Vol.11.Issue.1.2023 (Jan-March)
©KY PUBLICATIONS

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com

Email:editorbomsr@gmail.com

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

RESEARCH ARTICLE

Pseudo Regular I-spaces and Pseudo Regular U-spaces

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DOI:10.33329/bomsr.11.1.18

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ABSTRACT

In this paper Pseudo- regular I- spaces and Pseudo -regular U- spaces have been defined and a few important properties of these spaces have been proved.

Keywords: Pseudo regular I-spaces, Pseudo regular U-spaces

Mathematics Subject Classification: 54D10, 54A40, 54G12, 54D15, 54A05, 54C08.

I. INTRODUCTION

A topological space X is said to be **regular** if for each non- empty closed set F of X and any point $x \in X$, such that $x \notin F$ (i.e. x is the external point of F), there exist two disjoint open sets V and W such that $x \in V$ and $F \subseteq W$. In paper [1] a pseudo regular topological spaces has been defined by replacing a closed subset F by a compact subset K in the definition of a regular space. Here the concept of Pseudo regularity has been extended to I- spaces and U- spaces. These spaces were introduced and studied in [2] and [3].In this paper we have given examples of pseudo regular I- spaces and pseudo regular U – spaces which are not regular. A number of important theorems regarding these spaces have been established.

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II. PRELIMINARIES

We begin with some basic definitions and examples related to I- spaces and U- spaces.

I-Spaces

Definition 2.1: Let X be a non- empty set. A collection I of subsets of X is called an **I- structure** on X if

(i) $X, \Phi \in$ I and (ii) $G_1, G_2, G_3, \cdots\cdots, G_n \in$ I implies $G_1 \cap G_2 \cap G_3 \cap \cdots\cdots \cap G_n \in$ I . Then (X, I) is called an **I-space**.

The members of I are called **I-open** set and the complement of I- open set is called **I- closed** set.

Example 2.1: Let X = {a, b, c, d}, **I** = {X, Φ, {a},{b},{a, b}, {c},{a, c}, {b, c, d}}.

Here (X, I) is a I- space but not a topological space and nor a U- space.

Example 2.2: Let X = R, I = Finite unions of the sets in C , where C = $\{R, \Phi\} \cup \{[a,b] | a, b \in R\}$

Then I is an I- structure on R. **Thus (R, I) is an I- space**.

Definition 2.2: Let (X, **I**) be an I – space. An I- **open cover** of subset K is a collection {G_α } of I- open sets such that K $\subseteq \bigcup_\alpha G_\alpha$.

Definition 2.3: An I-space X is said to be **I- compact** if for every I-open cover of X has a finite sub-cover.

A subset K of a I- space X is said to be **I- compact** if every I-open cover of K has finite sub- cover.

Thus, if (X, **I**) be an I- space, and A $\subseteq$X, then A is said to be **I- compact** if for each $\{I_\alpha | I_\alpha \in$ **I** } such that A

$\subseteq \bigcup_\alpha I_\alpha$ *, there exit* $I_{\alpha_1}, I_{\alpha_2}, \cdots\cdots, I_{\alpha_n}$ *such that* $A \subseteq I_{\alpha_1} \cup I_{\alpha_2} \cup \cdots\cdots \cup I_{\alpha_n}$, *for some* $n \in N$.

Example 2.3: K is I- compact if K contains only intersections of the form [a, a] = {a}, and these must be finite in number. Thus K must be a finite set.

For, let K be a compact subset of R, for some n$\in$N and x$\in$R with x$\notin$K.

We know that I = $\{R, \Phi\} \cup \{[a,b] | a, b \in R\}$. Since K is I- compact,

$K = \{x_1, x_2, \cdots\cdots, x_n\}$, *where* $x_i \neq x_j, \forall i, j,$ *for some positive integer n.*

If $K \subseteq R, K \supseteq [a,b]$, *for some* $a, b \in R, a < b$, then K is not compact, since [a, b] is not so.

U-Spaces

Definition 2.4 A U-**structure** on a nonempty set X is a collection **U** of subsets of X having the following properties:

(i) Φ and X are in **U** ,

(ii) Any union of members of **U** is in **U**.

The ordered pair (X, **U**) is called a **U-space**. A U-space which is not a topological space is called a **proper U-space**. The members of **U** are called **U-open set** and the complement of a U-open set is called a **U- closed set**.

A U- structure and a U-space have been called a supra-topology and a supra-topological space respectively by some authors (see [4], [5], [6], [7])

In general we have

Topological space $\Rightarrow$ U-space, $\not\Leftarrow$ Topological space $\Leftarrow$ U-space

Example 2.4: Let X = {a, b, c, d}, U = {X, Φ, {a, b}, {a, c}, {a, b, c}}. **Here (X, U) is a U-space but not a topological space.**

Definition 2.5: A U-space X is said to be **U- compact** if for every U-open cover of X has a finite sub-cover.
A subset K of a U- space X is said to be **U- compact** if every U-open cover of K has finite sub- cover.

Example 2.5: Let X = **N**, U = {**2N, 4N, 8N, 16N, ……. 2^n N, ……,N**, Φ}. Then X is U- compact.
Let $\Phi \neq$ A $\subseteq$X and **C** be a U open cover of A. Let n_0 be smallest +ve integer such that **2^{n_0} N**$\in$ **C.**
Then A$\subseteq$**2^{n_0} N.** So {**2^{n_0} N**} is a finite sub-cover of **C.** Therefore any subset K of X, K$\neq$ X is U- compact.

Definition 2.6: A U- space X is called **U- regular space** if for any U- closed set F of X and any point x$\in$ X, such that x$\notin$F (i.e. x is the external point of F) there exist two disjoint U-open sets G and H such that x $\in$G and F$\subseteq$H.

For a U- space, 'Hausdorff' and regular are independent concepts.

Example of a U- space which is regular but not U-Hausdorff space

Example 2.6: Let X = {a, b, c, d}, U = {X, ϕ, {a},{d},{a, d},{a, b, c}, {b, c, d}}. Then (X, U) is a U-space. Here the U- closed sets are X, ϕ, {a},{d},{b, c},{a, b, c},{b, c, d}. We easily sec that X is a U- regular space but it is not U- Hausdorff space, since b and c cannot be separated by disjoint U- open sets. Also X is not a topological space.

III. PSEUDO REGULAR I - SPACES

Definition 3.1: An I- space X is said to be **pseudo regular** if for every I- compact subset K of X and for every x$\in$ X, x$\notin$K, there exist two I-open sets $I_1, I_2 \in$ **I** with $K \subseteq I_1, x \in I_2,\ I_1 \cap I_2 = \Phi$.

Example 3.1: (**Example of an I-pseudo regular space**)
Let $[x_1-\alpha_1, x_1+\alpha_1], [x_2-\alpha_2, x_2+\alpha_2], \cdots\cdots\cdots, [x_n-\alpha_n, x_n+\alpha_n]$ be n closed intervals with each $\alpha_l < \frac{1}{2}|x_i - x_j|$ and $\alpha_l < |x - x_i|$ for each I, j, l.
Let

$$\alpha = \frac{1}{2}\min\{\alpha_1, \alpha_2, \cdots\cdots \alpha_n\}. \ Then [x-\alpha, x+\alpha] \cap ([x_1-\alpha_1, x_1+\alpha_1] \cup \cdots\cdots \cup [x_n - \alpha_n\ x_n + \alpha_n]) = \Phi$$

. Now $x \in [x-\alpha, x+\alpha], K \subseteq [x_1-\alpha_1, x_1+\alpha_1] \cup \cdots\cdots \cup [x_n-\alpha_n, x_n+\alpha_n]$. Then **(R, I) is pseudo regular**.

Example 3.2: **(X, I) is I-pseudo regular but not I-regular**.

Proof: We first show that for a < b, **[a, b] is not I-compact**.
For this, we note that $\left\{\left[a-\frac{1}{n}, b\right] \middle| n \in N\right\} \cup \{[a, a]\}$ is an I- cover of [a, b], since
$\bigcup_{n=1}^{\infty}\left[a-\frac{1}{n}, b\right] = (a, b]$. However, it does not have a finite sub-cover. **Hence [a, b] is not I-compact.**

Moreover, if K$\subseteq$ r with K$\supseteq$ [a, b], then K is not compact.

For, if $\{[c_\alpha, d_\alpha] \alpha \in A\}$ is an I- cover of K – [a, b], then

$\{[c_\alpha, d_\alpha] | \alpha \in A\} \cup \left\{\left[a - \frac{1}{n}, b \middle| n \in N\right]\right\} \cup \{[a, a]\}$ is an I-cover of K. Obviously, it does not have a finite sub-cover. **Hence K is not I- compact.**

Thus, if K is a I- compact non-empty subset of r, then K cannot contain any [a, b], where a < b.
Hence K must be finite union of sets of the form [a, a] = {a}.
Let K be a non- empty compact set in X, and let $x \in$ X with $x \notin$ K. Then, K may be written as
$K = \{x_1, x_2, \cdots\cdots, x_n\}$. *Here* $x_i \in X, \forall i,\ x_i \neq x_j, \forall i, j$. *Now let* $x \neq x_i,\ \forall i$. *Let* $\delta = \min\{|x - x_i| i = 1,2,3,\cdots\cdots n\}$
Let $x_{i_0} = \min\{x_1, x_2, \cdots\cdots, x_n\},\ x_{i_1} = \max\{x_1, x_2, \cdots\cdots, x_n\}$

Case I: *Let* $x < x_i, \forall i$. *Let* $I_1 = \left[x - \frac{\delta}{3}, x + \frac{\delta}{3}\right], I_2 = \left[x_{i_0} - \frac{\delta}{3}, x_{i_1} + \frac{\delta}{3}\right]$. *Then* $I_1, I_2 \in$ **I** and $x \in I_1, K \subseteq I_2$ *and* $I_1 \cap I_2 = \Phi$.

Case II: Let $x_i < x < x_j$, for some I and j. Let x_{i_2} the largest of the x_i's with x_{i_2} < x and let x_{i_3} be the smallest of the x_i's with x < x_{i_3}.

$$Let\ I_1 = \left[x - \frac{\delta}{3}, x + \frac{\delta}{3}\right]\ and\ I_2 = \left[x_{i_0} - \frac{\delta}{3}, x_{i_2} + \frac{\delta}{3}\right] \cup \left[x_{i_3} - \frac{\delta}{3}, x_{i_1} + \frac{\delta}{3}\right].\ Then\ I_1, I_2 \in \mathbf{I}$$

and $x \in I_1, K \subseteq I_2$ *and* $I_1 \cap I_2 = \Phi$. **Thus** (X, **I**) **is pseudo regular**.

But (X, **I**) is not regular. Because F = $(-\infty,1) \cup (2,\infty) = [1,2]^c$ is an I- closed set in X, and $\frac{1}{2} \notin F$.
Then, x and F cannot be separated by disjoint I- open sets, since the only I-open set containing F is R which also contains $\frac{1}{2}$. **Therefore (X, I) is I-pseudo regular but not I-regular**.

Example 3.3: Let X = $R \cup \{i, 2i\}$, and **I** = Usual topology **U** R *on* $\{R \cup \{i, 2i\}, X, \Phi\}$.
I- closed sets in X: $\{F \cup \{i, 2i\} | F$ *closed in R with respect to* $U_R\}$, and X, Φ.
Then, (X, I) is I- regular but not I- pseudo- regular.

Proof: Let F_1 be a non- empty closed set in X. Let $x \in X - F_1$.

(i) If F_1 = R, then $x \notin F_1 \Rightarrow x = i\ (or\ 2i)$. Then, R and {i, 2i} separate F_1 and i (or 2i).

(ii) If $F_1 = F \cup \{i, 2i\}$, for some closed set F in R , $\Phi \neq F \neq R$. *Then* $x \in R$, *and* $x \notin F$. Since R is regular w. r. t. **U** R , F and x can be separated by two disjoint I-open sets in R , and hence open in X.

(iii) If $F_1 = \{i, 2i\}$, *i.e.,* $F = \Phi$, *then* $x \notin F$, *implies* $x \in R$. So, {i, 2i} and R separate {i, 2i} and x.

(iv) If F_1 = X, i.e., F = R, then there does not exist x such that $x \notin F_1$. Hence there is no question of separately x from F_1. **Hence (X, I) is I- regular**.

Now K = [0,1] $\cup$ {i} is a I-compact subset of X, since [0,1] is a compact subset of the topological space R w. r. t. **U** R . Now, 2i $\notin$K. K and 2i cannot be separated by disjoint I- sets, since the only I- sets containing 2i are X and {I, 2i}, both of which contains i. Hence **(X, I) is not I- pseudo- regular.**

Theorem 3.1: **Every I- closed subset of a compact I- space is I-compact**.

Proof: Let F be an I- closed subset of an I- space X. Let **C** be an I-open cover of F.

Then **C**₁= **C** $\cup$ {X - F}is an I- open cover of X, since X is I- compact. **C**₁ has a finite sub-cover, say, $\{I_1, I_2, \cdots\cdots, I_n\}$. At most one of these, say, I_{n_0} = X – F. Then $\{I_1, I_2, \cdots\cdots, I_n\} - \{I_{n_0}\}$ is a finite subcover of **C**. Hence F is I- compact.

Theorem 3.2: **Every I- compact I- pseudo- regular I- space is I-regular**.

Proof: Let X be a pseudo- regular I- space. Let F be a I- closed subset of X and let $x \in X$ *with* $x \notin F$ by Theorem – 2.1, F is I- compact. By the pseudo –regularity of X, F and x can be separated by disjoint I- open sets I_1 *and* I_2. Hence X is I- regular.

Definition 3.2: An I- space **(X, I)** is called **Hausdorff** if for each pair of distinct points $x_1, x_2 \in X$, there are disjoint I- open sets I_1, I_2 *in* X *such that* $x_1 \in I_1,\ x_2 \in I_2$..

Theorem 3.3: **Every I- compact subset K of a Hausdorff I- space X is I- closed**.

Proof: Let X be a Hausdorff I- space, and let K be a I- compact subset of X. Let $x_0 \in X, x_0 \notin K$. Then, for each $y \in K$, there exist I- open sets $I_{x_0,y}$, $J_{x_0,y}$ such that $x_0 \in I_{x_0,y}$, $y \in J_{x_0,y}$ and

$I_{x_0,y} \cap J_{x_0,y} = \Phi$.

Then **I** = $\{J_{x_0,y} \mid y \in K\}$ is an I- open cover of K. Since K is I- compact, there exist $\{J_{x_0,y_1}, J_{x_0,y_2}, \cdots\cdots, J_{x_0,y_n}\}$, for some + ve integer n, such that

$K \subseteq J_{x_0,y_1} \cup J_{x_0,y_2} \cup \cdots\cdots \cup J_{x_0,y_n} = J_{x_0}$.

Now, $x_0 \in I_{x_0,y}, \forall\, y \in K$. Hence $x_0 \in I_{x_0,y_1} \cap I_{x_0,y_2} \cap \cdots\cdots \cap I_{x_0,y_n} = I_{x_0}$. which is an I- open set in X. Also, $I_{x_0} \cap J_{x_0} = \Phi$. So, $I_{x_0} \subseteq K^c$. Thus, x_0 is an I – interior point of K^c. Since x_0 is arbitrary point of K^c, this implies that K^c is I-open. **Hence K is I-closed.**

Theorem 3.4: **Every I- Hausdorff I- regular space is I- pseudo- regular.**

Proof: Let X be an I- Hausdorff I- regular I- space. Let K be an I-compact subset of X, and let $x \in X$ *with* $x \notin K$ by Theorem 2.3, K is I- closed. Since X is regular, K and x can be separated by disjoint I- open sets. Thus, X is I- pseudo- regular.

IV. PSEUDO REGULAR U - SPACES

Definition 4.1: A X U-space will be called **pseudo regular** if for every U- compact subset K of X and for every x $\in$X, x$\notin$K, there exist U- open sets G₁and G₂ in X with $G_1 \cap G_2 = \phi$ such that x$\in$ G₁ and K $\subseteq$G₂.

Example 4.1: Let X = Z, and The U- structure generated by

$\{\{Z,\phi\} \cup \{(-\infty, a)] \| a \in Z\} \cup \{[(b,\infty) \| b \in Z\}\}$

, U = $\langle \{\{Z,\phi\} \cup \{(-\infty, a)] \| a \in Z\} \cup \{[(b,\infty) \| b \in Z\}\} \rangle$. **Then X is pseudo regular U- space.**

Proof: The subsets of X are:

$(-\infty, a)], a \in Z \cdots\cdots(1),\ \ [(b,\infty), b \in Z \cdots\cdots(2),\ \ [(c,d)],\ c,d \in Z \cdots\cdots(3).$

The sets in (3) are finite, and so, compact.

If $A = (-\infty, a)],$ *for some* $a \in Z,$ then any U- open cover C of A must contain (-∞, a')] where a $\le a'$. Then {(-∞, a')]} is a finite subcover of C. Thus, the sets in (1) are compact.

Similarly, we can show that the sets in (2) are compact.

Thus, every nonempty subset of X is compact.

Let K be a compact subset of X and $x\in X$ with $x \notin K$.

Case-I: If K is a set of the form (1) i.e., $K=(-\infty,a)]$, *for some* $a\in Z$, then $x\in Z$, x > a, ($x\geq a$ *if* $K=(-\infty,a)$))

Let choose G_1 = [(a + 1, ∞) if K = (-∞, a], G_1 = [a, ∞) if K = (-∞, a), and G_2 = (-∞,a)] if K = (-∞, a], G_2 = (-∞, a) if K = (-∞, a). Thus, in each case $x\in G_1$, $K\subseteq G_2$ and $G_1\cap G_2=\phi$.

Case- II: If K is a set of the form (2), i.e. K= [(b, ∞), for some $b\in Z$, then $x\leq b$, or x < b, according as K = (b, ∞), or [b, ∞).

Let choose G_1= (-∞,b], G_2 = (b, ∞), if K = (b, ∞); G_1= (-∞,b), G_2 = [b, ∞) if K = [b, ∞)

Then for each case, $x\in G_1$, $K\subseteq G_2$ and $G_1\cap G_2=\phi$.

Case- III: Now let K= [(c,d)], for some c, $d\in Z$. Then, (i) $x\leq c$ and / or $x\geq d$, if K = (c, d), (ii) x < c, and / or $x\geq d$, if K = [c, d), (iii) $x\leq c$, and / or x > d, if K = (c, d], (iv) x < c, and / or x > d if K= (c, d]

For (i), we choose G_1 = (-∞,c] $\cup$ (d, ∞) , G_2 = (c, d), for (ii), we choose G_1 = (-∞,c) $\cup$ [d, ∞) , G_2 = [c, d), for(iii), we choose G_1 = (-∞,c] $\cup$ (d, ∞) , G_2 = (c, d],for(iv), we choose G_1 = (-∞,c) $\cup$ (d, ∞) , G_2 = [c, d].

Then, for each case $x\in G_1$, $K\subseteq G_2$ and $G_1\cap G_2=\phi$. **Thus, X is pseudo regular U- space.**

Example 4.2: **A U –space X may be regular but not pseudo regular.**

Proof: Let X = {a, b, c, d} and U = {X, ϕ,{a, b}, {c, d},{a, c, d},{a}}.

Here closed sets are {c, d},{a, b},{b, c, d},{b}, X, ϕ. Then (X, U) is U- regular. For, {a, c} is compact, $b\notin$ {a, c}, but {a, c} and b cannot be separated by disjoint U- open sets. Here (X, U) is not pseudo regular.

Example 4.3: **A U- space X may not be regular but pseudo regular.**

Proof: Let X = R, U = $\langle U_0\cup\xi\rangle$, where U_0 is the usual U – structure on R, and $\xi=\{\{x_0\}|x_0\in R-Q\}$, Q is closed, since R-Q is U- open. But Q cannot be separated from any in point, since the only U- open set containing Q is R. **X is not regular**. The compact sets of X are (-∞, a], [b, ∞), [a, b] such that a, $b\in$ R. Any U-open cover C of (-∞, a] must contains a U- open set of the form (-∞, a') where a' > a. Then {(-∞, a')}is a finite sub cover of C. So (-∞, a] is U- compact. Similarly, [b, ∞) is U- compact. Then for each a, $b\in$ R, for each a< b, [a, b] is U- compact, by Theorem 3.4 of [N.S.S, 2014, On Hausdorff and compact U-spaces], since [a, b] = $[(-\infty, a]\cup[b,\infty)]^c$ is a closed and is a subspaces.

Let K be any compact in X and let $x\in X$, $x\notin K$. Then K is of the form (-∞, a], or [b, ∞), or[a, b].

Case- I: Let K = (-∞, a]. Then, x > a. Let $G_1=\left(\frac{x+a}{2},\infty\right)$ and $G_2=\left(-\infty,\frac{x+a}{2}\right)$ .Then $x\in G_1$, $K\subseteq G_2$, and $G_1\cap G_2=\phi.$

Case- II: Let K **=** [b, ∞), Then, x< b. We take $G_1=\left(-\infty,\frac{b-x}{2}\right)$ and $G_2=\left(\frac{b-x}{2},\infty\right)$ .Then $x\in G_1$, $K\subseteq G_2$, and $G_1\cap G_2=\phi.$

Case – III: Let K = [a, b]. Then either x < a or x > b. If x$\leq$a, then we take $G_1=\left(-\infty, \frac{x+a}{2}\right)$ and $G_2 = \left(\frac{x+a}{2}, \infty\right)$ and if x > b, we take $G_1=\left(\frac{b+x}{2}, \infty\right)$ and $G_2 = \left(-\infty, \frac{b+x}{2}\right)$. Then, for both the cases x $\in G_1$, K$\subseteq G_2$ and $G_1 \cap G_2 = \phi$. **Thus K is U- pseudo regular**.

Result and Discussion: It has been proved that for both U- spaces and I- spaces regularity does not imply pseudo regularity and also pseudo regularity does not imply regularity. i.e., Regularity and Pseudo regularity are independent concept. Some connection between regular and pseudo regular I-spaces and U- spaces have been established.

Conclusion: It is the beginning of study of pseudo regular I- spaces and U- spaces and will be continue later.

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