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RESEARCH ARTICLE

Diophantine equations, $(x^n + y^n + z^n = w^2)$ & $(x^n - y^n - z^n = m^2)$

Seiji Tomita[1], Oliver Couto[2]
[1]Computer Engineer, Tokyo software co. Inc., Japan.
Email: fermat@m15.alpha-net.ne.jp
[2]University of Waterloo, Waterloo, Ontario, Canada
Email: matt345@celebrating-mathematics.com
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ABSTRACT

In this paper the authors have considered, Diophantine equations, $(x^n + y^n + z^n = w^2)$ & $(x^n - y^n - z^n = m^2)$ for $n = (2,3,4,5,6,7)$. While Andrew Bremner & Maciej Ulas ref. (2) in their year 2011 paper discussed above equation for n=6, this paper has done a systematic approach for degree 2 to 7. Though the authors tried to parameterize the above equations for (n=2 to 7) by algebraic means, we could give parametric solution for n=5 by only elliptic curve method. For degree six only numerical solutions are shown as arrived at by Bremner & Ulas (2).

Equation (A):

(1) $$x^n + y^n + z^n = w^2$$

2a)

$$n = 2$$

$$x^2 + y^2 + z^2 = w^2$$

$$Take, x = 12, y = -3 \ \& \ z = 4,$$

$$w = (x + y + z) \ \& \ we\ get{:}(12, -3,4)^2 = (12 - 3 + 4)^2 = (13)^2$$

$$x^2 + y^2 + z^2 = (x + y + z)^2 = w^2$$

Hence we get after simplification:

$$(xy + yz + zx) = 0 \quad ---(1)$$

$$We\ parametrize\ (1)\ at\ x = 12 + t,\ y = -3 + t\ ,\ z = 4 + kt$$

$$we\ get\ t = -\frac{9k + 17}{2k + 1}$$

$$Hence, \quad x = 5(3k - 1),\ y = -5(3k + 4),\ z = (9k^2 + 9k - 4)$$

For k=1 we get:

$$(x, y, z) = (10, -35, -14)$$

$$(x, y, z)^2 = (10,35,14)^2 = (39)^2$$

$$Also,\ z = -\frac{xy}{x+y}$$

$$For(a, b, c) = (3,2,-65)\ we\ get, (6,10,15) = (19)^2$$

Equation(2b):

(B): $x^n - y^n - z^n = m^2$

degree, n=2

$$(x^2 - y^2 - z^2) = m^2 \qquad ----(2)$$

$$Above\ has\ solution:$$

$$(x, y, z, m) = [(2m^2 + n^2), (2m^2), (n^2), (2mn)]$$

$$For\ (m, n) = (3,2) \qquad we\ get: (11)^2 - (9)^2 - (2)^2 = (6)^2$$

3a)

For n=3,

$$x^3 + y^3 + z^3 = w^2$$

$$We\ take\ (x, y, z) = (3,2,1)\ \&\ w = (x + y + z)$$

$$Hence: (3,2,1)^3 = (3 + 2 + 1)^2$$

$$(x^3 + y^3 + z^3) = (x + y + z)^2$$

We fix, (z=1) & we get after simplification:

$$(x^2 - xy + y^2) = (x + y + 2) \quad -------(2)$$

$$We\ parametrize\ (2) at, (x, y) = [(3 + t), (2 + kt)]$$

$$We\ get: t = (-3)/(k^2 - k + 1)\ \&\ we\ have:$$

$$(x, y, z, w) = [m(3k)(k - 1), m(k - 2)(2k - 1), (m)(k^2 - k + 1), m^2(k - 1)(2k - 1)]$$

where $m = (k^2 - k + 1)$

$$k = 3\ we\ have: (x, y, z) = [\left(\frac{18}{5}\right), \left(\frac{5}{7}\right), (1)]$$

Hence numerical solution is:

$$(126,35,49)^3 = (1470)^2$$

The above (3a) equation can be also written as:

Equation (B):

3b) $(x^3 + y^3 + z^3) = (x^3 - (-y)^3 - (-z)^3)$

$$= (x + y + z)^2(x^3 - y^3 - z^3) = m^2\ has\ solution:$$

$$x, y, z, m) = [(p^2 + 3q^2)^2, (p^2 - 3q^2)^2, (12p^2q^2), (6pq(p^4 - 9q^4))]$$

$$Numerical\ solution\ is: [(49)^3 - 48^3 - 1^3 = (84)^2]$$

4) $X^4 + Y^4 + Z^4 = W^2$

$$X^4 + Y^4 + Z^4 = W^2.$$

Diophantus [1] found the parametric solution as follows.

Let $Y = a$, $Z = b$, $W = X^2 - k$.

Then,

$$X^2 = \frac{a^4 + b^4 - k^2}{2k}$$

Taking k=$a^2 + b^2$, $we\ get\ X^2 = \frac{a^2b^2}{a^2+b^2}\ since\ a^2 + b^2$ must be a perfect qure number, let's take (a,b)= $m^2 + n^2, 2mn)$. Thus we ger a parametric solution

$$X = 2mn(m^2 - n^2)$$

$$Y = (m^2 - n^2)(m^2 + n^2)$$

$$Z = 2mn(m^2 + n^2)$$

$$W = m^8 + 14m^4n^4 + n^8)$$

For (m,n)=(3,2) we get:

$(60,65,156)^4 = (24961)^2$

New solution is for : $x^4 + y^4 + z^4 = w^2$

Let (p,q,r,s) be solution of above equation:

$(px + 1)^4 + (qx)^4 + (rx)^4 = (sx^2 + kx + 1)^2$............(1)

And let, $(p, q, r, s) = (2mn(m^2 - n^2), (m^2 - n^2)(m^2 + n^2), 2mn(m^2 + n^2), m^8 + 14m^4n^4 + n^8)$

$$k = -(16m^3n^3)\frac{(n^2-m^2)^3}{m^8+14m^4*n^4+n^8}$$

$$x = -4(n^{10} - m^2n^8 + 14m^4n^6 - 14m^6n^4 + m^8n^2 - m^{10})mnp$$

Where, $p=1/(n^{16}-8m^2n^{14}+12m^4n^{12}+8m^6n^{10}+230m^8n^8+8m^{10}n^6+12m^{12}n^4-8m^14n^2+m^{16})$

$Substitute\ k\ an\ x\ to\ (1), hence\ we\ obtain\ below\ parametric\ solution.$

$x = m^{16} - 4m^4n^{12} + 128m^6n^{10} + 6m^8n^8 + 128m^{10}n^6 - 4m^{12}n^4 + n^{16}$

$$y = 4(-m^2 + n^2)(m^2 + n^2)(n^{10} - m^2n^8 + 14m^4n^6 - 14m^6n^4 + m^8n^2 - m^{10})mn$$

$$z = -8m^2n^2(m^2 + n^2)(n^{10} - m^2n^8 + 14m^4n^6 - 14m^6n^4 + m^8n^2 - m^{10})$$

$$w = (n^{32} + 120m^4n^{28} - 256m^6n^{26} + 2332m^8n^{24} + 768m^{10}n^{22} + 5960m^{12}n^{20} - 512m^{14}n^{18} + 48710m^{16}n^{16} - 512m^{18}n^{14} + 5960m^{20}n^{12} + 768m^{22}n^{10} + 2332m^{24}n^8 - 256m^{26}n^6 + 120m^{28}n^4 + m^{32})$$

4b)

$$(x^4 - y^4 - z^4) = (m)^2$$

we take, $m = (x^2 - (y - z)^2)$

We get after simplification:

$$x = \left(\frac{y^2 - yz + z^2}{y - z}\right)$$

We parametrize above at: $(x, y, z) = (7 + kt, 3 + t, 2 + t)$

we get, $t = (k - 5)$

$$(x, y, z) = [(t^2 + 5t + 7), (3 + t), (2 + t)]$$

for t=3 we get: $(31)^4 - 6^4 - 5^4 = (960)^2$

5a) $(X)^5 + (Y)^5 + (Z)^5 = W^2$

$$X^5 + Y^5 + Z^5 = W^2 \qquad (5a)$$

We prove that there are infinitely many integer solutions of $X^5 + Y^5 + Z^5 = W^2$. Let $X = t - a$, $Y = -t + 1$, $Z = a$ then equation (5a) becomes to the quartic equation $Q : v^2 = (5 - 5a)t^4 + (-10 + 10a^2)t^3 + (-10a^3 + 10)t^2 + (5a^4 - 5)t + 1$.

The quartic equation is bi-rationally equivalent to an elliptic curve.

$$E: Y^2 + (5a^4 + 5)YX + (-20 + 20a^2)Y = X^3 + \left(-10a^3 + \frac{15}{4} - \frac{25}{4}a^8 + \frac{25}{2}a^8\right)X^2 + (-20 + 20a)X - 200a^3 - 400a^4 - 75 + 75a + 125a^8 - 125a^9 + 125a^5$$

E has a point

$$P(X, Y) = \left(\frac{40a^3 - 15 - 25a^8 - 50a^3}{4}, \frac{-175a^4 - 125a^{12} - 375a^8 + 200a^3 + 5 - 80a^2}{4}\right)$$

According to the Nagell-Lutz theorem, the point $P(X,Y)$ is not a point of finite order, hence we can obtain infinitely many parametric solutions of (5a). We show one of the solutions of equation (6) using group law.

For instance, doubling the point P,(2p), we get a quartic point $2Q(t,v)$

$$t = \frac{(a + 1)(5a5 + 5a4 - 5a - 1)}{f}$$

$v = -(\frac{1}{f^2})(1875a^{20} + 7500a^{19} + 11250a^{18} + 7500a^{17} - 5625a^{16} - 25000a^{15} - 28000a^{14} - 9000a^{13} + 12750a^{12} + 26000a^{11} + 17700a^{10} + 1200a^9 - 6050a^8 - 10200a^7 - 4400a^6 + 2920a^5 + 735a^4 - 860a^3 - 390a^2 - 60a - 101)$

$f = (25a^{10} + 50a^9 + 25a^8 - 50a^6 - 40a^5 + 50a^4 + 40a^3 - 15a^2 - 50a - 19)$

Finally, we get a parametric solution of (5a).

$X = -(-80a^6 - 30a^5 - 10a^2 + 29a + 8 + 25a^{11} + 50a^{10} + 25a^9 - 50a^7 - 15a^3)f$,

$Y = (-90a^6 - 120a^5 + 25a^2 - 2a + 10a^4 - 11 + 25a^{10} + 50a^9 + 25a^8 + 40a^3)f,$

$Z = af^2,$

$W = (1875a^{20} + 7500a^{19} + 11250a^{18} + 7500a^{17} - 5625a^{16} - 25000a^{15} - 28000a^{14} - 9000a^{13} + 12750a^{12} + 26000a^{11} + 17700a^{10} + 1200a^9 - 6050a^8 - 10200a^7 - 4400a^6 + 2920a^5 + 735a^4 - 860a^3 - 390a^2 - 60a - 101)f^3$

where, $f = (25a^{10} + 50a^9 + 25a^8 - 50a^6 - 40a^5 + 50a^4 + 40a^3 - 15a^2 - 50a - 19)$

Numerical solution is: $(48)^5 + (-30)^5 + (-18)^5 = (15120)^2$

5b) $(X)^5 - (Y)^5 - (Z)^5 = (M)^2$

Consider the below equation:

$X^5 + aY^5 + aZ^5 = M^2$... (1)

Substitute $X = p, Y = t - p, Z = -t$ to equation (1), we obtain

$(M)^2 = -5apt^4 + 10ap^2t^3 - 10ap^3t^2 + 5ap^4t + p^5 - ap^5$.............................. (2)

Let a=-p+1, U=t, and V=W, then we obtain equation (3).

$V^2 = (-5p + 5p^2)U^4 + (-10p^3 + 10p^2)U^3 + (10p^4 - 10p^3)U^2 + (-5p^5 + 5p^4)U + p^6$ ... . (3)

Since quartic equation (3) *has a rational solution* $Q(U, V) = (0, p^3)$, *this quartic equation* (3) *is birationally equivalent to an elliptic curve below*:

$Y^2 + (-5p^2 + 5p)YX + (-20p^6 + 20p^5)Y$

$= X^3 + (\frac{15}{4}p^4 + \frac{5}{2}p^3 - \frac{25}{4}p^2)X^2 + (20p^7 - 20p)X + (25p^{11} - 75p^{12} + 175p^{10} - 125p^9)$

Transformation is given by:

$$U = \frac{4p^3K + 15p^7 + 10p^6 - 25p^5}{2F}$$

$V = (\frac{18}{2}Y^2) * (625p^7X - 500p^8X + 90p^7X^2 + 60p^6X^2 + 8p^3X^3 - 150p^5X^2 + 385p^{11}X - 200p^{14} + 600p^{15} - 1400p^{13} + 1000p^{12} + 140p^{10}X - 650p^9X - 10p^9Y + 110p^8Y - 350p^7Y + 250p^6Y)$

$X = (\frac{1}{U^2}) * (2p^3V + 2p^6 - 5Up^5 + 5Up^4)$

$Y = \frac{8p^6V + 8p^9 - 20Up^8 + 20Up^7 + 15U^2p^7 + 10U^2p^6 - 25U^2p^5}{2U^3}$

The point corresponding to point Q is:

$$P(X,Y) = (-\frac{15}{4}p^4 - \frac{5}{2}p^3 + \frac{25}{4}p^2, \frac{5}{4}p - \frac{55}{4}p^5 + \frac{175}{4}p^4 - \frac{125}{4}p^3).$$

This point P is of infinite order, and the multiples (nP),

n = 2,3, ...give infinitely many points.

Hence we can obtain infinitely many integer solutions for equation (1).

Case : n=2

$$X = p(19p^2 - 10p - 25)^2$$
$$Y = -p(11p^2 + 30p - 25)(19p^2 - 10p - 25)$$
$$Z = -8p^2(p - 5)(19p^2 - 10p - 25)$$
$$M = p^3(19p^2 - 10p - 25)^3(-1875 + 2500p - 1450p^2 + 980p^3 + 101p^4)$$
$$p \text{ is arbitrary}$$
$$For, \quad p = 2 \text{ we get}, \quad a = (-p + 1) = -1$$
$$(X, Y, Z) = (1922, 4898, -2976) \ \& \ M = 1616102168$$

Hence we have:

$$(X)^5 - (Y)^5 - (Z)^5 = (M)^2$$

$$(1922)^5 - (-4898)^5 - (2976)^5 = (1616102168)^2$$

Another numerical solution is:

$$(48, -30, -18)^5 = (15120)^2$$

6a) $(X)^6 + (Y)^6 + (Z)^6 = (W)^2$

$$(X)^6 + (Y)^6 + (Z)^6 = (W)^2 \quad (7)$$

According to Bremner and Ulas[2], (7) has an infinite many integer solutions.

They gave the below numerical solutions of, x6 + y6 + z6 = (w)2 in the range 0 < x < y < z with x + y + z < 30000 and (x,y,z) = 1. There is no parametric solution available & is an open problem.

Table (A):

x	y	z	$(x^6 + y^6 + z6 = w^2)$
42	81	100	w=1134865
42	873	3596	w=46505412377
4728	5306	10617	w= 1210664898377
85	90	168	w=4836493
2043	2184	2518	w=20883327517
5340	6626	9765	w=987341285501
140	213	390	w=60163597
792	3759	5038	w=138465240337
1689	10528	14886	w=3498954949801
207	324	350	w=55441585
1515	3262	5160	w=141747483853
588	8224	26097	w= 17782152244433
278	369	378	w=76831633

3087	3404	4482	w=102604114673
834	17094	21373	w=10966834991269
715	924	1230	w=2053967149
2975	4950	7902	w=508783710817
1182	14644	24597	w=15209227541197
694	945	1308	w=2414891825
4410	5463	8270	w=594854319097
5802	13469	29316	w=25313949479269

6b) $X^6 - Y^6 - Z^6 = M^2$

$$X^6 - Y^6 - Z^6 = M^2 \tag{6b}$$

According to Bremner and Ulas [2], only one numerical solution is known for (6b), $(X,Y,Z,)$ = (57, 44, 28, 162967) for $(X + Y + Z) < 5000$. It has not been proven that the equation has infinite solutions. Also there is no parametric solution available & it is an open problem.

In passing, we note that even though there is no parametric solution for above, $X^6 - Y^6 - Z^6 = M^2$

There is a parametric solution for, $x^6 - y^6 - z^6 = 3(m)^2$

And is given by:

$$(x, y, z, m) = [(m^2 + n^2), (m^2 - n^2), (2mn), (2mn)(m^4 - n^4)]$$

$$For\ (m, n) = (3,2)\ we\ get:$$

$$(13,12,5)^6 = 3(780)^2$$

7a) $(X)^7 + (Y)^7 + (Z)^7 = (W)^2$

$$X^7 + Y^7 + Z^7 = W^2. \tag{9}$$

We prove that there are infinitely many integer solutions of ($X^7 + Y^7 + Z^7 = W^2$) Let $X = a + b$, $Y = -a$, $Z = -b$. Then, LHS of (9) can be factorized as follows.

$X^7 + Y^7 + Z^7 = 7ab(a + b)(b^2 + ab + a^2)^2$.

Hence, $7ab(a + b)$ has to be a perfect square number. Let's consider the system equation

$7ab = u^2$, $a + b = v^2$

Let $a = 7m^2$, $b = n^2$, and then we get: u=7mn,v=k^{2+7},uv=14k (k^{4-49})

$$7m^2 + n^2 = v^2 = (k^2 + 7)^2 \tag{10}$$

Hence, we get a parametric solution of (7a) where:

k is an arbitrary integer.

$X = (7 + k^2)^2$

$Y = -28k^2$,

$Z = -(-7 + k^2)^2$,

$W = 14k(-7 + k^2)(7 + k^2)(2401 + 686k^4 + k^8)$

For k=2,*we get* $(121)^7+(-112)^7+(-9)^7=(12596892)^2$

Thus, $X^7 + Y^7 + Z^7 = W^2$ has an infinite many integer solutions.

7b) $(X)^7 -(Y)^7 -(Z)^7 =(M)^2$

$[(p+q)^7 - p^7 - q^7] = [7pq(p + q)(p^2 + pq + r^2)^2] = m^2$ ---(11)

We take (p,q)=(u^2, $7v^2$) Hence (p+q)=($u^2 + 7v^2$)

(p+q)=($u^2 + 7v^2$)= r^2 --(12)

We parametrize (11) at (u,v,r)=(1,3,8) & we get:

$$p = (k^2 - 16k + 36)^2$$
$$q = 7(3k^2 - 16k + 20)^2$$

w=$[7(-2+k)(-40+12k)(36-16k+k^2)(16-11k+2k^2) * (13148416 - 37105664k + 46477312k^2 - 33646592k^3 + 15350496k^4 - 4506880k^5 + 829696k^6 - 87424k^7 + 4033k^8)]$

For, $k=0$ & after removing the common factors we get:

$(256, -175, -81)^7 = (258859440)^2$

Numerical solution is, $[(16)^7 - 9^7 - 7^7 = (16212)^2]$

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