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RESEARCH ARTICLE

FRACTIONAL FORMULAE OF EXTENDED MULTIINDEX BESSEL-MAITLAND FUNCTION IN TERMS OF HYPERGEOMETRIC FUNCTIONS

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ABSTRACT

This article deals with the study of the generalized multi-index Bessel-Maitland function $\mathbb{J}^{(\alpha_j)_m,\ \gamma,\ \ c}_{(\beta_j)_m,\ k,\ \ b}(\mathrm{z})$ where $(j = 1, 2,\cdots, m)$ with relation of pathway fractional operator and with extended Caputo fractional derivative operator which plays a ubiquitous role in wide range of diverse fields (such as acoustic field, electromagnetism, heat, hydrodynamics, wave motion, elasticity and optical science) which are expressed in terms of generalized Wright hypergeometric function rΨs[z]. We also discuss some special cases of our main result by choosing some particular values of the parameters in $\mathbb{J}^{(\alpha_j)_m,\ \gamma,\ \ c}_{(\beta_j)_m,\ k,\ \ b}(\mathrm{z})$. The result obtained here is also reduced to the known result of J. Choi as special cases.

MSC: 33C20, 33B15.

Keywords: Generalized multi-index Bessel function, generalized hypergeometric function, Fox-H function and Integrals.

1. Introduction and Preliminaries

The Bessel function has gained importance and popularity due to its applications in the problem of wave propagation, cylindrical coordinate system, heat conduction in cylindrical object and static potential etc. In the recent years, some generalizations(unification) and number of integral transforms of Bessel functions have been given by many mathematicians and physicist as well as engineers for example: Choi et al. [3], Kiryakova [12] Khan and Ghayasuddin [10], Khan et al. [9]. Lately, Choi et al. [3] and Agarwal [4] introduced and studied various properties of generalized multi-index Bessel function and then discussed an interesting unified integrals formulas involving the generalized multi-index Bessel function. For the present study, we consider the following definitions:

Definition: The generalized Wright hypergeometric function rΨs$[x]$ is also called Fox- Wright function (see [22], [23]) is defined as:

$$ {}_r\Psi_s[x] = {}_r\Psi_s \begin{bmatrix} (\gamma_1, \acute{\gamma}_1)\cdots & (\gamma_r, \acute{\gamma}_s); \\ (l_1, \acute{l}_1)\cdots & (l_r, \acute{l}_s); \end{bmatrix} x \quad (1.1)$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(\gamma_1 + \acute{\gamma}_1 k),\cdots\Gamma(\gamma_r + \acute{\gamma}_s k)x^k}{\Gamma(l_1 + \acute{l}_1 k),\cdots\Gamma(l_r + l_s k)k!} \quad (1.2)$$

$$= H_{r,s+1}^{1,r}\left[-x \middle| \begin{matrix} (1-\gamma_1, \acute{\gamma}_1)\cdots & (1-\gamma_r, \acute{\gamma}_r) \\ (0,1)(1-l_1, \acute{l}_1)\cdots & (1-l_s, \acute{l}_s) \end{matrix}\right] \quad (1.3)$$

where $H_{r,s+1}^{1,r}[z]$ is a Fox-H function [6] and the coefficients $\acute{\gamma}_1, \cdots \acute{\gamma}_r$; $\acute{l}_1 \cdots \acute{l}_s \in R^+$ such that $1+\sum_{j=1}^{s} \acute{l}_j - \sum_{i=1}^{r} \acute{\gamma}_i$ for suitable bounded value of $|x|$.

When $\acute{\gamma}_1 =, \cdots, = \acute{\gamma}_r = 1, \acute{l}_1 =, \cdots, = \acute{l}_s = 1$ in (1.1), Fox-Wright function reduce to simpler in generalized hypergeometric function ${}_rF_s$[23]

$$ {}_r\Psi_s\left[\begin{matrix} (\gamma_1, \acute{\gamma}_1), \cdots, (\gamma_r, \acute{\gamma}_s); \\ (l_1, \acute{l}_1), \cdots, (l_r, \acute{l}_s); \end{matrix} x\right] = \frac{\Gamma(\gamma)_1, \cdots, \Gamma(\gamma)_r}{\Gamma(l)_1, \cdots, \Gamma(l)_s} {}_rF_s(\gamma_1, \cdots, \gamma_r; l_1, \cdots, l_r; x). \quad (1.4)$$

Now we introduce and studies the extension of generalized multi-index Bessel-Maitland function, is called extended multi-index Bessel-Maitland function as follows:

Definition. Let $\alpha_j, \beta_j, \gamma, b, c \in \mathbb{C}, (j = 1,2,..,m) be\ such\ that\ \sum_{j=1}^{m} \mathbb{R}(\alpha_j) > \max\{0; \mathbb{R}(k) - 1\}; k > 0,\ \mathbb{R}(\beta_j) > 0, \mathbb{R}(\gamma) > 0$ then,

$$\mathbb{J}_{(\beta_j)_m,\ k,\ \ b}^{(\alpha_j)_m,\ \gamma,\ \ c}(z) = \sum_{n=0}^{\infty} \frac{c^n(\gamma)_{km}}{\prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j n + \frac{1+b}{2}\right)} \frac{z^n}{n!},, \qquad (m \in \mathrm{N}). \quad (1.5)$$

where $(\gamma)_n$ is the Pochhammer symbol defined as:

$$(\gamma)_n = \begin{cases} 1 & ; n = 0 \\ \gamma(\gamma+1), \dots, (\gamma+n-1); & n \in \mathbb{N} \end{cases}$$

$$= \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$$

Special Cases: The following special cases of the multi-index Bessel-Maitland function

$\mathbb{J}_{(\beta_j)_m, k,\ b}^{(\alpha_j)_m, \gamma,\ c}(z)$ are given according to their particular values of the parameters $\alpha_j, \beta_j, \gamma, k, b, c$ (j = 1,2,$\cdots$,m)

(i) On setting $b = c = 1$ and $z \to -z$ in (1.5), then we get generalized multi-index Bessel function where $\alpha_j, \beta_j, \gamma, b, c \in \mathbb{C}, (j = 1,2,..,m), \sum_{j=1}^{m} \mathbb{R}(\alpha_j) > \max\{0; \mathbb{R}(k) - 1\}; k > 0,\ \mathbb{R}(\beta) > -1, \mathbb{R}(\gamma) > 0\}$,

$$\mathbb{J}_{(\beta_j)_m\ \ k,\ \ 1}^{(\alpha_j)_m,\ \gamma,\ \ 1}(z) = \mathbb{J}_{(\beta_j)_m,\ \ k}^{(\alpha_j)_m,\ \ \gamma}(z) \quad (1.6)$$

(ii) On setting $b = c = 1$ in (1.5) then we get generalized multi-index Mittag-Laffler function [17] which is defined as:

$$\mathbb{J}^{(\alpha_j)_m, \ \gamma, \ \ 1}_{(\beta_j)_m \ \ k, \ \ 1}(\mathrm{z}) = \mathbb{E}^{\gamma,k}_{(\alpha_j,\beta_j)^m_{j=1}} \tag{1.7}$$

where$(\alpha_j, \beta_j, \gamma, k, z \in \mathbb{C}, \mathbb{R}\left(\beta_j > 0\right), \mathbb{R}(\sum_{j=1}^{m} \alpha_j) > \max\{0, \mathbb{R}(k) - 1\}$

(iii) Let m = c = 1, b = -1, then multi-indices Bessel function reduces to Srivastava and Tomovski functions [18].

$$\mathbb{J}^{(\alpha, \ \gamma, \ 1)}_{(\beta, \ k, \ 1)}(\mathrm{z}) = E^{\gamma,k}_{\alpha,\beta}(z) \tag{1.8}$$

(iv) Let $m = \alpha = 1, k = 0, \beta_1 = v$ and $z \to \frac{z^2}{4}$ in (1.5), then multi-index Bessel-Maitland function reduces to the Bessel function of the first kind [20]

$$\mathbb{J}^{1,\gamma}_{\nu,0}\left(\frac{z^2}{4}\right) = \left(\frac{2}{z}\right)\mathbb{J}_\nu(z) \tag{1.9}$$

Definition. The classical Caputo fractional derivative operator which is defined by Kilmaz et al. [11].

$$D^\mu_z\left\{f(z)\right\} = \frac{1}{\Gamma(m-\mu)}\int_0^z (z-t)^{m-\mu-1}\frac{d^m}{dt^m}f(t)dt, \tag{1.10}$$

where $m - 1 < \mathbb{R}(u) < m, (m = 1,2,\cdots)$ and $\mathbb{R}(p) > 0$

Definition The extended Caputo fractional derivative is defined as:

$$D^\mu_z\left\{f(z)\right\} = \frac{1}{\Gamma(m-\mu)}\int_0^z (z-t)^{m-\mu-1} exp\left(\frac{-p}{t(1-t)}\right)\frac{d^m}{dt^m}f(t)dt, \tag{1.11}$$

where $m - 1 < \mathbb{R}(u) < m$ $(m = 1,2,\cdots)$ and $\mathbb{R}(p) > 0, v > 0$

Definition. The extension of extended Caputo fractional derivative operator is defined as:

$$D^\mu_z\left\{f(z):p,\nu\right\} = \sqrt{\frac{2pz^2}{\pi}}\frac{1}{\Gamma(m-\mu)}\int_0^z t^{-\frac{1}{2}}(z-t)^{m-\mu-\frac{3}{2}}K_{\nu+\frac{1}{2}}\left(\frac{pz^2}{t(z-t)}\right)\frac{d^m}{dt^m}f(t)dt, \tag{1.12}$$

where *$m - 1 < \mathbb{R}(u) < m, (m = 1,2,\cdots)$ and $\mathbb{R}(p) > 0, v > 0$.*

Lemma. The following formula hold true for $m - 1 < R(\mu)$ and $R(\mu) < R(\eta), R(\mu) > 0$

$$\mathcal{D}^\mu_z\{z^n; p, v\} = .\frac{\Gamma(\eta+1)\beta_v(\eta-m+1, m-\mu; p)}{\Gamma(\eta-\mu+1)\beta(\eta-m+1, m-\mu)}\, z^{n-\mu}. \tag{1.13}$$

Recently pathway fractional integral operator involving the various special function have been considered by many authors (see for references [11],[14],[19],[20]).

Definition Let $f(x)$ is Lebesgue measurable function and for $\eta \in \mathbb{C}$, $\mathbb{R}(\eta) > 0, d > 0$ and pathway parameter $\lambda < 1$ then the pathway fractional integration operator [14] is defined as:

$$(\mathrm{P}^{\eta,\lambda,d}_{a^+}f)(x) = x^n\int_0^{\left[\frac{x}{d(1-\lambda)}\right]}\left[1 - \frac{d(1-\lambda)t}{x}\right]^{\frac{x}{(1-\lambda)}} f(t)dt \tag{1.14}$$

Let $[a,b] \subseteq \mathbb{R}$, The left sided and right sided Riemann-Liouville integral $I^\eta_{a^+}f$ and $I^\eta_{b_-}f$ of order $\eta \in \mathbb{C}$ $(\mathbb{R}, (\eta) > 0)$ are defined respectively by,

$$(\mathrm{I}^\eta_{a^+}f)(x) = \frac{1}{\Gamma(n)}\int_a^x \frac{f(t)}{(x-t)^{1-\eta}}dt(x > a, \mathbb{R}(\eta) > 0) \tag{1.15}$$

and

$$(\mathrm{I}_{b^-}^{\eta} f)(x) = \frac{1}{\Gamma(\eta)} \int_x^b \frac{f(t)}{(t-x)^{1-\eta}} dt (x < b, \mathbb{R}(\eta) > 0) \tag{1.16}$$

Note: On setting α = 0,a = 1 and $\eta \rightarrow \eta - 1$, $\mathbb{R}(\eta) > 0$, then the pathway fractional integration operator (1.14) reduces to left-sided Riemann-Liouville fractional integral given as follows:

$$(\mathrm{P}_{0_+}^{\eta-1,0,1} f)(t) = \Gamma(\eta)(\mathrm{I}_{0_+}^{\eta} f)(t) \tag{1.17}$$

For $0 \le a < t < b \le \infty$, $\mathbb{R}(\eta) > 0, \sigma > 0, \alpha \in \mathbb{C}$, one of the Erdelyi-Kober type fractional integral operator 2[7] defined as:

$$(\mathrm{I}_{0^+,\sigma,\alpha}^{\eta} f(t)) = \frac{\sigma(t)^{-\sigma(\eta+\alpha)}}{\Gamma(\eta)} \int_a^t \frac{\tau^{\sigma\alpha+\sigma-1} f(\tau) d\tau}{(t^{\sigma-}\tau^{\sigma})^{1-\eta}} \tag{1.18}$$

Pathway fractional integration operator is closely related to Erdelyi-Kober operator which is given as:

$$(\mathrm{P}_{0_+}^{\eta-1,0,1} f)(t) = \Gamma(\eta) t^{\eta} (\mathrm{I}_{0_{+,1,0}}^{\eta} f)(t) \ (\mathbb{R} > 0) \tag{1.19}$$

Now on setting $f(t) = t^{\beta-1}$ in (1.14), we get the following relation.

Lemma. Let $\beta, \eta \in \mathbb{C}, \mathbb{R}(\eta), \mathbb{R}(\beta) > 0$, α <1, $\mathbb{R}\left(\frac{\eta}{1-\alpha}\right) > -1$ then we have the following result:

$$\left\{p_{0^+}^{(\eta,\alpha)} \, t^{\beta-1}\right\}(x) = \frac{t^{\eta+\beta}\Gamma(\beta)\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\beta}\Gamma(1+\left(\frac{\eta}{1-\alpha}\right)+\beta)} \tag{1.20}$$

Lemma. Let $\beta, \eta \in \mathbb{C}, \mathbb{R}(\eta), \mathbb{R}(\beta) > 0$, $\lambda < 1$, $\mathbb{R}\left(\frac{\eta}{1-\alpha}\right) > -1$,then we have the following result:

$$\left\{p_{0^+}^{(\eta,\lambda,d)} \, t^{\beta-1}\right\}(x) = \frac{x^{\eta+\beta}\Gamma(\beta)\Gamma(1+\frac{\eta}{1-\lambda})}{[d(1-\lambda)]^{\beta}\Gamma(1+\left(\frac{\eta}{1-\lambda}\right)+\beta)}. \tag{1.21}$$

2. Pathway fractional integral of multi-index Bessel-Maitland function.

In this section we consider composition of pathway fractional integration operator given by (1.14) with the multi-index Bessel- Maitland function (1.5) and obtained our result in term of Fox-Wright function (1.1). Also, we discuss some related and useful corollaries by using some suitable parameters.

Theorem 2.1. *Let* $\alpha_j, \beta_j (j = 1,2, \ldots m)\gamma, \eta, b, c \in \mathbb{C}, \alpha < 1, \mathbb{R}(\eta) > 0, \mathbb{R}(\beta) > 0, \mathbb{R}\left(\frac{\eta}{1-\alpha}\right) > -1$*,*

(j = 1,2,⋯,m) be such that $\sum_{j=1}^{m} \mathbb{R}(\alpha_j) > max\{0; \mathbb{R}(k) - 1\}; k > 0$, $\mathbb{R}(\beta_j) > 0$, $\mathbb{R}(\gamma) > 0$*, then following relation holds:*

$$\left\{p_{0^+}^{(\eta,\alpha,d)} \left(t^{\beta-1} \, \mathbb{J}_{(\beta_j)_m,q,b}^{(\alpha_j)_m,\tau,c}(zt^v)\right)\right\}(u) = \frac{u^{\eta+\beta}\Gamma(1+\frac{\eta}{1-\alpha})}{\Gamma(\gamma)[d(1-\alpha)]^{\beta}} \times$$

$$ {}_2\psi_{m+1}\left[\begin{matrix} (\gamma, k) & & (\beta, v) \\ \left(\beta_j + \frac{b+1}{2}, \alpha_j\right)_{j=1}^{m} & & (1 + \frac{\eta}{1-\alpha} + \beta \, , v) \end{matrix} \Big| \, \frac{zuc}{d(1-\alpha)}\right]. \tag{2.1}$$

Proof. Let us denote left-hand side of (2.1) by $\mathbb{I}$ Applying the definition of (1.5) and replace $z \rightarrow zt^v$, we get

$$\mathbb{I} = \left\{p_{0^+}^{(\eta,\alpha,d)}\ (t^{\beta-1}\,\mathbb{J}_{(\beta_j)_m,q,b}^{(\alpha_j)_{m,\tau,c}}(zt^v)\right\}(u)$$

$$\mathbb{I} = \sum_{n=0}^{\infty} \frac{c^n(\gamma)kn}{\prod_{j=1}^{m}\Gamma\left(\alpha_j n + \beta_j n + \frac{b+1}{2}\right)}\frac{z^n}{n!}\left\{p_{0^+}^{(\eta,\alpha,d)}\ (t^{\beta+nv-1}\right\}$$

After simplifying and using (1.21), we get

$$\mathbb{I} = \frac{u^{\eta+\beta}\Gamma(1+\frac{\eta}{1-\alpha})}{\Gamma(\gamma)[d(1-\alpha)]^{\beta}}\sum_{n=0}^{\infty}\frac{c^n(\gamma)_{kn}\Gamma(\beta+n)}{\prod_{j=1}^{m}\Gamma\left(\alpha_j n+\beta_j n+\frac{b+1}{2}\right)\Gamma(1+\frac{\eta}{1-\alpha}+\beta+n)}\frac{z^n}{n!}\left(\frac{cu}{[d(1-\alpha)]}\right)^n$$

After using the definition of (1.1) we get our result. This completes the proof of theorem.

Corollary 2.1. *Let* $\eta, \alpha_j, \beta_j\gamma, b, c \in \mathbb{C}, \alpha < 1, d = 1, \mathbb{R}(\beta), \mathbb{R}(\gamma) > 0, \mathbb{R}\left(\frac{\eta}{1-\alpha}\right) > -1$*, be such that* $\mathbb{R}(\beta_j) > 0$ *, then following relation hold:*

$$\left\{p_{0^+}^{(\eta,\alpha)}\ (t^{\beta-1}\,\mathbb{J}_{(\beta_j)_m,q,b}^{(\alpha_j)_{m,\tau,c}}(zt^v)\right\}(u) = \frac{u^{\eta+\beta}\Gamma(1+\frac{\eta}{1-\alpha})}{\Gamma(\gamma)[(1-\alpha)]^{\beta}}\times$$
$${}_2\psi_{m+1}\left[\begin{matrix}(\gamma,k) & (\beta,v) \\ \left(\beta_j+\frac{b+1}{2},\alpha_j\right)_{j=1}^{m} & (1+\frac{\eta}{1-\alpha}+\beta\,,v)\end{matrix}\,\Big|\,\frac{zuc}{(1-\alpha)}\right].$$

Corollary 2.2. *Let* $c = m = 1, b = -1\ and\ \eta, \alpha, \beta, \gamma \in \mathbb{C}, \mathbb{R}(\eta), \mathbb{R}(\beta), \mathbb{R}(\gamma), > 0, \alpha < 1$*,* $\mathbb{R}\left(\frac{\eta}{1-\alpha}\right) > -1$ *,then following relation hold:*

$$\left\{p_{0^+}^{(\eta,\alpha,d)}\ (t^{\beta-1}\,\mathbb{E}_{(\alpha,\beta)}^{(\gamma,k)}(zt^v)\right\}(u)$$
$$= \frac{u^{\eta+\beta}\Gamma(1+\frac{\eta}{1-\alpha})}{\Gamma(\gamma)[d(1-\alpha)]^{\beta}}\times {}_2\psi_1\left[\begin{matrix}(\gamma,k) & (\beta,v) \\ (\beta+\frac{b+1}{2},\alpha) & (1+\frac{\eta}{1-\alpha}+\beta\,,v)\end{matrix}\,\Big|\,\frac{zu}{d(1-\alpha)}\right]$$

Corollary 2.3. *Let* $c = 1, b = -1\ and\ \eta, \alpha_j, \beta_j\gamma, b, c \in \mathbb{C}, \mathbb{R}(\eta), \mathbb{R}(\beta), \mathbb{R}(\gamma), > 0, \alpha < 1$*,* $\mathbb{R}\left(\frac{\eta}{1-\alpha}\right) > -1$*,* $\mathbb{R}(\beta_j) > 0$ *then following relation hold:*

$$\left\{p_{0^+}^{(\eta,\alpha,d)}\ (t^{\beta-1}\mathbb{E}_{(\beta_j)_{m,k}}^{(\alpha_j)_{m,\gamma}}\ (zt^v)\right\}(u)$$
$$= \frac{u^{\eta+\beta}\Gamma(1+\frac{\eta}{1-\alpha})}{\Gamma(\gamma)[d(1-\alpha)]^{\beta}}\times {}_2\psi_{m+1}\left[\begin{matrix}(\gamma,k) & (\beta,v) \\ \left(\beta_j,\alpha_j\right)_{j=1}^{m} & (1+\frac{\eta}{1-\alpha}+\beta\,,v)\end{matrix}\,\Big|\,\frac{zu}{d(1-\alpha)}\right]$$

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Corollary 2.4. *Let* $b = c = 1\ and\ z \to -z, \eta, \alpha_j, \beta_j\gamma, b, c \in \mathbb{C}, \mathbb{R}(\eta)\mathbb{R}(\beta)\mathbb{R}(\gamma), > 0, \alpha < 1$*,* $\mathbb{R}\left(\frac{\eta}{1-\alpha}\right) > -1$*,* $\mathbb{R}(\beta_j) > 0$*, then following relation hold:*

$$\left\{p_{0^+}^{(\eta,\alpha,d)}\ (t^{\beta-1}\mathbb{J}_{(\beta_j)_{m,q}}^{(\alpha_j)_{m,\tau}}\ (zt^v)\right\}(u) = \frac{u^{\eta+\beta}\Gamma(1+\frac{\eta}{1-\alpha})}{\Gamma(\gamma)[d(1-\alpha)]^{\beta}}\times$$
$${}_2\psi_{m+1}\left[\begin{matrix}(\gamma,k) & (\beta,v) \\ \left(\beta_j+1,\ \alpha_j\right)_{j=1}^{m} & (1+\frac{\eta}{1-\alpha}+\beta\,,v)\end{matrix}\,\Big|\,\frac{zu}{d(1-\alpha)}\right].$$

Corollary 2.5. *Let* $\alpha = 0, a = 1, \eta \to \eta - 1, \alpha_j, \beta_j, b, c \in \mathbb{C}, \mathbb{R}(\gamma), \mathbb{R}(\eta) > 0$, *then following relation hold:*

$$\left\{p_{0^+}^{(\eta,0,1)} \left(t^{\beta-1}\mathbb{J}_{(\beta_j)_{m,q,b}}^{(\alpha_j)_{m,\tau,c}} (zt^v)\right\}(u) = \frac{u^{\eta+\beta-1}\Gamma(\eta)}{\Gamma(\gamma)} \times {}_2\psi_{m+1}\left[\begin{matrix}(\gamma,k) & (\beta,v) \\ \left(\beta_j + \frac{b+1}{2}, \alpha_j\right)_{j=1}^m & (\eta+\beta, v)\end{matrix} \middle| zu\right].$$

Corollary2.6 *Let* $\gamma, k = 1, \alpha_j \to \frac{1}{\alpha_j} (j = 1,2, \dots\dots, m), \eta, \alpha \in \mathbb{C}, \mathbb{R}(\gamma), \mathbb{R}(\beta) > 0$

, then following relation holds:

$$\left\{p_{0^+}^{(\eta,\alpha,d)} \left(t^{\beta-1}\mathbb{E}_{(\beta_j)_m}^{(\frac{1}{\alpha_j})_m} (zt^v)\right\}(u) = \frac{u^{\eta+\beta}\Gamma(1+\frac{\eta}{1-\alpha})}{\Gamma(\gamma)[d(1-\alpha)]^\beta} \times {}_2\psi_{m+1}\left[\begin{matrix}(1,1) & (\beta,v) \\ \left(\beta_j \ \frac{1}{\alpha_j}\right)_{j=1}^m & (1+\frac{\eta}{1-\alpha}+\beta, v)\end{matrix} \middle| \frac{zu}{d(1-\alpha)}\right].$$

Corollary 2.7. *Let* $b = c = m = 1, k = 0, \alpha_1 = 1, \beta_1 = v$ *and* $z \to \frac{z^2}{4}, \mathbb{R}(\gamma) > 0, \ \alpha < 1, \beta \in \mathbb{C}$ *then following relation hold:*

$$\left\{p_{0^+}^{(\eta,\alpha,d)} \left(t^{\beta-1}\left(\frac{2}{z}\right)^v\right)\mathbb{J}_v(zt^v)\right\}(u) = \frac{u^{\eta+\beta}\Gamma(1+\frac{\eta}{1-\alpha})}{\Gamma(\gamma)[d(1-\alpha)]^\beta} \times {}_2\psi_{m+1}\left[\begin{matrix}(\gamma,0) & (\beta,v) \\ (\beta+1,\alpha) & (1+\frac{\eta}{1-\alpha}+\beta, v)\end{matrix} \middle| \frac{zu}{d(1-\alpha)}\right].$$

Caputo Fractional differential operator of multi-index Bessel-Maitland function

In this section we consider composition of Caputo fractional operator given by (1.12) and (1.13) with the multi-index Bessel Maitland function (1.5) and obtained our result in term of Fox-Wright function (1.1). Also, we discuss some useful result on these as given in corollaries by using some particular values of the parameters.

Theorem 3.1 Let $\mathbb{R}(\mu), \mathbb{R}(\gamma) > 0, \alpha_j, \beta_j (j = 1,2, \dots, m), b, c \in \mathbb{C}, m - 1 < \mathbb{R}(\mu) < m, \mathbb{R}(\mu) < \mathbb{R}(\eta), \mathbb{R}(\gamma) > 0, \mathbb{R}(m-\mu) > 0$, then following relation hold:

$$D_z^\mu\left\{\left(\mathbb{J}_{(\beta_j)_{m,q,b}}^{(\alpha_j)_{m,\tau,c}} (z,p,v)\right)\right\}(u) = \frac{\beta_v(n-m+1, m-\mu; p)}{\Gamma(\gamma)\Gamma(m-\mu)} \times {}_2\psi_{r+1}\left[\begin{matrix}(\gamma,k) & (1,1) \\ \left(\beta_j + \frac{b+1}{2}, \alpha_j\right)_{j=1}^r & (1-m, 1)\end{matrix} \middle| z^{1-\mu}\right] \tag{3.1}$$

Proof. Let us denote left-hand side of (3.1) by I and applying the definition of (1.5), we get

$$\mathbb{I} = D_z^\mu\left\{\left(\mathbb{J}_{(\beta_j)_{m,q,b}}^{(\alpha_j)_{m,\tau,c}} (z,p,v)\right)\right\}(u)$$

$$\mathbb{I} = \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{kn}}{\prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j n + \frac{b+1}{2}\right) n!} \{D_z^{\mu} \, (z^n; p, v)\}$$

Simplifying the above and using the result of (1.13), we get

$$\mathbb{I} = \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{c^n \Gamma(\gamma+kn)\Gamma(\beta+n)\Gamma(n+1) B_v(n-m+1, m-\mu; p)}{\Gamma\left(\alpha_j n+\beta_j n+\frac{b+1}{2}\right)\Gamma(n-m+1)\Gamma(m-\mu)} \frac{z^{n-\mu}}{n!}$$

In view of the definition (1.1) we get required result. This completes the proof of theorem.

Corollary 3.1. *Let* $p = 1, \mathbb{R}(\mu) > 0, \alpha_j, \beta_j (j = 1,2, \dots \dots, m), b, c \in \mathbb{C}, m-1 < \mathbb{R}(\mu) < m, \mathbb{R}(\mu) < \mathbb{R}(\eta), \mathbb{R}(\gamma) > 0, \mathbb{R}(m-\mu) > 0$, *then following relation hold.*

$$D_z^{\mu}\left\{\left(\mathbb{J}_{(\beta_j)_{m,q,b}}^{(\alpha_j)_{m,\tau,c}} \, (z, v)\right)\right\}(u) = \frac{1}{\Gamma(\gamma)} \times {}_2\psi_{r+1}\left[\begin{matrix} (\gamma, k) & (1-m, 1) \\ \left(\beta_j + \frac{b+1}{2}, \; \alpha_j\right)_{j=1}^{r} & (m\,, 1) \end{matrix} \middle| z^{1-\mu}\right].$$

Corollary 3.2. *Let* $b = 1, c = 1, \alpha_j, \beta_j (j = 1,2, \dots \dots, m), b, c \in \mathbb{C}, m-1 < \mathbb{R}(\mu) < m, \mathbb{R}(\mu) < \mathbb{R}(\eta), \mathbb{R}(\gamma) > 0, \mathbb{R}(\mu) > 0, \mathbb{R}(m-\mu) > 0$, *then following relation hold.*

$$D_z^{\mu}\left\{\left(\mathbb{E}_{(\beta_j)_{m,k}}^{(\alpha_j)_{m,\gamma}} \, (z, p, v)\right)\right\}(u)$$
$$= \frac{\beta_v(n-m+1, m-\mu; p)}{\Gamma(\gamma)\Gamma(m-\mu)} \times {}_2\psi_{r+1}\left[\begin{matrix} (\gamma, k) & (1,1) \\ (\beta_j, \alpha_j)_{j=1}^{r} & (1-m\,, 1) \end{matrix} \middle| z^{1-\mu}\right]$$

Corollary 3.3. Let $b = -1, c = 1, m = 1, \mathbb{R}(\mu) > 0, \alpha_j, \beta_j (j = 1,2, \dots \dots, m), b, c \in \mathbb{C}, m-1 < \mathbb{R}(\mu) < m, \mathbb{R}(\mu) < \mathbb{R}(\eta), \mathbb{R}(\gamma) > 0, \mathbb{R}(m-\mu) > 0$, *then following relation hold.*

$$D_z^{\mu}\left\{\left(\mathbb{E}_{(\beta,k)}^{(\alpha,\gamma)}(z, p, v)\right)\right\}(u) = \frac{\beta_v(n-m+1, m-\mu; p)}{\Gamma(\gamma)\Gamma(m-\mu)} \times {}_2\psi_2\left[\begin{matrix} (\gamma, k) & (1,1) \\ (\beta, \alpha) & (1-m\,, 1) \end{matrix} \middle| z^{1-\mu}\right]$$

Corollary 3.4. *Let* $b = c = 1, z \to -z, \mathbb{R}(\mu) > 0, \alpha_j, \beta_j (j = 1,2, \dots \dots, m), b, c \in \mathbb{C}, m-1 < \mathbb{R}(\mu) < m, \mathbb{R}(\mu) < \mathbb{R}(\eta), \mathbb{R}(\gamma) > 0, \mathbb{R}(m-\mu) > 0$, *then following relation hold.*

$$D_z^{\mu}\left\{\left(\mathbb{J}_{(\beta_j)_{m,q}}^{(\alpha_j)_{m,\tau}} \, (z, p, v)\right)\right\}(u)$$
$$= \frac{\beta_v(n-m+1, m-\mu; p)}{\Gamma(\gamma)\Gamma(m-\mu)} \times {}_2\psi_{r+1}\left[\begin{matrix} (\gamma, k) & (1,1) \\ (\beta_j, \alpha_j)_{j=1}^{r} & (1-m\,, 1) \end{matrix} \middle| z^{1-\mu}\right]$$

Corollary 3.5. *Let* $\gamma = k = 1, \alpha_j \to \frac{1}{\alpha_j}, (j = 1,2, \dots \dots, m), \mathbb{R}(\mu) > 0, \mathbb{R}(\gamma) > 0, \alpha_j, \beta_j \in \mathbb{C}, m-1 < \mathbb{R}(\mu) < m, \mathbb{R}(\mu) < \mathbb{R}(\eta), \mathbb{R}(\gamma) > 0, \mathbb{R}(m-\mu) > 0$, *then following relation holds:*

$$D_z^{\mu}\left\{\left(\mathbb{E}_{(\beta_j)_m, \; 1}^{\left(\frac{1}{\alpha_j}\right)_m, \; 1}(z, p, v)\right)\right\}(u)$$
$$= \frac{\beta_v(n-m+1, m-\mu; p)}{\Gamma(m-\mu)} \times {}_2\psi_{r+1}\left[\begin{matrix} (1,1) & (1,1) \\ \left(\beta_j + \frac{b+1}{2}, \alpha_j\right)_{j=1}^{r} & (m\,, 1) \end{matrix} \middle| z^{1-\mu}\right]$$

Remark: On setting $\gamma = k = 1, \alpha_j \to \frac{1}{\alpha_j}, (j = 1,2, \dots\dots, m), \mathbb{R}(\mu), \mathbb{R}(\gamma) > 0, \alpha_j, \beta_j \in \mathbb{C}, m-1 < \mathbb{R}(\mu) < m, \mathbb{R}(\mu) < \mathbb{R}(\eta), \mathbb{R}(\gamma) > 0, \mathbb{R}(m-\mu) > 0,$

then following relation hold:

$$D_z^{\mu}\left\{\left(\mathbb{E}_{(\beta_j)_m,\ 1}^{\left(\frac{1}{\alpha_j}\right)_m,\ 1}(z,v)\right)\right\}(u) = \frac{1}{\Gamma(\gamma)} \times {}_2\psi_{r+1}\left[\begin{matrix} (\gamma, k) & (1-m, 1) \\ \left(\beta_j + \frac{b+1}{2},\ \alpha_j\right)_{j=1}^{r} & (m, 1) \end{matrix} \middle| z^{1-\mu}\right].$$

Corollary 3.6. *Let* $\beta, b, c \in \mathbb{C}, \alpha < 1, \mathbb{R}(\gamma) > 0, b = c = m = \alpha_1 = 1, k = 0, \beta_1 = v$ *and* $z \to \frac{z^2}{4}$

then following relation hold:

$$\left\{D_z^{\mu}\ \left(t^{\beta-1}\left(\frac{2}{z}\right)^{v}\right)\mathbb{J}_v(zt^v)\ \right\}(u) = \frac{u^{\eta+\beta}\Gamma(1+\frac{\eta}{1-\alpha})}{\Gamma(\gamma)[d(1-\alpha)]^{\beta}} \times$$
$${}_2\psi_{m+1}\left[\begin{matrix} (\gamma, 0) & (\beta, v) \\ (\beta+1, \alpha) & (1+\frac{\eta}{1-\alpha}+\beta\ , v) \end{matrix} \middle| \frac{zu}{d(1-\alpha)}\right].$$

Concluding Remark: We conclude in this investigation by remarking that the result obtained hare are in general in character and useful in deriving various integral formulas in the theory of pathway fractional integral formula. In this paper we have presented composition formula of the pathway fractional integration and Caputo fractional integration operator in form of Fox-Wright function. We can also write these result in form of various functions like as Fox H-function, Merger G-function etc.

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