varepsilon^2): \left[t_{10}\eta_{10,tt} + \eta_{20,t}\right]_{(t_0,0)} = 0 \tag{2.3.3b}$$

$$\begin{aligned}(\varepsilon^2\bar{\xi}): [t_{11}\eta_{10,tt} + t_{10}\eta_{10,t\tau} + t_{01}t_{10}\eta_{10,ttt} + t_{10}t_0\eta_{10,tt\tau} + t_{10}\eta_{11,tt} + t_{01}\eta_{20,tt} \\ + t_0\eta_{20,t\tau} + \eta_{21,t} + t_{10}\eta_{10,t\tau} + t_{10}\eta_{10,\tau\tau} + \eta_{20,\tau}]\Big|_{(t_0,0)} = 0\end{aligned} \tag{2.3.3c}$$

To solve the first part of (2.3.3a), we substitute (2.2.17c) and simplify to get

$$\cos t_0 - \cos Qt_0 = 0 \tag{2.3.4a}$$

An approximate value of t_0 is obtained by maintaining the first few terms in the Taylor series expansion of (2.3.4a) to get

$$t_0 \cong \pm\left(\frac{12}{1+Q^2}\right)^{\frac{1}{2}} \tag{2.3.4b}$$

From the second part of (2.3.3a) we obtain after simplifying

$$t_{01} = 2h_1Q^2 + \frac{Q\sin Qt_0}{Q\sin Qt_0 - \sin t_0} \tag{2.3.4c}$$

Similarly, we solve (2.3.3b) to obtain

$$t_{10} = -\frac{\left[K_1\left(Q\bar{Z}_1h_1\right)^2 g_{28} + K_2\left(Q\bar{Z}_2Sf_1\right)^2 g_{29} + \bar{Z}_1h_1g_{30}\right]}{\bar{Z}_1h_1\left[Q\sin Qt_0 - \sin t_0\right]} \tag{2.3.4d}$$

where g_i, i = 28, …,30 represent the terms associated with $K_1\left(Q\bar{Z}_1h_1\right)^2$, $K_2\left(Q\bar{Z}_2Sf_1\right)^2$ and $\bar{Z}_1h_1$ respectively in the solution of (2.3.3b).

We evaluate the maximum displacement η_a by first evaluating and substituting the respective terms in (2.3.2) to obtain

$$\begin{aligned}\eta_a = \eta(t_a,\tau_a) = \varepsilon\left[\bar{Z}_1h_1g_{59} + \bar{\xi}\{t_0\bar{Z}_1h_1g_8g_{60}\}\right] + \varepsilon^2[\ K_1\left(Q\bar{Z}_1h_1\right)^2 g_{53} \\ + K_2\left(Q\bar{Z}_2Sf_1\right)^2 g_{54} + \bar{Z}_1h_1g_{55} + \bar{\xi}\{\ t_{10}\bar{Z}_1h_1g_8g_{60} + t_{01}t_{10}\bar{Z}_1h_1g_{61} \\ + t_{10}\bar{Z}_1h_1g_8{}^2g_{62} + t_{01}\{\{K_1\left(Q\bar{Z}_1h_1\right)^2 g_{28} + K_2\left(Q\bar{Z}_2Sf_1\right)^2 g_{29} + \bar{Z}_1h_1g_{30}\}\} \\ + K_1\left(Q\bar{Z}_1h_1\right)^2 g_{56} + K_2\left(Q\bar{Z}_2Sf_1\right)^2 g_{57} + \bar{Z}_1h_1g_{58}\ \}\ ]\end{aligned} \tag{2.3.5}$$

where g_i, i = 31, …, 62 are the terms associated with $K_1\left(Q\bar{Z}_1h_1\right)^2$, $K_2\left(Q\bar{Z}_2Sf_1\right)^2$ and $\bar{Z}_1h_1$ respectively and their combination with the time parameters.

Also the condition for ς_c is

$$\varsigma_{,t}(t_c,\tau_c) + \bar{\xi}\varsigma_{,\tau}(t_c,\tau_c) = 0 \tag{2.3.6a}$$

Let

$$t_c = \hat{t}_0 + \bar{\xi}\hat{t}_{01} + \ldots + \varepsilon\left[\hat{t}_{10} + \bar{\xi}\hat{t}_{11} + \ldots\right] + \varepsilon^2\left[\hat{t}_{20} + \xi\hat{t}_{21} + \ldots\right] + \ldots \quad (2.3.6b)$$

$$\tau_c = \bar{\xi}t_c = \bar{\xi}\left\{\hat{t}_0 + \bar{\xi}\hat{t}_{01} + \ldots + \varepsilon\left[\hat{t}_{10} + \bar{\xi}\hat{t}_{11} + \ldots\right] + \varepsilon^2\left[\hat{t}_{20} + \xi\hat{t}_{21} + \ldots\right] + \ldots\right\} \quad (2.3.6c)$$

Using (2.2.29b) and expanding in Taylor series about $(\hat{t}_0, 0)$ we get

$$\begin{aligned}\varsigma_c = \varsigma(t_c, \tau_c) = \varepsilon\left[\varsigma_{10} + \bar{\xi}\left\{\hat{t}_{01}\varsigma_{10,t} + \hat{t}_0\varsigma_{10,\tau} + \varsigma_{11}\right\}\right]_{(t_0,0)} + \varepsilon^2[\hat{t}_{10}\varsigma_{10,t} + \varsigma_{20} \\ + \bar{\xi}\{\hat{t}_{11}\varsigma_{10,t} + \hat{t}_{10}\varsigma_{10,\tau} + \hat{t}_{01}\hat{t}_{10}\varsigma_{10,tt} + \hat{t}_0\hat{t}_{10}\varsigma_{10,t\tau} + \hat{t}_{10}\varsigma_{11,t} + \hat{t}_{01}\varsigma_{20,t} \\ + \hat{t}_0\varsigma_{20,\tau} + \varsigma_{21}\}]_{(\hat{t}_0,0)} + O\left(\varepsilon\bar{\xi}^2\right) + O\left(\varepsilon^2\bar{\xi}^2\right)\end{aligned} \quad (2.3.6d)$$

Similarly, to determine the time parameters expressed in (2.3.6d), we evaluate the terms in the equation and compare coefficients of terms of ε , ε^2 and $\varepsilon^2\bar{\xi}$. We solve respectively to get;

$$\hat{t}_0 = \pm\left(\frac{12}{1+R^2}\right)^{\frac{1}{2}}, \quad \hat{t}_{01} = -\frac{\left[\hat{t}_0\hat{l}_2 + \hat{l}_3 + \hat{l}_4\right]}{\hat{l}_1}, \quad \hat{t}_{10} = -\frac{\left[\hat{l}_5 + \bar{Z}_1\hat{L}_1\right]}{\hat{l}_1} \quad (2.3.7a)$$

where

$$\hat{l}_1 = f_1\left(R\sin R\hat{t}_0 - \sin\hat{t}_0\right), \; \hat{l}_2 = R^2 f_1\cos R\hat{t}_0, \; \hat{l}_3 = Rf_1\sin R\hat{t}_0, \; \hat{l}_4 = 2R^2 f_1^{\,2}\left(\sin\hat{t}_0 - R\sin R\hat{t}_0\right)$$

$$\begin{aligned}\hat{l}_5 &= -g_9 l_{20}^{\ *}\sin R\hat{t}_0 + \frac{2l_{12}\sin 2\hat{t}_0}{R^2-4} - \frac{(1-R)l_{13}\sin(1-R)\hat{t}_0}{2R-1} + \frac{(1+R)l_{14}\sin(1+R)\hat{t}_0}{2R+1} \\ \hat{L}_1 &= -g_9 L_{20}^{\ *}\sin R\hat{t}_0 + \frac{2L_{12}\sin 2\hat{t}_0}{R^2-4} - \frac{(1-R)L_{13}\sin(1-R)\hat{t}_0}{2R-1} + \frac{(1+R)L_{14}\sin(1+R)\hat{t}_0}{2R+1} \\ &+ \frac{(Q-R)L_{15}\sin(Q-R)\hat{t}_0}{Q(2R-Q)} + \frac{(Q+R)L_{15}\sin(Q+R)\hat{t}_0}{Q(2R+Q)} + \frac{(1-Q)L_{16}\sin(1-Q)\hat{t}_0}{R^2-(1-Q)^2} \\ &\frac{(1+Q)L_{16}\sin(1+Q)\hat{t}_0}{R^2-(1+Q)^2}\end{aligned} \quad (2.3.7b)$$

Substituting (2.3.7a) into (2.3.6d) we obtain after simplifying

$$\begin{aligned}\varsigma_c = \varsigma(t_c, \tau_c) = \varepsilon\left[\bar{Z}_2 S\hat{l}_{13} + \bar{\xi}\left\{\hat{t}_0\bar{Z}_2 S\hat{l}_3\right\}\right] + \varepsilon^2\left[\ \bar{Z}_2 S\hat{l}_{14} + \bar{Z}_1\bar{Z}_2 S\hat{L}_6 + \bar{\xi}\left\{\ \hat{t}_{10}\bar{Z}_2 S\hat{l}_3\right.\right. \\ + \hat{t}_{01}\hat{t}_{10}\bar{Z}_2 S\hat{l}_1 + \hat{t}_0\hat{t}_{10}\bar{Z}_2 S\hat{l}_2 + \hat{t}_{10}\bar{Z}_2 S\hat{l}_4 + \hat{t}_{01}\left(\bar{Z}_2 S\hat{l}_5 + \bar{Z}_1\bar{Z}_2 S\hat{L}_1\right) + \hat{t}_0\left(\bar{Z}_2 S\hat{l}_{11} + \bar{Z}_1\bar{Z}_2 S\hat{L}_4\right) \\ \left.\left.\bar{Z}_2 S\hat{l}_{15} + \bar{Z}_1\bar{Z}_2 S\hat{L}_7\ \right\}\ \right] + O\left(\varepsilon\bar{\xi}^2\right) + O\left(\varepsilon^2\bar{\xi}^2\right)\end{aligned} \quad (2.3.8)$$

where $\hat{l}_i$ and $\hat{L}_k$, i = 2, …, 15 , k = 4, 6, 7 are constants arising from the above

Substitution and simplification similar to those of (2.3.7b).

Consequently, total or net maximum displacement ξ_a is obtained from (2.3.6d) and (2.3.8) as

$$\xi_a = \eta_a + \varsigma_c = \varepsilon C_1 + \varepsilon^2 C_2 + \ldots \quad (2.3.9a)$$

where

$$C_1 = \left[\bar{Z}_1 h_1 g_{01} + \bar{Z}_2 Sl_{01} + \bar{\xi}\left\{\bar{Z}_1 h_1 g_{02} + \bar{Z}_2 Sl_{02}\right\}\right] \tag{2.3.9b}$$

$$\begin{aligned} C_2 = \big[\ & K_1(Q\bar{Z}_1 h_1)^2 g_{03} + K_2\left(QQ\bar{Z}_2 Sf_1\right)^2 g_{04} + \bar{Z}_1 h_1 g_{05} + \bar{Z}_2 Sl_{03} \\ & + \bar{Z}_1 \bar{Z}_2 SL_{01} + \bar{\xi}\{\ K_1\left(Q\bar{Z}_1 h_1\right)^2 g_{06} + K_2\left(Q\bar{Z}_2 Sf_1\right)^2 g_{07} + \bar{Z}_1 h_1 g_{08} \\ & + \bar{Z}_2 Sl_{04} + \bar{Z}_1 \bar{Z}_2 SL_{02}\ \}\ \big] + O\left(\varepsilon\bar{\xi}^2\right) + O\left(\varepsilon^2\bar{\xi}^2\right) \end{aligned} \tag{2.3.9c}$$

where

$$g_{01} = g_{59},\ l_{01} = \hat{l}_{13},\ g_{02} = t_0 g_8 g_{60},\ l_{02} = \hat{t}_0 \hat{l}_3,\ g_{03} = g_{53},\ g_{04} = g_{54},\ g_{05} = g_{55}$$

$$l_{03} = \hat{l}_{14},\ L_{01} = \hat{L}_6,\ g_{06} = t_{01} g_{28} + t_0 g_{37} + g_{56},\ g_{07} = t_{01} g_{29} + t_0 g_{38} + g_{57}$$

$$g_{08} = t_{10} g_8 g_{60} + t_{01} t_{10} g_{61} + t_{10} g_8^{\ 2} g_{62} + t_{01} g_{30} + t_0 g_{39} + g_{58},\ L_{02} = \hat{t}_{01}\hat{L}_1 + \hat{t}_0 \hat{L}_4 + \hat{L}_7$$

$$l_{04} = \hat{t}_{10}\hat{l}_3 + \hat{t}_{01}\hat{t}_{10}\hat{l}_1 + \hat{t}_0 \hat{t}_{10}\hat{l}_2 + \hat{t}_{10}\hat{l}_4 + \hat{t}_{01}\hat{l}_5 + \hat{t}_0 \hat{l}_{11} + \hat{l}_{15}$$

According to Budiansky and Hutchinson [1 – 3] and Ette [10], the condition for dynamic buckling is

$$\frac{dI}{d\xi_a} = 0 \tag{2.3.10}$$

Using the method of reversal of series by Amazigo [16], we have

$$\varepsilon = d_1 \xi_a + d_2 \xi_a^{\ 2} + d_3 \xi_a^{\ 3} + \ldots \tag{2.3.11}$$

We now substitute for ξ_a from (2.3.9a) into (2.3.11) and equate the coefficients of ε and ε^2 respectively to get

$$d_1 = \frac{1}{C_1} \quad \text{and} \quad d_2 = -\frac{C_2}{C_1^{\ 3}} \tag{2.3.12}$$

Since $C_1, C_2, \ldots$, depend on I , we apply (2.3.10) on (2.3.11) to obtain

$$\xi_{aD} = \frac{C_1^{\ 2}}{2C_2} \tag{2.3.13}$$

where ξ_{aD} is the maximum displacement at buckling and the right hand side of (2.3.13) is evaluated at I_D. By evaluating (2.3.11) at buckling we have

$$\varepsilon = d_1 \xi_{aD} + d_2 \xi_{aD}^{\ 2} + d_3 \xi_{aD}^{\ 3} + \ldots \tag{2.3.14}$$

We now substitute d_1 and d_2 from (2.3.12) and ξ_{aD} from (2.3.13) into (2.3.14) and simplify to get

$$\varepsilon_D = \frac{C_1}{4C_2} \tag{2.3.15}$$

where ε_D is the value of ε at buckling.

On substituting ε from (2.2.7), we have

$$I_D\left(\frac{\omega_1}{\omega_0}\right)^2 = \frac{C_1}{4C_2} \quad (2.3.16)$$

$$\therefore\ I_D = \frac{C_1 Q^{-2}}{4C_2} \quad (2.3.17)$$

3. RESULTS AND DISCUSSION

3.1 ANALYSIS OF RESULT

For $K_1 = 0.28,\ K_2 = 0.30,\ \bar{Z}_1 = 0.03,\ \bar{Z}_2 = 0.02$ and $\bar{\xi} = 0.01, 0.02, \ldots, 0.10$, the corresponding values of I_D were computed from (2.3.17). Hence figure 2 below shows the relationship between the dynamic buckling impulse load I_D and light viscous damping $\bar{\xi}$ of the discretized spherical cap.

3.2 DISCUSSION OF RESULT

The result in (2.3.9a – c) is carefully arranged as to display the contributions of each of the terms in the governing differential equations (2.1.1) to (2.1.3). For example the terms multiplying $K_1,\ K_2,\ \bar{Z}_1\bar{Z}_2$ indicate the contributions to dynamic buckling of the terms $K_1\xi_1^{\ 2}$, $K_2\xi_2^{\ 2}$ and $\xi_1\xi_2$ respectively in the equations (2.1.2) and (2.1.3). Similarly, the terms multiplying $\bar{Z}_1 h_1$ and $\bar{Z}_2$, in the same vein, indicate the contributions of the terms $\xi_1\xi_0$ and $\xi_2\xi_0$ respectively. If we assume that $\bar{Z}_2$ i.e. the non – axisymmetric imperfection equals zero, then we have the following result from (2.3.16).

$$I_D\left(\frac{\omega_1}{\omega_0}\right)^2 = \frac{\bar{Z}_1 h_1\left(g_{01} + \bar{\xi} g_{02}\right)}{K_1\left(Q\bar{Z}_1 h_1\right)^2 g_{03} + \bar{Z}_1 h_1 g_{05} + \bar{\xi}\left\{K_1\left(Q\bar{Z}_1 h_1\right)^2 g_{06} + \bar{Z}_1 h_1 g_{08}\right\}} \quad (3.2.1)$$

We observe the following from (3.2.1)

(a) The effect of the coupling is zero.

(b) The effect of the quadratic term $K_2\xi_2^{\ 2}$ is also zero.

(c) The effects of the coupling term $\xi_2\xi_0$ is zero.

(d) The effect of the coupling term $\xi_1\xi_0$ is non – zero.

(e) This means that the only major non – linear term that influences buckling is the quadratic term $K_1\xi_1^{\ 2}$.

However, if we set (Danielson’s assumption) as in [14], we have the following result as obtained from (2.3.16)

$$I_D\left(\frac{\omega_1}{\omega_0}\right)^2 = \frac{\bar{Z}_2 S\left(l_{01} + \bar{\xi} l_{02}\right)}{K_2\left(Q\bar{Z}_2 S f_1\right)^2 g_{04} + \bar{Z}_2 S l_{03} + \bar{\xi}\left\{K_2\left(Q\bar{Z}_2 S f_1\right)^2 g_{07} + \bar{Z}_2 S l_{04}\right\}} \quad (3.2.2)$$

From (3.2.2), we observe the following

(f) The effect of the coupling term $\xi_1\xi_2$ is again zero.

(g) The effect of the quadratic term $K_1\xi_1^{\ 2}$ is zero.

(h) The effect of the coupling term $\xi_1\xi_0$ is also zero.

(i) The effect of the coupling term $\xi_2\xi_0$ is non zero.

(j) The effect of the quadratic term $K_2\xi_2^{\ 2}$ is non zero and it is this term that dominates the buckling process.

We make the following additional observations:

(k) The only condition in which the effect of the coupling term $\xi_1\xi_2$ is felt is if none of the imperfections $\bar{Z}_1$ and $\bar{Z}_2$ is set equal to zero.

(l) Once we set an imperfection equal to zero, the effect of the coupling of the mode that is in the shape of the neglected imperfection, with any other mode, be it buckling mode or pre – buckling mode, is automatically equal to zero.

(m) We note that setting $\bar{Z}_1 = 0$ automatically nullifies the effect of $K_1\xi_1^{\ 2}$; the converse is however not true. In the same way, setting $\bar{Z}_2 = 0$, nullifies the effect of $K_2\xi_2^{\ 2}$.

(n) All these deductions confirm those obtained by Ette [13].

(o) If we set $\bar{\xi} = 0$ in (2.3.16), we obtain the same result obtained by Ette [13] for the undamped case.

Finally, we observe that damping a system gives additional dynamic stability to any elastic structure in the dynamic buckling process.

4. CONCLUSION

From the foregoing, we have successfully demonstrated that damping enhances the dynamic stability of elastic structures in the dynamic buckling process. Figure 2 reveals a steady rise in the dynamic buckling impulse load I_D with increase in the light viscous damping $\bar{\xi}$. This corroborates the results of earlier investigations in the subject area. However, it is our candid opinion that further work should be carried out in this area using other forms of dynamic loading and engineering structures.

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TABLE 1: DYNAMIC BUCKLING IMPULSE LOAD I_D WITH LIGHT VISCOUS DAMPING $\bar{\xi}$

$\bar{\xi}$	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
I_D	3.20E-04	3.69E-04	4.19E-04	4.68E-04	5.17E-04	5.67E-04	6.16E-04	6.66E-04	7.15E-04	7.65E-04

FIGURES

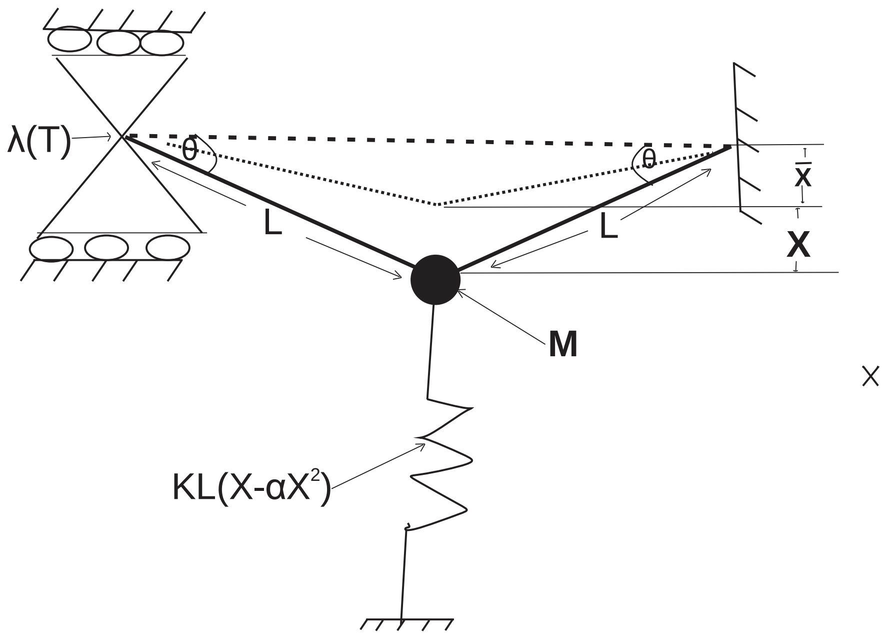

Figure 1: A simple quadratic - elastic model structure

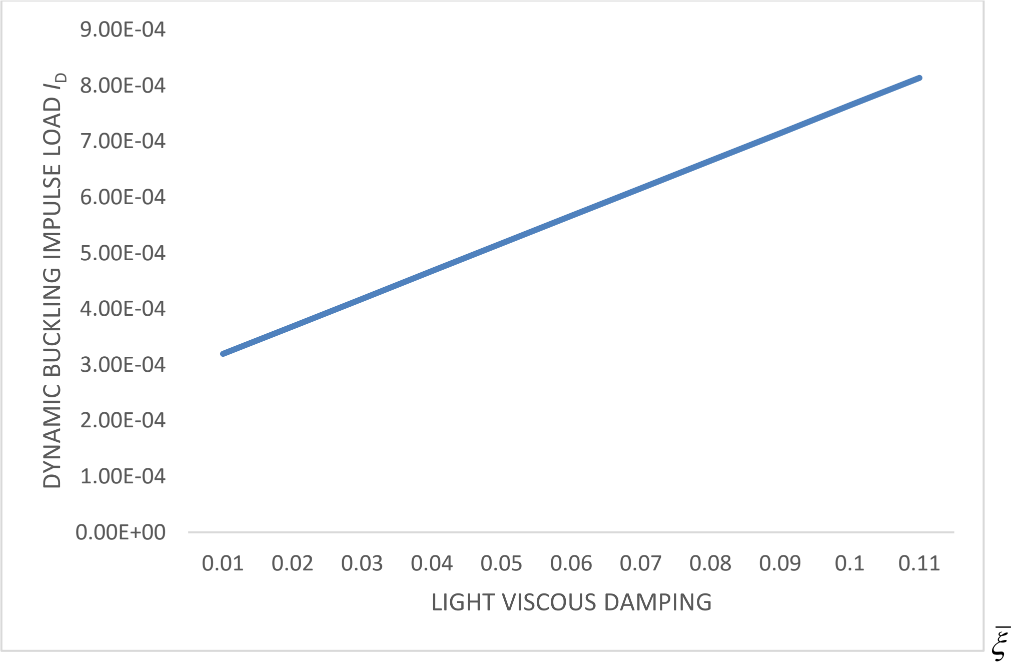

FIGURE 2: A GRAPH OF DYNAMIC BUCKLING IMPULSE LOAD I_D AGAINST LIGHT VISCOUS DAMPING $\bar{\xi}$.

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