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RESEARCH ARTICLE

# A NEW ELLIPSE PERIMETER APPROXIMATION FORMULA THAT REDUCES THE RELATIVE ERROR TO THE ORDER OF $10^{-8}$

**K. Idicula Koshy**
Professor of Mathematics (Retired)
Kerala Agricultural University
Cheruthuruthil House
Oonnukal P.O.(via) Omallur, Kerala
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K. Idicula Koshy

**ABSTRACT**

In this article, the author introduces **a new formula** for Ellipse Perimeter Approximation. It contains two indices: one is a numerical constant and the other is a polynomial. Though the numerical constant can take values greater than 1, values greater than 2 give much reduced relative error. In this article we take 2.11 as the numerical constant. The polynomial index is obtained by curve fitting method. It is observed that the more the degree of the polynomial, the higher the accuracy of approximation. Therefore, we consider 6$^{th}$ degree polynomials, whose coefficients are slightly modified, if found useful. Following this method, we have arrived at a Maximum Absolute Relative Error less than 0.0982 parts per million (ppm) for any ellipse (that is: less than 0.0982 millimeter per kilometer), **thereby achieving an Absolute Relative Error less than 0.1 millimeter per kilometer for the first time.**

**Keywords:** Ellipse, Aspect Ratio, Relative Error, ppm, Simpson's Rule, Curve Fitting.

## 1. Introduction

The ellipse, whose rectangular cartesian equation is $(x/a)^2 + (y/b)^2 = 1$, is named here as the standard ellipse. 'a ≠ 0' and 'b' ($a \geq b \geq 0$) are the major and minor radii of the ellipse. Its perimeter P (a, b) is given by the definite integral:

$$P(a,b) = \int_0^{2\pi} \sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)}\, d\theta$$

where (a cos θ, b sin θ), 0 ≤ θ < 2 π, is a parametric point on the ellipse. Due to the symmetry of the ellipse w. r. t. its axes, P (a, b) = 4* Q (a, b), where Q (a, b) is the perimeter of the standard ellipse in the first quadrant. Therefore, the first-quadrant perimeter is given by the definite integral:

$$Q(a,b) = \int_0^{\pi/2} \sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)}\, d\theta$$

Then, Q (a, 0) = a, Q (a, a) = πa /2, P (a, 0) = 4*a and P (a, a) = 2π*a.

However, the integral for P (a, b) or Q (a, b) given above could not be evaluated till now by any direct integration method. Hence, an exact formula for Ellipse Perimeter is not available. Therefore, Numerical Integration Methods are adopted to approximate Q (a, b). We adopt here Simpson's 1/3-Rule for numerical integration of Q (a, b) due to its very low relative error. The Quarter Perimeter values Q (100, b; Sim) used in Table 1 in this article are obtained by this method, by dividing the interval of integration [0, π /2] into 500 equal sub-intervals, so that the length h of each sub-interval is equal to π/1000. The Absolute Relative Error, then, is to the order of $h^4$ (E. Kreiszig 2010) which is less than $10^{-10}$.

## 2. Terminology and Notations

Conventional notations and terminologies related to the **standard ellipse** are used in this article. 'a $\neq 0$' and 'b' denote the lengths of the semi-major axis (**major radius**) and the semi-minor axis (**minor radius**) of the standard ellipse. The ratio (b/a) is called the **Aspect Ratio,** which takes values in the closed interval [0, 1]. An ellipse perimeter formula associated with the name of a mathematician is identified in this article by adding the name-indicative characters after the parameter 'b'. For example, **Q (a, b; Sim)** indicates the Quarter Perimeter values obtained by Simpson's (1/3) Rule. **Q (a, b; p, r)** denotes the Quarter Perimeter formulae-class, which the author introduces in this article. Parts per million is abbreviated as **ppm.** Curve Fitting means 'The Least Square Method' of curve fitting.

## 3. Evolution of the New Formula for Ellipse Perimeter Approximation.

In a published article (Koshy 2019), it was established that the quarter perimeter Q (a, b) follows Lagrange's first order linear partial differential equation in 'a' and 'b', namely, $a\frac{\partial Q}{\partial a} + b\frac{\partial Q}{\partial b}$ = Q. Two independent particular solutions of this partial differential equation are: $(a^p + b^p)^{(1/p)}$, $p \geq 1$ and $\sqrt{ab}$ . Therefore, any solution for Q (a, b), has to be a function of $(a^p + b^p)^{(1/p)}$, $p \geq 1$ and $\sqrt{ab}$.

Hence, we introduce a new formula for the quarter perimeter by defining:

$$Q(a, b; p, r) := (a^p + b^p)^{(1/p)} + r(a, b) * \sqrt{ab}\ ,\ p \geq 1 \qquad \ldots\ldots \qquad (1)$$

Where r (a, b) is of zero dimension. Obviously, Q (a, 0; p, r) = a; but Q (a, a) = $\frac{\pi}{2}a$ requires that r (a, a): = $\left\{\frac{\pi}{2} - 2^{\left(\frac{1}{p}\right)}\right\}$. It is observed that taking r (a, b): = $\{\frac{\pi}{2} - 2^{\left(\frac{1}{p}\right)}\}\ \{\frac{\sqrt{ab}}{0.5(a+b)}\}^{k(a,b)}$ improves the accuracy without any contradiction. Hence, we define the final form of the new formula for ellipse quarter perimeter approximation as follows:

$$Q(a, b; p, k) := (a^p + b^p)^{(1/p)} + \left\{\frac{\pi}{2} - 2^{\left(\frac{1}{p}\right)}\right\} * \{\frac{\sqrt{ab}}{0.5(a+b)}\}^{k(a,b)} * \sqrt{ab},\ p \geq 1 \qquad \ldots\ldots \qquad (2)$$

Formula (2), which has two indices p and k (a, b), is in fact, a family of formulae. To get a particular formula, we have to assign a value to p and then find the befitting k (a, b)..

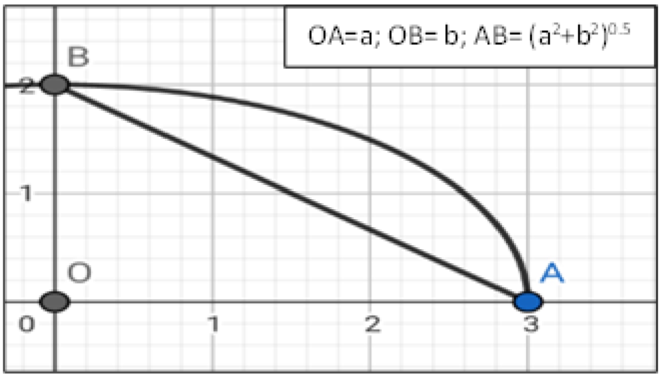



Fig. 1: The standard ellipse in the first quadrant

From the first quadrant graph of the standard ellipse given above, it is clear that $(a^2+b^2)^{0.5} \leq Q\,(a, b) \leq (a + b)$. Also, the inequality $(a^2+b^2)^{0.5} \leq Q\,(a, b; p, k) \leq (a + b)$ is true, if p is assigned any numerical value in the closed interval [1, 2]. However, the absolute relative error need not be to the order of $10^{-8}$. For example, if $2^{\left(\frac{1}{p}\right)} = \frac{\pi}{2}$, that is, if p = ln (2) /ln ($\frac{\pi}{2}$) = $p_0 \cong 1.53$ ..., then Q (a, b; $p_0$, r) is the YNOT formula of Roger Maertens, which yields maximum absolute relative error more than $3.6*10^{-3}$. Therefore, we look for $Q\,(a, b; p, k)$ with $p > 2$. We are going to show below in Result 3 that Q (a, b; p, k) with p = 2.11 together with a befitting k (a, b) can reduce the absolute relative error to less than 0.0982 millimeter per kilometer (Result 3). This is done through Results 1 and 2.

Putting p = 2.11, Formula (2) becomes:

$$Q\,(a, b; 2.11, k) := (a^{2.11} + b^{2.11})^{(1/2.11)} + \left\{\frac{\pi}{2} - 2^{\left(\frac{1}{2.11}\right)}\right\} * \left\{\frac{\sqrt{ab}}{0.5(a+b)}\right\}^{k(a,b)} * \sqrt{ab}, \quad .... \quad (3)$$

which is the main formulae for Ellipse Perimeter Approximation considered here.

## 4. Materials and Methods.

All computations are done in MS Excel. Q (a, b; Sim) values used here are derived with step-width h = $\pi$/1000. First, an index is obtained at each x = b/a by equating Q (a, b; 2.11, k) with Q (a, b, Sim). Then, using these discrete set of index values, the 6th degree polynomial is obtained by curve fitting method. Minor changes are made in the coefficients of the polynomial, if found useful to minimize the error. The modified polynomial, which is a function of b/a, is denoted by k (a, b), and, it is used to evaluate Q (a, b; 2.11, k). The Relative Error is computed taking Q (a, b; Sim) as basis. The coefficient of determination $R^2$ between the discrete theoretical values and corresponding discrete estimated values is found to be very close to 1.

## 5. Results

**Result 1** (Interval [0,1] kept as such)

$$Q\,(a, b; 2.11, k) := (a^{2.11}+b^{2.11})^{(1/2.11)} + \left(\frac{\pi}{2} - 2^{\left(\frac{1}{2.11}\right)}\right) * \left\{\frac{\sqrt{ab}}{0.5(a+b)}\right\}^{k(a,b)} * \sqrt{ab} \quad . ... \quad (3)$$

with k (a, b) = $2.6393+1.6519x - 2.2633x^2+2.0497x3 -1.4894x^4 + 0.7333x^5 - 0.1635x^6$ ... (1a) where x = b/a ∈ [0, 1], yields maximum absolute relative error less than $1.69*10^{-6}$.

In order to reduce the maximum absolute relative error further, next we split the interval [0,1] into two subintervals: [0, 0.10] and [0.10, 1] and assign separate polynomials k (a, b) of degree 6 to each subinterval. Hence,

**Result 2** (Interval [0,1] divided into two subintervals [0, 0.10] and [0.10, 1])

$$Q(a, b; 2.11, k) := (a^{2.11}+b^{2.11})^{(1/2.11)} + \left(\frac{\pi}{2} - 2^{\left(\frac{1}{2.11}\right)}\right) * \left\{\frac{\sqrt{ab}}{0.5(a+b)}\right\}^{k(a,b)} * \sqrt{ab} \quad .... \quad (3)$$

with k (a, b) = $2.63975+1.6526x - 2.3066x^2 +2.2597x^3 - 1.9044x^4 +1.1027x^5 - 0.2863x^6$), .... (2a)

where x = b/a ∈ [0.10, 1], yields Maximum Absolute Relative Error less than $4.62*10^{-7}$.

and

with k (a, b) = $2.6472+0.671x+41.225x^2 -978.08x^3+12098x^4-77662x^5+202310x^6$, .... (2b)

where x = b/a ∈ [0, 0.10], yields Maximum Absolute Relative Error less than $8.48*10^{-8}$.

*Therefore, the combined maximum absolute relative error is less than $4.62*10^{-7}$ for all ellipses.*

There are two values for Q (a, b; 2.11, k) at the common boundary b/a = 0.10; the lower value, at b/a = 0.10 on [0.10, 1], may be taken as the value of Q (a, b; 2.11, k) at b/a = 0.10.

**Result 3** (Interval [0,1] divided into three subintervals [0, 0.10], [0.10, 0.40] and [0.40, 1])

$$Q(a, b; 2.11, k) := (a^{2.11}+b^{2.11})^{(1/2.11)} + \left(\frac{\pi}{2} - 2^{\left(\frac{1}{2.11}\right)}\right) * \left\{\frac{\sqrt{ab}}{0.5(a+b)}\right\}^{k(a,b)} * \sqrt{ab} \quad .... \quad (3)$$

with k (a, b) = $2.6492+1.5423x - 1.7904x^2+1.0219x^3 - 0.2986x^4 +0.0315x^5+0.0016x^6$) .... (3a)

where x = b/a ∈ [0.40, 1], yields Maximum Absolute Relative Error less than $1.56*10^{-8}$

**and** with k (a, b) = $2.6363+1.7334x-3.0386x^2 +5.6188x^3-10.293x^4+12.064x^5-6.1961x^6$ .... (3b)

where x = b/a ∈ [0.10, 0.40], yields Maximum Absolute Relative Error less than $2.74*10^{-8}$

**and** with k (a, b) = $2.6472+0.6709x+41.225x^2 -978.08x^3+12098x^4-77662x^5+202310x^6$ .... (3c)

where x = b/a ∈ [0, 0.10], yields Maximum Absolute Relative Error less than $9.82*10^{-8}$

Table 1: Ellipse Quarter Perimeter Approximation by Modified Curve Fit Method.
Result: Absolute Relative Error less than 0.0982 ppm

| a | b | b/a | Q (a, b; Sim) | k = 2.6492 +1.5423*x - 1.7904*x²+1.0219*x³ - 0.2986*x⁴+0.0315*x⁵+ 0.0016*x⁶; x = b/a ∈[0.4, 1] | Q (a, b; 2.11, k): (a^2.11+b^2.11) ^ (1/2.11) + (pi/2- 2^ (1/2.11) * ((GM/AM) ^k) *(a*b) ^ 0.5 | Relative Error in Q (a, b; 2.11, k) based on Q (a, b; Sim) |
|---|---|---|---|---|---|---|
| 100 | 100 | 1 | 157.0796326795 | 3.1575000000 | 157.0796326795 | 3.61876444E-16 |
| 100 | 99 | 0.99 | 156.2952211988 | 3.1574830943 | 156.2952211996 | 5.12006474E-12 |
| 100 | 98 | 0.98 | 155.5128030354 | 3.1574296651 | 155.5128030378 | 1.56273635E-11 |
| 100 | 97 | 0.97 | 154.7324086029 | 3.1573386227 | 154.7324086068 | 2.51030340E-11 |
| 100 | 96 | 0.96 | 153.9540689771 | 3.1572088480 | 153.9540689815 | 2.80929034E-11 |
| 100 | 95 | 0.95 | 153.1778159151 | 3.1570391928 | 153.1778159182 | 1.98923260E-11 |
| ROWS DELETED | | | | | | |
| 100 | 58 | 0.58 | 126.2949227137 | 3.1091657695 | 126.2949207948 | -1.51941204E-08 |
| 100 | 57 | 0.57 | 125.6309848359 | 3.1062887520 | 125.6309829031 | -1.53850576E-08 |
| 100 | 56 | 0.56 | 124.9712260432 | 3.1032988960 | 124.9712241058 | -1.55022144E-08 |
| 100 | 55 | 0.55 | 124.3157350111 | 3.1001934811 | 124.3157330802 | ***-1.55319030E-08*** |
| 100 | 54 | 0.54 | 123.6646032119 | 3.0969697381 | 123.6646013004 | -1.54572246E-08 |
| 100 | 53 | 0.53 | 123.0179250376 | 3.0936248488 | 123.0179231606 | -1.52579944E-08 |
| 100 | 52 | 0.52 | 122.3757979294 | 3.0901559455 | 122.3757961047 | -1.49109774E-08 |
| 100 | 51 | 0.51 | 121.7383225154 | 3.0865601103 | 121.7383207635 | -1.43905880E-08 |
| ROWS DELETED | | | | | | |
| 100 | 42 | 0.42 | 116.2287558112 | 3.0479789028 | 116.2287558001 | -9.57216398E-11 |
| 100 | 41 | 0.41 | 115.6441503067 | 3.0429419548 | 115.6441504923 | 1.60545240E-09 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 100 | 40 | 0.4 | 115.0655629783 | 3.0377425536 | 115.0655632753 | ***2.58095054E-09*** |
| a | b | b/a | Q (a, b; Sim) | k = 2.6363+1.7334x-3.0386x$^2$ + 5.6188x$^3$-10.293x$^4$ + 12.064x$^5$-6.1961x $^6$ ; x = b/a ∈ [0.1, 0.4] | Q (a, b; 2.11, k): (a^2.11+b^2.11) ^ (1/2.11) + (pi/2- 2^ (1/2.11) * ((GM/AM) ^k) *(a*b) ^ 0.5 | Relative Error in Q (a, b; 2.11, k) based on Q (a, b; Sim) |
| 100 | 40 | 0.4 | 115.0655629783 | 3.0377425344 | 115.0655632918 | ***2.72407136E-09*** |
| 100 | 39 | 0.39 | 114.4931472778 | 3.0323780405 | 114.4931467599 | -4.52338633E-09 |
| 100 | 38 | 0.38 | 113.9270627145 | 3.0268431489 | 113.9270618178 | -7.87081439E-09 |
| 100 | 28 | 0.28 | 108.6546463399 | 2.9612798630 | 108.6546434270 | -2.68082014E-08 |
| 100 | 27 | 0.27 | 108.1708799880 | 2.9536077036 | 108.1708770262 | ***-2.73800817E-08*** |
| 100 | 26 | 0.26 | 107.6959938396 | 2.9457137481 | 107.6959909023 | -2.72736146E-08 |
| 100 | 25 | 0.25 | 107.2302721895 | 2.9375927490 | 107.2302693289 | -2.66770347E-08 |
| ROWS DELETED | | | | | | |
| 100 | 12 | 0.12 | 102.1742247323 | 2.8084087794 | 102.1742230756 | -1.62141035E-08 |
| 100 | 11 | 0.11 | 101.8785506041 | 2.7963618798 | 101.8785490306 | -1.54440738E-08 |
| 100 | 10 | 0.1 | 101.5993545025 | 2.7839579439 | 101.5993527129 | -1.76145503E-08 |
| a | b | b/a | Q (a, b; Sim) | k= 2.6472+0.6709x+41.225x$^2$ - 978.08x$^3$+12098x$^4$ - 77662x$^5$+202310x$^6$ ; x = b/a ∈[0, 0.1] | Q (a, b; 2.11, k): (a^2.11+b^2.11) ^ (1/2.11) + (pi/2- 2^ (1/2.11) * ((GM/AM) ^k) *(a*b) ^ 0.5 | Relative Error in Q (a, b; 2.11, k) based on Q (a, b; Sim) |
| 100 | 10 | 0.1 | 101.5993545025 | 2.7839500000 | 101.5993581304 | 3.57080968E-08 |
| 100 | 9 | 0.09 | 101.3375183618 | 2.7711624449 | 101.3375283050 | ***9.81194546E-08*** |
| 100 | 8 | 0.08 | 101.0940281651 | 2.7580206310 | 101.0940240652 | ***-4.05547022E-08*** |
| 100 | 7 | 0.07 | 100.8699983194 | 2.7444320858 | 100.8700043950 | 6.02319905E-08 |
| ROWS DELETED | | | | | | |
| 100 | 2 | 0.02 | 100.0959790450 | 2.6709834694 | 100.0959787778 | -2.66962098E-09 |
| 100 | 1 | 0.01 | 100.0274635978 | 2.6571668361 | 100.0274640249 | 4.26965791E-09 |
| 100 | 0 | 0 | 100.0000000000 | 2.6472000000 | 100.0000000000 | 4.26325641E-16 |
| | | | | Combined Max. Rel. Error | 9.81194546E-08 | |
| | | | | Combined Min. Rel. Error | -4.05547022E-08 | |

*Therefore, the combined Maximum Absolute Relative Error is less than 9.82*10$^{-8}$ for all ellipses (Table 1).* (*The deleted rows in Table 1 can be filled in using the related formula*).

In Table 1, there are two values for Q (a, b; 2.11, k) at each of the common boundary points b/a = 0.10 and b/a = 0.40. The lower value at each point may be taken as the value of Q (a, b; 2.11, k) there.

There may be other numerical values of p, for which Q (a, b; 2.11, k) yields very close maximum absolute relative error. For example, it is verified that, if p = 2.1, by the same procedure as above, the maximum absolute relative error can be brought to less than 9.98*10$^{-8}$. However, for some other values of p (e. g: p = 2), even a modified best fit polynomial k (a, b) could not reduce the Absolute Relative Error to the order of 10$^{-8}$.

## 6. Discussion/Comments

The Quarter Perimeter Approximation Formula introduced in this article is purely empirical and is based on the author's discovery that such a formula is a function of $(a^p + b^p)^{(1/p)}$, for some p $\geq 1$and $\sqrt{ab}$ (Koshy 2019). The quarter perimeter as per Formula (3) can be calculated using a scientific calculator, if the major and minor radii a and b are known and the k (a, b) attached to b/a is correctly chosen.

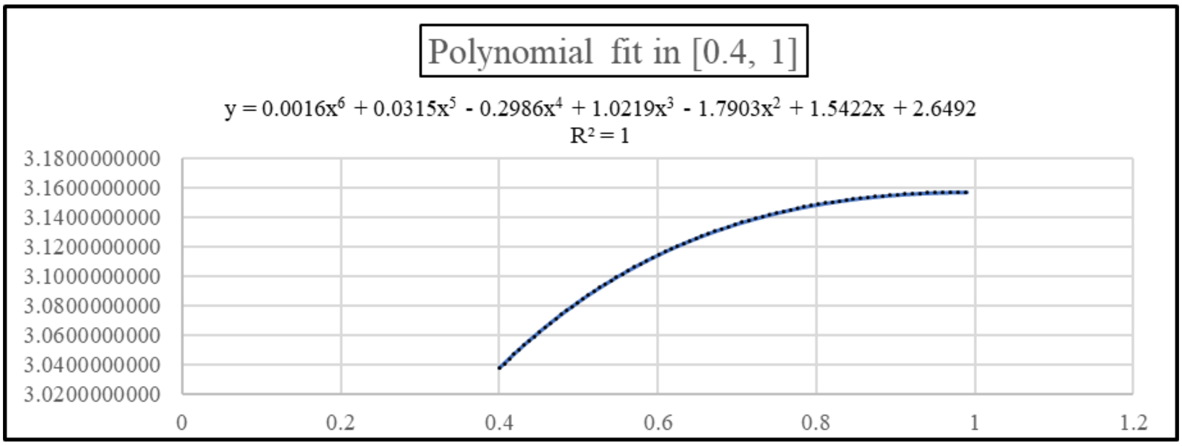
The author has critically examined several Ellipse Perimeter Approximation Formulae known after several eminent Mathematicians: Kepler, Euler, Seki, Muir, Maertens (YNOT formula), Rivera, Lindner, Zafary, Cantrell etc. and, of course, the formulae of the Great Indian Mathematical Genius Srinivasa Ramanujan. None of these formulae gives Maximum Absolute Relative Error anywhere near that obtained in Result (3) above, with p =2.11.

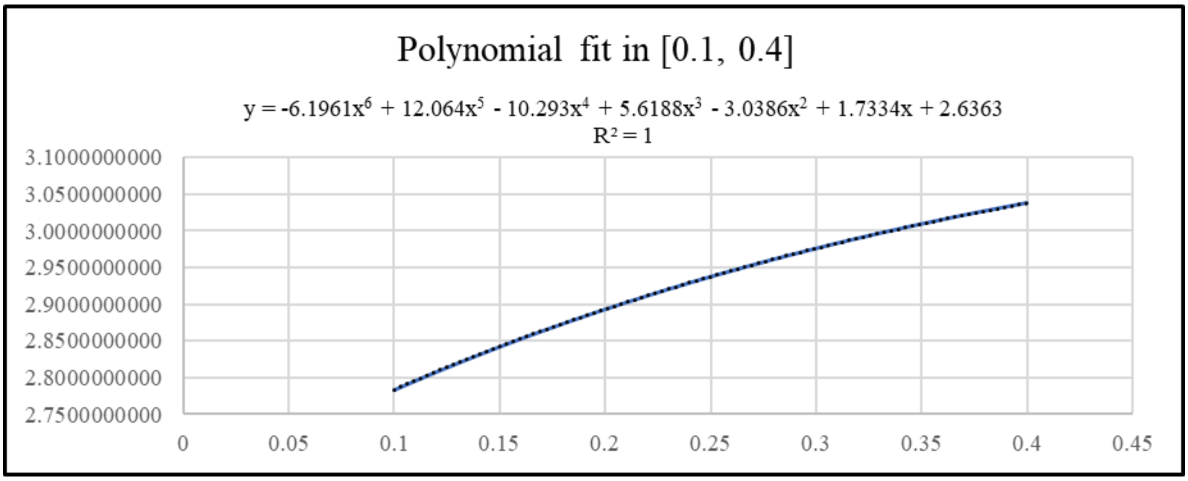
Ramanujan's Formula II: P (a, b; Ram): = $\pi\ (a+b)\{1+\frac{3h^2}{(10+\sqrt{4-3h^2})}\}$,

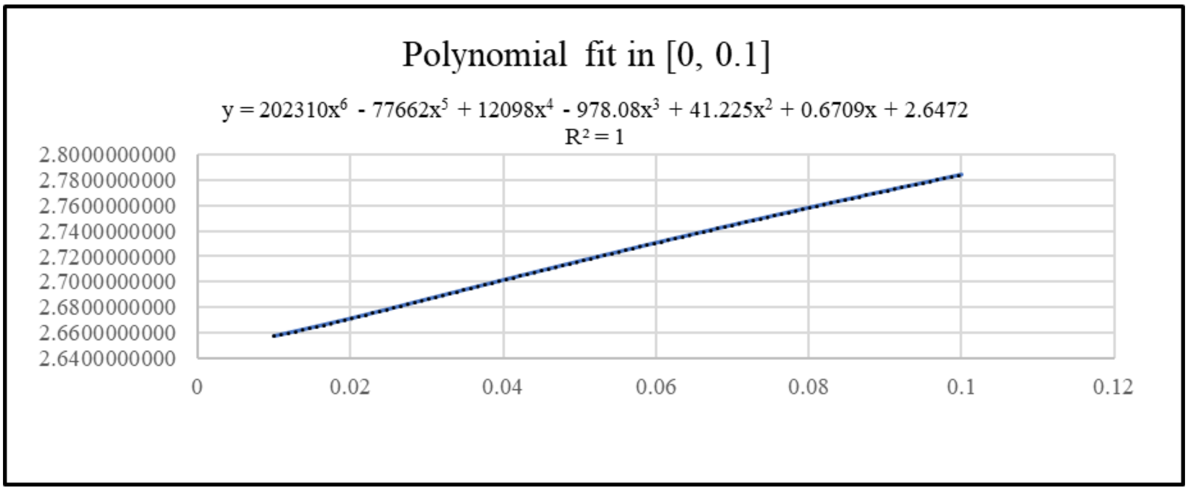
where h = (a-b)/ (a + b), generates only very small Absolute Relative Error for ellipses of high and medium Aspect Ratios. However, the Absolute Relative Error steadily increases to the order of $10^{-5}$, $10^{-4}$ etc., for b: $0 <$ (b/a) $\leq 0.1$; and P (a, 0; Ram) = 4a is true only if $\pi = 22/7$, which is incorrect.

Hence, if very high accuracy is the prime objective, Formula (3) is better than all other formulae known so far for the Approximation of the Quarter Perimeter of the Ellipse.

Actual Polynomials of Result 3 in the interval [0,1]



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