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On Stereographic Reflected Gamma Distribution with Application

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ABSTRACT

This paper presents a novel circular model termed the stereographic reflected gamma distribution, which is constructed using inverse stereographic projection. Trigonometric moments are derived, and population characteristics are examined. Through analysis, it is determined that this model provides the most accurate fit for modeling 100 ants data from Fisher (1993), when compared to other stereographic circular models such as the stereographic reflected gamma distribution, stereographic logistic distribution, and stereographic double exponential distribution.

1. Introduction

Several circular models, which wrap certain life-testing models around a unit circle, were derived by Dattatreya Rao et al. (2007) and Girija (2010). One method for constructing circular models involves applying stereographic projection to linear models. Upon reviewing the literature, it becomes evident that little attention has been given to constructing circular models induced by inverse stereographic projection. Minh and Farnum (2003) introduced a novel approach for generating probability distributions through stereographic projection, mapping each point on the unit circle to a point on the real line. Building upon this, Toshihiro Abe et al. (2010) developed symmetric unimodal distributions using inverse stereographic projection. Dattatreya Rao et al. (2011) (2016) utilized stereographic projection on the Cardioid model and logistic model to produce Cauchy-type models and circular versions of the logistic distribution and also introduced a differential approach to circular models. Various methods for constructing circular models are discussed in Jammalamadaka and Sen Gupta (2001) and Girija (2010). In recent years Phani et al. (2012, 2013, 2017, 2019, and 2020) derived some circular and semicircular distributions based on stereographic projection. In the wrapping method, density functions are expressed as infinite series, posing challenges for computation. A modified inverse stereographic projection, from the real line to the circle, serves as the basis for constructing new circular models.

The Gamma distribution holds a significant place in actuarial science, particularly in life-testing scenarios where the reflected gamma distribution is often employed. This paper endeavours to develop a stereographic version of the reflected gamma distribution by employing inverse stereographic projection, and it further derives its characteristic function. The primary focus of this study lies in assessing the goodness-of-fit of the newly proposed parametric model, particularly concerning a specific dataset involving the movements of ants, as documented by Fisher (1993). Additionally, the paper aims to identify the most suitable model by comparing various stereographic circular models, including the stereographic reflected gamma distribution, stereographic logistic distribution, and stereographic double exponential distribution, which has demonstrated well fits.

2. Construction of Circular Models through Inverse Stereographic Projection

Probability distributions (both circular and linear) can be generated by applying stereographic projection, which yields one to one correspondence between the points on the unit circle and those on the real line. Inverse stereographic projection is defined by a one to one mapping given by $J(\phi) = y = v\tan\left(\frac{\phi}{2}\right)$, where $y \in (-\infty, \infty)$, $\phi \in [-\pi, \pi)$, $and\ v > 0$. Suppose y is randomly chosen on the interval $(-\infty, \infty)$. Let $F(y)$ and $f(y)$ denote the cumulative distribution and the probability density functions of the random variable Y respectively. Then $J^{-1}(y) = \phi = 2\tan^{-1}\left\{\frac{y}{v}\right\}$ is a random point on the unit circle. Let $G(\phi)$ and $g(\phi)$ denote the cumulative distribution and the probability density functions of this random point ϕ respectively. Then $G(\phi)$ and $g(\phi)$ can be derived in terms of $F(y)$ and $f(y)$ using the following lemma.

Lemma 2.1 If $v > 0$, $J^{-1}(y) = \phi = 2\tan^{-1}\left(\frac{y}{v}\right)$ increases monotonically from $-\pi$ to π as y increases from $-\infty$ to ∞.

Theorem 2.2: For $v > 0$,

$$G(\phi) = F\left(v\tan\left(\frac{\phi}{2}\right)\right) = F\left(y(\phi)\right) \tag{2.1}$$

$$g(\phi) = v\left(\frac{1+\tan^2\left(\frac{\phi}{2}\right)}{2}\right) f\left(v\tan\left(\frac{\phi}{2}\right)\right) \tag{2.2}$$

The characteristic function of a circular model with the probability density function $g(\phi)$ is defined as $\varphi_\tau(\phi) = \int_0^{2\pi} e^{i\tau\phi} g(\phi)\, d\phi$, $\tau \in$. The characteristic function of a stereographic circular model can be obtained in terms of respective linear model. Lukacs (1970) proved the following related to the characteristic function of linear model which is applied here in the case of stereographic circular models

Let Y be a random variable with distribution function $F(y)$ and suppose that $S(y)$ is a finite and single-valued function of y . The characteristic function of $\varphi_Y(t)$ of the random variable $Z = S(y)$ is then given by

$$\varphi_Z(t) = E\left(e^{itY}\right) = E\left(e^{itS(Y)}\right) = \int_{-\infty}^{\infty} e^{itS(y)} dF(y).$$

By applying the above the characteristic function of a stereographic circular model is proposed in Theorem 2.3.

Theorem 2.3 (Phani et al (2012)) : If $g(\phi)$ and $G(\phi)$ are the pdf and the cdf of the stereographic circular model and $f(y)$ and $F(y)$ are the pdf and the cdf of the respective linear model, then the characteristic function of a stereographic circular model is $\varphi_{Y_S}(\tau) = \varphi_{2\tan^{-1}\left(\frac{y}{v}\right)}(\tau), \tau \in$

2. Stereographic Reflected Gamma Distribution

A random variable Y on the real line is said to have reflected gamma distribution with scale parameter $\lambda > 0$, shape parameter $c > 0$ and location parameter α if the probability density and cumulative distribution functions of Y are given by

$$f(y) = \frac{|y-\alpha|^{c-1}}{2\lambda^c \Gamma(c)} \exp\left(\frac{-|y-\alpha|}{\lambda}\right),\ \lambda, c > 0,\ -\infty < y < \infty \text{ and} -\infty < \alpha < \infty \tag{3.1}$$

and

$$F(y) = \frac{1}{2}\left[1 + \frac{\Gamma_{\left(\frac{y}{\lambda}\right)}(c)}{\Gamma(c)}\right],\ \lambda, c > 0,\ -\infty < y < \infty \tag{3.2}$$

respectively.

Then by applying inverse stereographic projection given by a one to one mapping $y = v\tan\left(\frac{\phi}{2}\right), v > 0, -\pi \le \phi < \pi$, which leads to a three parametric symmetric circular model on unit circle. We call this distribution as stereographic reflected gamma distribution and it is denoted by SRG (σ, c, μ).

A random variable Y_S on a unit circle is said to have stereographic reflected gamma distribution with scale parameters $\sigma > 0$, $c > 0$ and location parameter μ denoted by SRG(σ, c, μ). If its probability density and cumulative distribution functions are respectively given by

$$g(\phi) = \frac{1}{2\sigma^c \Gamma(c)(1+\cos(\phi))} \left| \tan\left(\frac{\phi}{2}\right) - \mu \right|^{c-1} \exp\left(-\frac{1}{\sigma} \left| \tan\left(\frac{\phi}{2}\right) - \mu \right| \right),$$

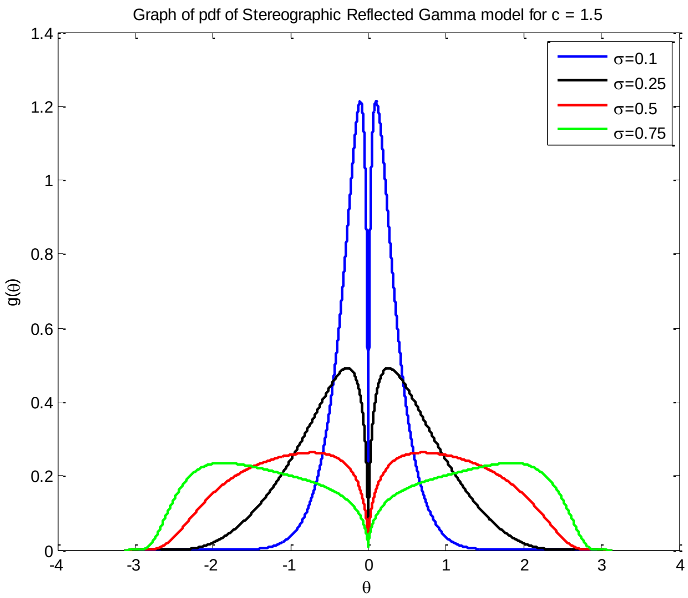
where $c,\ \sigma = \frac{\lambda}{v} > 0,\ \mu = \frac{\alpha}{v}, -\pi \le \phi < \pi$, and (3.3)

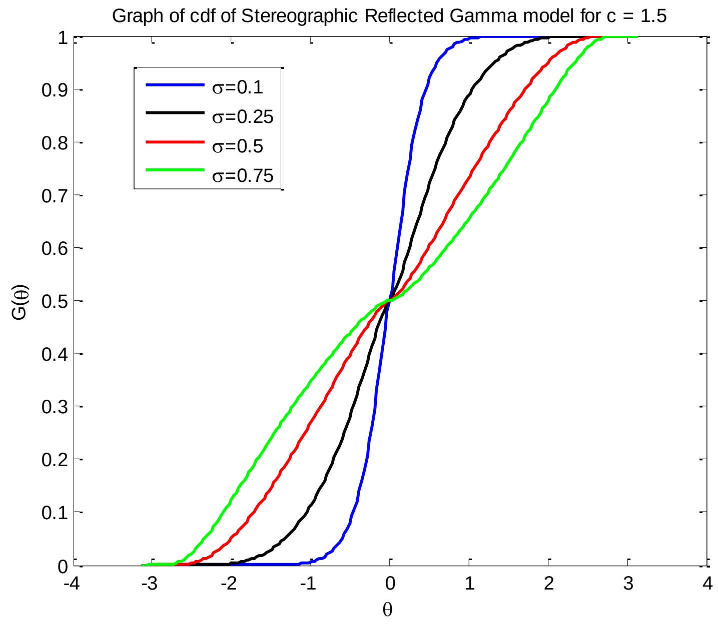
$$G(\phi) = \begin{cases} \frac{1}{2}\left(1 - \frac{\Gamma_{y_1}(c)}{\Gamma(c)}\right), \text{ where } y_1 = -\frac{1}{\sigma}\tan\left(\frac{\phi}{2}\right) \text{ for } \phi \in (-\pi, 0) \\ \frac{1}{2}\left(1 + \frac{\Gamma_{y_2}(c)}{\Gamma(c)}\right), \text{ where } y_2 = \frac{1}{\sigma}\tan\left(\frac{\phi}{2}\right) \text{ for } \phi \in [0, \pi) \end{cases}, \quad c, \sigma > 0, \ -\pi \le \phi < \pi, \tag{3.4}$$

Clearly g satisfies the conditions of circular distribution [Jammalamadaka and Sengupta (2001)]

1. $g(\phi) \ge 0$ for every $-\pi \le \phi < \pi$
2. $g(\phi + 2\pi k) = g(\phi),\ k \in Z$
3. $\int_{-\pi}^{\pi} g(\phi)\, d\phi = 1$

Graphs of probability density function and cumulative distribution function of stereographic reflected gamma distribution for various values of σ and c are presented here.





The Characteristic function of Stereographic Reflected Gamma Distribution

$$\varphi_{X_S}(p) = \int_{-\pi}^{\pi} e^{ip\phi} g(\phi)\, d\phi$$

$$= \frac{1}{2\sigma^c \Gamma(c)} \int_{-\pi}^{\pi} \frac{e^{ip\phi}}{\left(1+\left(\cos(\phi)\right)\right)} \left|\tan\left(\frac{\phi}{2}\right)\right|^{c-1} e^{-\frac{1}{\sigma}\left|\tan\left(\frac{\phi}{2}\right)\right|} d\phi$$

$$= \frac{1}{\sigma^c \Gamma(c)} \int_{0}^{\pi} \frac{e^{ip\phi}}{\left(1+\left(\cos(\phi)\right)\right)} \left|\tan\left(\frac{\phi}{2}\right)\right|^{c-1} e^{-\frac{1}{\sigma}\left|\tan\left(\frac{\phi}{2}\right)\right|} d\phi$$

$$= \frac{1}{2\sigma^c \Gamma(c)} \int_{-\pi}^{\pi} \frac{\cos(p\phi)}{\left(1+\left(\cos(\phi)\right)\right)} \left|\tan\left(\frac{\phi}{2}\right)\right|^{c-1} e^{-\frac{1}{\sigma}\left|\tan\left(\frac{\phi}{2}\right)\right|} d\phi$$

Trigonometric moments

The trigonometric moments of the proposed distribution are given by $\{\varphi_\tau : \tau = \pm 1, \pm 2, \pm 3, \ldots\}$, where $\varphi_\tau = \alpha_\tau + \beta_\tau$, with $\alpha_\tau = E(\cos \tau\phi)$ and $\beta_\tau = E(\sin \tau\phi)$ being the τ^{th} order cosine and sine moments of the random angle ϕ, respectively. Because the stereographic reflected gamma distribution is symmetric about $\mu = 0$, it follows that the sine moments are zero. Thus $\varphi_\tau = \alpha_\tau$.

Theorem 3.1: Under the pdf of stereographic reflected gamma distribution with $\mu = 0$, the first four $\alpha_\tau = E(\cos \tau\phi)$, $\tau = 1, 2, 3, 4$, are given as follows:

$$\alpha_1 = 1 - \frac{1}{\sqrt{\pi}\,\sigma^c \Gamma(c)} G_{13}^{31}\left(\frac{1}{4\sigma^2} \middle| \begin{matrix} -\frac{c}{2} \\ -\frac{c}{2}, 0, \frac{1}{2} \end{matrix} \right)$$

$$\alpha_2 = 1+\frac{4}{\sqrt{\pi}\sigma^c\Gamma(c)}G_{13}^{31}\left(\frac{1}{4\sigma^2}\left|\begin{matrix}-\frac{c}{2}-1\\ -\frac{c}{2},\,0,\frac{1}{2}\end{matrix}\right.\right)-\frac{4}{\sqrt{\pi}\sigma^c\Gamma(c)}G_{13}^{31}\left(\frac{1}{4\sigma^2}\left|\begin{matrix}-\frac{c}{2}\\ -\frac{c}{2},\,0,\frac{1}{2}\end{matrix}\right.\right)$$

$$\alpha_3 = 1-\frac{16}{3\sqrt{\pi}\sigma^c\Gamma(c)}G_{13}^{31}\left(\frac{1}{4\sigma^2}\left|\begin{matrix}-\frac{c}{2}-2\\ -\frac{c}{2},\,0,\frac{1}{2}\end{matrix}\right.\right)+\frac{24}{\sqrt{\pi}\sigma^c\Gamma(c)}G_{13}^{31}\left(\frac{1}{4\sigma^2}\left|\begin{matrix}-\frac{c}{2}-1\\ -\frac{c}{2},\,0,\frac{1}{2}\end{matrix}\right.\right)$$

$$-\frac{9}{\sqrt{\pi}\sigma^c\Gamma(c)}G_{13}^{31}\left(\frac{1}{4\sigma^2}\left|\begin{matrix}-\frac{c}{2}\\ -\frac{c}{2},\,0,\frac{1}{2}\end{matrix}\right.\right)$$

$$\alpha_4 = 1+\frac{32}{3\sqrt{\pi}\sigma^c\Gamma(c)}G_{13}^{31}\left(\frac{1}{4\sigma^2}\left|\begin{matrix}-\frac{c}{2}-3\\ -\frac{c}{2},\,0,\frac{1}{2}\end{matrix}\right.\right)+\frac{80}{\sqrt{\pi}\sigma^c\Gamma(c)}G_{13}^{31}\left(\frac{1}{4\sigma^2}\left|\begin{matrix}-\frac{c}{2}-1\\ -\frac{c}{2},\,0,\frac{1}{2}\end{matrix}\right.\right)$$

$$-\frac{64}{\sqrt{\pi}\sigma^c\Gamma(c)}G_{13}^{31}\left(\frac{1}{4\sigma^2}\left|\begin{matrix}-\frac{c}{2}-2\\ -\frac{c}{2},\,0,\frac{1}{2}\end{matrix}\right.\right)-\frac{16}{\sqrt{\pi}\sigma^c\Gamma(c)}G_{13}^{31}\left(\frac{1}{4\sigma^2}\left|\begin{matrix}-\frac{c}{2}\\ -\frac{c}{2},\,0,\frac{1}{2}\end{matrix}\right.\right)$$

Where $\int_0^\infty x^{2\nu-1}\left(u^2+x^2\right)^{Q-1}e^{-\mu x}dx=\frac{u^{2\nu+2Q-2}}{2\sqrt{\pi}\Gamma(1-Q)}G_{13}^{31}\left(\frac{\mu^2u^2}{4}\left|\begin{matrix}1-\nu\\ 1-Q-\nu,0,\frac{1}{2}\end{matrix}\right.\right)$

(3.5)

for $\left|\arg u\pi\right|<\frac{\pi}{2},\mathrm{Re}\,\mu>0$ and $\mathrm{Re}\,\nu>0$ and $G_{13}^{31}\left(\frac{\mu^2u^2}{4}\left|\begin{matrix}1-\nu\\ 1-Q-\nu,0,\frac{1}{2}\end{matrix}\right.\right)$ is called as Meijer's **G-**

function (Gradshteyn and Ryzhik, 2007).

4. Application

For the purpose of verifying goodness of fit the following movements of ants data set is considered.

Data Set : Directions chosen by 100 ants in response to an evenly illuminated black target placed [Fisher (1993)]

Direction (in degrees)

330, 290, 60, 200, 200, 180, 280, 220, 190, 180, 180, 160, 280, 180, 170, 190, 180, 140, 150, 150, 160, 200, 190, 250, 180, 30, 200, 180, 200, 350, 200, 180, 120, 200, 210, 130, 30, 210, 200, 230, 180, 160, 210, 190, 180, 230, 50, 150, 210, 180, 190, 210, 220, 200, 60, 260, 110, 180, 220, 170, 10, 220, 180, 210, 170, 90, 160, 180, 170, 200, 160, 180, 120, 150, 300, 190, 220, 160, 70, 190, 110, 270, 180, 200, 180, 140, 360, 150, 160, 170, 140, 40, 300, 80, 210, 200, 170, 200, 210, 190.

The above data set is used to verify goodness of fit of stereographic reflected gamma model.

The data plot is shown in figure 4.1

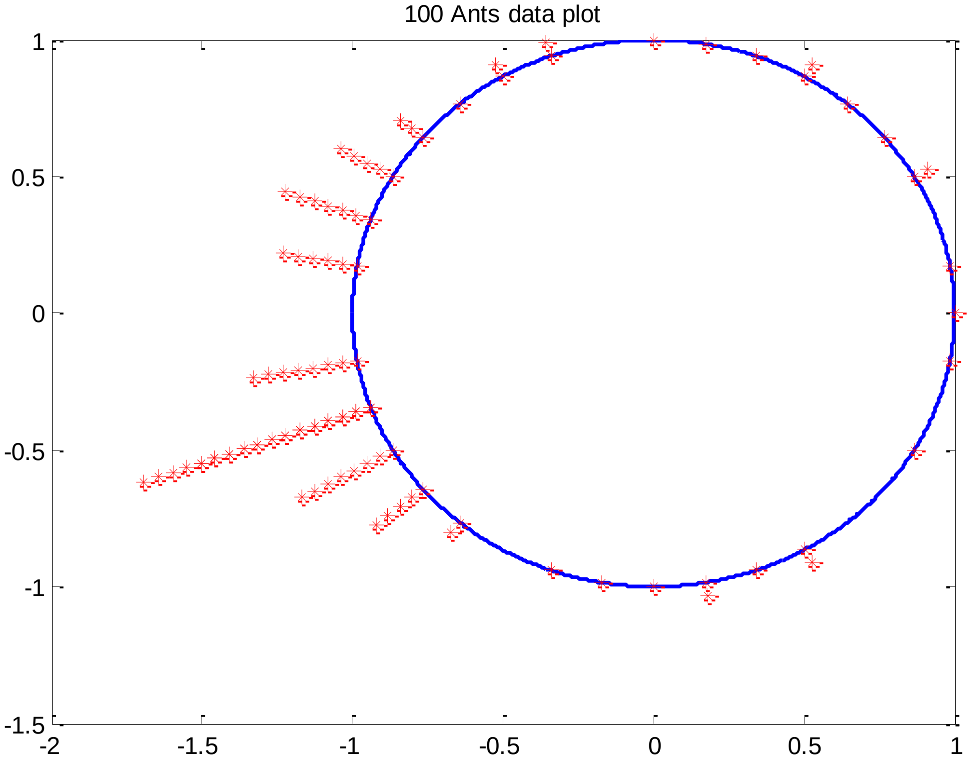


Figure 4.1

To adapt a model to given data, it's necessary to first estimate the model's parameters. Circular models are defined by their mean direction and concentration parameter. These parameters can be determined using various statistical methods documented in literature. In this study, we employ the following approach for parameter estimation.

Let $\phi_1, \phi_2, \phi_3, \ldots, \phi_n$ be a set of circular observations.

The mean direction is

$$\hat{\mu} = \begin{cases} \tan^{-1}(A/B) & \text{if } \ A > 0\,,\ B > 0 \\ \tan^{-1}(A/B) + \pi & \text{if } \ B < 0 \\ \tan^{-1}(A/B) + 2\pi & \text{if } \ A < 0\,,\ B > 0 \end{cases}$$

and the mean resultant length (estimate of the concentration parameter) is $\hat{\sigma} = \sqrt{B^2 + A^2}$, where $B = \frac{1}{n}\sum \cos\phi_i$ and $A = \frac{1}{n}\sum \sin\phi_i$

To demonstrate the modeling behavior of stereographic reflected gamma distribution, the shape parameter *c* is to be estimated. It can be estimated by invoking fmincon, the built-in MATLAB function. The estimates are $\mu = -3.0868,\ \sigma = 0.6101$ **and** $\hat{c} = 0.8042$ **.**

We consider stereographic reflected gamma, stereographic logistic (Dattatreya Rao et al (2011) and stereographic double exponential models (Phani (2013) to verify goodness-of-fit for movements of ants data of size *n* = 100. As this is a large sample, we apply Kuiper's and modified Watson's U^2 tests. The statistic of Watson's U^2 test for large sample [Mardia and Jupp (2000), p. 105] is

$W^2 = \frac{V_n^2}{\pi^2}$ Where V_n the statistic of Kuiper's test [Mardia and Jupp (2000)] **.**

The statistics of the two goodness-of-fit tests viz., Kuiper's and Watson's U^2 tests are computed for the said circular models and are tabulated.

Table 4.1 Statistics of the Kuiper's and Watson's U ^{2} tests of goodness-of-fit tests

	Stereographic Logistic distribution	Stereographic Reflected Gamma distribution	Stereographic Double Exponential model
Sample Size *n* =100	$\mu = -3.0868,$ $\sigma = 0.6101$	$\mu = -3.0868,$ $\sigma = 0.6101$ $\hat{c} = 0.8042$	$\mu = -3.0868,$ $\sigma = 0.6101$
Kuiper's Test	7.9811	8.4098	8.2945
Watson's U^2Test	0.0623	0.0692	0.0673

On the lines of algorithm in Devaraj (2012) the cut off points for the data set of sample size *n* = 100 are computed using MATLAB techniques.

Table 4.2 The cut of points for the data set of sample size *n* = 100

LOS Tests	1%	5%	10%
Kuiper's Test	0.6987 - 2.1225	0.7924 - 1.8348	0.8474 - 1.7437
Watson's U^2 - Test	0.0163 - 0.2976	0.0231 - 0.2231	0.0270 - 0.1888

Based on the cutoff points derived from a sample size of n = 100, the statistical analysis conducted indicates that the dataset on ant movements adheres to all three circular models: stereographic reflected gamma, stereographic logistic, and stereographic double exponential models, across all levels of significance (i.e., 1%, 5%, and 10%) as determined by the modified Watson's U2 test for estimated parameters.

Choosing the Best Model

When multiple parametric circular models demonstrate a good fit for a given dataset, determining the most appropriate model relies on the following criteria:

i) Maximum Log Likelihood (MLL) (Law and Kelton, 1991)

This method consists of maximizing the likelihood function, *L*, given by

$$L = \prod_{i=1}^{n} f\left(\phi_i\right)$$

Where $f\left(\phi\right)$ is the probability density function of the selected distribution, and $\phi_1, \phi_2, \phi_3, \ldots, \phi_n$ are the *n* data points in the sample to be fitted. More commonly, the log-likelihood function given by

$$\log(L) = \sum_{i=1}^{n} \log\left(f\left(\phi_i\right)\right)$$

is maximized. Since the log function is a strictly increasing monotonic function, the results from maximizing either function are identical.

ii) Akaike's Information Criteria (AIC)

A widely used information criterion is the Akaike Information Criterion (AIC). Proposed by Akaike in 1973, the concept of AIC aims to identify the model that minimizes the negative likelihood, adjusted for the number of parameters, as depicted in the equation.

$$AIC = 2k - 2\log(L)$$

In this context, L represents the likelihood derived from the fitted model, and k denotes the number of parameters within the model. AIC, in essence, strives to identify the model that provides the best approximation to the underlying, unknown data generating process and its practical uses.

Among a collection of candidate models for the dataset, the optimal model is determined as the one exhibiting the lowest AIC value.

iii) Bayesian Information Criteria (BIC)

Another commonly employed information criterion is the Bayesian Information Criterion (BIC). Unlike the Akaike Information Criterion, BIC is formulated within a Bayesian framework, serving as an estimate of the Bayes factor for two rival models (Schwarz, 1978; Kass and Raftery, 1995). The BIC is mathematically defined as follows:

$$BIC = -2\log(L) - k\log(n)$$

At first glance, BIC appears similar to AIC, with the key distinction lying in the second term, which now incorporates the sample size n. Models minimizing the Bayesian Information Criterion are chosen. From a Bayesian standpoint, BIC aims to identify the most probable model for a given dataset. Among a set of candidate models, the preferred model is determined as the one with the lowest BIC value.

From a data analysis perspective, the objective is to select the most suitable circular model that accurately represents the provided dataset among the well-fitting options. This challenge is addressed by applying the criteria of AIC, BIC, and MLL. The computations of measures of relative performances are outlined in Table 4.3.

Table 4.3: Measures of Relative Performance for Goodness-of-fit at $\mu = -3.0868,\ \sigma = 0.6101$ and $\hat{c} = 0.8042$

Distribution	Parameters	*MLL*	*AIC*	*BIC*
Stereographic Logistic model	$\mu = -3.0868$, $\sigma = 0.6101$	-246.2138	250.2138	501.6379
Stereographic Reflected Gamma model	$\mu = -3.0868$, $\sigma = 0.6101$ $\hat{c} = 0.8042$	-220.3863	224.3863	449.9829
Stereographic Double Exponential model	$\mu = -3.0868$, $\sigma = 0.6101$	-224.2926	228.2926	457.7955

Conclusion

After considering AIC, BIC, and MLL, it is unanimously determined that the stereographic reflected gamma model provides a superior fit compared to the stereographic logistic and stereographic double exponential models for the estimates. $\mu = -3.0868,\ \sigma = 0.6101$ **and** $\hat{c} = 0.8042$**.**

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