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RESEARCH ARTICLE

Sum of divisors function in terms of $r_k(n)$

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ABSTRACT

In this paper, we exhibit a connection between the sum of divisors function and the number of representations of a positive integer as a sum of squares.

Keywords: Sum of divisors function, Determinant, Colour partitions, Sum of squares.

1. Introduction

Gandhi [1, 2] deduced the following recurrence relation for colour partitions:

$$-\frac{n}{r}p_r(n) = \sum_{k=0}^{n} p_r(k)\,\sigma(n-k), \qquad r, n \geq 1, \tag{1}$$

involving the sum of divisors function $\sigma(n)$, where we can apply the Gould's method [3] to obtain the corresponding inversion:

$$\sigma(n) = -\frac{1}{r}\begin{vmatrix} np_r(n) & p_r(1) & p_r(2) & \cdots & \cdots & p_r(n-1) \\ (n-1)p_r(n-1) & 1 & p_r(1) & \cdots & \cdots & p_r(n-2) \\ (n-2)p_r(n-2) & 0 & 1 & \cdots & \cdots & p_r(n-3) \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 2p_r(2) & 0 & & 0 & \cdots & p_r(1) \\ p_r(1) & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}, \tag{2}$$

which is valid $\forall\, r$, then we select $r = 1$ and we remember the property [4, 5]:

$$p_1(n) = a_n = \begin{cases} 0 & if \quad n \neq \frac{N}{2}(3N+1), \\ (-1)^N & if \quad n = \frac{N}{2}(3N+1), \end{cases} N = 0, \pm 1, \pm 2, \ldots \tag{3}$$

therefore:

$$\sigma(n) = -\begin{vmatrix} na_n & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ (n-1)a_{n-1} & 1 & a_1 & a_2 & \cdots & a_{n-2} \\ (n-2)a_{n-2} & 0 & 1 & a_1 & \cdots & a_{n-3} \\ \vdots & 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2a_2 & 0 & 0 & 0 & \ddots & a_1 \\ a_1 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}, \quad n \geq 1. \tag{4}$$

For example, for $n = 4, 6$:

$$\sigma(4) = -\begin{vmatrix} 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ -2 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{vmatrix} = 7\ ;\ \sigma(6) = -\begin{vmatrix} 0 & -1 & -1 & 0 & 0 & 1 \\ 5 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ -2 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 12.$$

To employ (4) is useful to know the values [6]:

$$a_j = \begin{cases} 1, & j = 0, 5, 7, 22, 26, 51, 57, 92, 100, 145, 155, 210, 222 \ldots \\ -1, & j = 1, 2, 12, 15, 35, 40, 70, 77, 117, 126, 176, 187, \ldots \\ 0 & otherwise \end{cases} \tag{5}$$

We observe that the determinant (2) gives the sum of divisors function in terms of colour partitions, which is equivalent to the following expression obtained by Jha [7]:

$$\sigma(n) = n \sum_{j=1}^{n} \frac{(-1)^j}{j}\binom{n}{j} p_j(n), \qquad n \geq 1. \tag{6}$$

Besides, we know that any positive integer can be written in the form $n = 2^k\, m,\ \ k \geq 0$ such that *m* is odd, then [8]:

$$\sigma\left(2^k\, m\right) = \left(2^{k+1} - 1\right) \sigma(m), \tag{7}$$

thus (6) and (7) imply the connection:

$$\sigma(n) = (2n - m) \sum_{j=1}^{m} \frac{(-1)^j}{j}\binom{m}{j} p_j(m), \qquad n = 2^k\, m. \tag{8}$$

In Sec. 2 we obtain the relationship between the sum of divisors function and $r_k(n)$, the number of representations of n as a sum of k squares.

2.- Sum of divisors function in terms of $r_k(n)$.

If $D(j)=\sum_{odd\ d|j}\frac{1}{d}$ then it is easy to prove the relation:

$$D(n)=D\left(2^k\, m\right)=D(m)=\frac{1}{m}\,\sigma(m), \tag{9}$$

where m is an odd number. On the other hand, in [9, 10] it was showed the property:

$$D(m)=-\frac{1}{2}\sum_{j=1}^{m}\frac{(-1)^j}{j}\binom{m}{j}r_j(m), m \text{ is odd}, \tag{10}$$

then (7), (9) and (10) generate the interesting connection:

$$\sigma(n)=\frac{1}{2}(m-2n)\sum_{j=1}^{m}\frac{(-1)^j}{j}\binom{m}{j}r_j(m), \qquad n=2^k\, m. \tag{11}$$

Finally, the comparison of (8) and (11) gives the following identity between colour partitions and the number of representations of a positive integer as a sum of squares:

$$\sum_{j=1}^{m}\frac{(-1)^j}{j}\binom{m}{j}\left(2\,p_j(m)+r_j(m)\right)=0, \tag{12}$$

where m is an arbitrary odd number.

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