Vol.11.Issue.4.2023 (Oct-Dec)

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com

Email:editorbomsr@gmail.com

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

RESEARCH ARTICLE

Ellipse Perimeter Approximation by p-norm formula

Dr. K. Idicula Koshy
Professor of Mathematics (Retired), Kerala Agricultural University
Cheruthuruthil House
Oonnukal P.O.
(via) Omallur, Kerala
Email: kidiculakoshy@gmail.com

DOI:10.33329/bomsr.11.4.93

Dr. K. Idicula Koshy

ABSTRACT

The YNOT formula for the perimeter of the standard ellipse, having 'a' and 'b' respectively as major and minor radii, states that the perimeter **P (a, b)** is approximately equal to $\mathbf{4*(a^y + b^y)^{(1/y)}}$, where y = ln (2)/ln ($\pi/2$). Though the formula is simple for computation, its Relative Error yield is of order $\mathbf{10^{(-3)}}$, which is very high when compared with that of several other formulae for the purpose. In this article, we introduce a similar formula for Ellipse Perimeter Approximation, retaining the form of the formula as such, but replacing the **fixed index 'y' by a variable index 'p'.** Our new formula for ellipse perimeter approximation is: $\mathbf{P\ (a, b) \cong 4*\ (a^p + b^p)^{(1/p)}}$. We prefer to name the new formula as the **'p-norm Formula',** as in Mathematical Analysis, the expression $(\mathbf{a^p + b^p})^{(1/p)}$ **is known as the p-norm** of the ordered pair (a, b). The Absolute Relative Error due to the 'p-norm Formula' introduced here is less than $\mathbf{8.92*10^{(-6)}}$, much less than that put out by the YNOT formula.

Keywords: Ellipse, Aspect Ratio, Simpson's Rule, YNOT formula, p-norm, Relative Error

1. Introduction

The ellipse, whose rectangular cartesian equation is $(x/a)^2 + (y/b)^2 = 1$, is named here as the standard ellipse. ‘a $\neq$ 0’ and ‘b’ (a ≥ b ≥ 0) are the major and minor radii of the ellipse. Its perimeter P (a, b) is given by the definite integral:

$$P(a,b) = \int_0^{2\pi} \sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)}\, d\theta$$

where (a cos θ, b sin θ), 0 ≤ θ < 2 π, is a parametric point on the ellipse. Due to the symmetry of the ellipse w. r. t. its axes, P (a, b) = 4* Q (a, b), where Q (a, b) is the perimeter of the standard ellipse in the first quadrant. Therefore, the first-quadrant perimeter is given by the definite integral:

$$Q(a,b) = \int_0^{\pi/2} \sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)}\ \ d\theta$$

Then, Q (a, 0) = a, Q (a, a) = πa /2, P (a, 0) = 4*a and P (a, a) = 2π*a.

However, if ‘a’ $\neq$ ‘b’ $\neq$ 0, the integral for P (a, b) or Q (a, b) given above could not be evaluated till now by any of the known direct integration methods. Therefore, a closed formula is not available for Ellipse Perimeter, which makes it necessary to look for Numerical Integration Methods to approximate Q (a, b). We follow here Simpson’s 1/3-Rule method for the numerical integration of the integral for Q (a, b) due to its very low relative error. The Quarter Perimeter values **Q (100, b; Sim)** used in **Table 1** below are obtained by this method, by dividing the interval of integration [0, π /2] into 500 equal sub-intervals, so that the **length h** of each sub-interval is equal to π/1000. Then the Absolute Relative Error is of order of h^4 (E. Kreiszig 2010) which is less than 10^{-10}.

2. Terminology and Notations

Conventional notations and terminologies related to the standard ellipse are used in this article. ‘a $\neq$ 0’ and ‘b’ denote the lengths of the semi-major axis (**major radius**) and the semi-minor axis (**minor radius**) of the standard ellipse. The ratio (b/a) is called the **Aspect Ratio,** which takes values in the closed interval [0, 1]. **P (a, b)** and **Q (a, b)** denote respectively the total perimeter and the first-quarter perimeter of the standard ellipse; **P (a, b; p)** and **Q (a, b; p)** are their approximations using the p-norm formula. GM and AM are respectively the Geometric Mean and the Arithmetic mean of ‘a’ and ‘b’. **S** = **Q (a, b; Sim)** denotes the Quarter Perimeter values obtained by Simpson’s (1/3) Rule. QPM and ARE are abbreviations used for the Quarter Perimeter and Absolute Relative Error.

3. Result (p-norm formula)

$$\mathbf{Q\ (a, b) \cong Q\ (a, b; p) = (a^p + b^p)^{(1/p)}} \tag{1}$$

$$\text{where}\quad p = \ln(2)/\ln(\pi/2) + 1.45*(1 - (GM/AM)^k) \tag{2}$$

and
$$k = \begin{cases} 0.0046 * \ln\left(\frac{b}{a}\right) + 0.084, & \text{if } 0.2 \le \left(\frac{b}{a}\right) \le 1 \quad (3a) \\ 0.0056 * \ln\left(\frac{b}{a}\right) + 0.0856, & \text{if } 0.01 \le \left(\frac{b}{a}\right) \le 0.2 \quad (3b) \end{cases}$$

approximates the Quarter Perimeter of the standard ellipse with ARE less than $\mathbf{8.92*10^{(-6)}}$.

(The value of p recommended for (b/a) ∈(0, 0.01)) is discussed in Section 5.)

4. Materials and Methods.

All computations are done in MS Excel. Q (a, b; Sim) values used here are derived with step-width h = π/1000. Due to their very high accuracy, Q (a, b, Sim) values are used for computing not only the values of the index p in the p-norm formula, but also the Relative Error on account of the new formula. The step by step process of computation is given below.

i) Equate S = Q (a, b; Sim) with Q (a, b; p). Then $(a^p + b^p) = S^p$ or $(1 + (b/a)^p = (S/a)^p$ **.** Solving this equation, we get p. The p-values obtained thus starts with ln (2)/ln $(\pi/2) = 1.5349285356614$ … for (b/a) = 1 and ends with 1.6693302026113… for (b/a) = 0.01. (The p-values shown in Table 1 Part(A) column 5 are obtained using the software GeoGebra. The point of intersection of the functions f (x) = 1 + $(b/a)^x$ and g (x) = $(S/a)^x$ gives the value of p. These p-values are used for further computation).

ii) It is **discovered** that the difference d: = p – ln (2)/ln (π/2), is very closely approximated by the expression 1.45*(1- $(GM/AM)^k$), where GM and AM are respectively the Geometric Mean and the Arithmetic mean of 'a' and 'b'.
Then, k = ln (1- d/1.45) / ln (GM/AM).

iii) Next, it is found that the best-fit function representing the k-values are:

$$k = \begin{cases} 0.0046 * \ln\left(\frac{b}{a}\right) + 0.084, \text{if } 0.2 \le \left(\frac{b}{a}\right) \le 1 & (3a) \\ 0.0056 * \ln\left(\frac{b}{a}\right) + 0.0856, \text{if } 0.01 \le \left(\frac{b}{a}\right) \le 0.2 & (3b) \end{cases}$$

Therefore, the best and most suitable estimate for 'p' in the interval [0.01, 1] is

p = ln (2)/ ln $(\pi/2)$ + 1.45*(1- $(GM/AM)^k$), where k is as in (3a) and (3b) above.

iv) The linear correlation coefficient between the p - values (obtained by solving the equations f (x) = $(1+(b/a)^x$ & g(x) = $(S/a)^x$) and the estimated p - values given by p = ln (2)/ ln (π/2) + 1.45*(1- $(GM/AM)^k$) is more than 0.99999#

5. Discussion/Comments

Evidently, Q (a, a; p) = π/2*a for p = ln (2)/ ln $(\pi/2)$, and, Q (a, 0; p) = a, whatever be the value of p. The maximum absolute relative error is less than **$8.92*10^{(-6)}$ (**i. e. less than 9 mm per km). (Table 1). The maximum / minimum relative error occurs at (b/a) = 0.3 and 0.65.

The author has critically examined several Ellipse Perimeter Approximation Formulae known after several eminent Mathematicians: Kepler, Euler, Seki, Muir, Maertens (YNOT formula), Rivera, Lindner, Zafary, Cantrell etc. and, in particular, the second formula of the Great Indian Mathematical Genius Srinivasa Ramanujan [2]. None of these formulae gives Maximum Absolute Relative Error less than 9-millimeter per kilometer consistently for ellipses of all aspect ratios. **Therefore, the author's 'p-norm formula' introduced here claims the status of an important closed formula for EPM approximation.**

Also, either of the expressions for 'k' given in 3 (a) / 3 (b) can be used in getting 'p' at (b/a) = 0.2. For, the author has verified that the 'p' with 'k' as in equation 3(a) maintains the Relative Error of order $10^{(-6)}$ for all (b/a) in [0.16, 1] and that with 'k' as in 3(b) maintains the same order of Relative Error for all (b/a) in [0.01, 0.32]. The change of the estimated expression for 'k' in Table 1, from 3 (a) to 3 (b) at (b/a) = 0.2 was made for easy recollection of the point of transition.

The Title Row of Table 1 Part (A) gives the expressions for 'd' in terms of 'p', and also 'k' in terms of 'd'. In Table 1 Part (B), their estimates are indicated by the suffix 'e' and these estimates are used there for computation of Q (a, b; p) and the Relative Error. Suffixes are used for clarity of procedure. Henceforth, the main Result in Section 3 can be directly used for computation of EPM of any ellipse, having (b/a) ∈ [0.01, 1]

Again, for (b/a): 0 < (b/a) < 0.01, (that is: for very flat ellipses), the expression 3 (b) may not yield correct value for 'p' due to the presence of log (b/a). Therefore, taking

p = 1.66934 + 33*(0.01-b/a) is a way out to get the value of Q (a, b; p). This 'p' ensures that, in the interval (0, 0.01), 'p' is increasing, 'p' lies strictly between 1.6693302026113 and 2.0 and that

Q (a, b; 2) = $(a^2+b^2)^{(1/2)}$ is a lower bound for every Q (a, b; p).

We recall that for $1 \leq p \leq q \Rightarrow$ Q (a, b; p) $\geq$ Q (a, b; q).

Obviously, the author's 'p-norm formula' introduced in this article is a new and high accuracy formula for Ellipse Perimeter Approximation.

Table 1: p- norm formula for Quarter Perimeter Approximation of the Standard Ellipse: Part (A)

a	b	b/a	S = Q (100, b; Sim.)	p: $1+(b/a)^p = (S/a)^p$	d = p - ln(2)/ln(π/2)	GM/AM	k: d = 1.45*(1 - $(GM/AM)^k$)
100	99	0.99	156.295221198759	1.5349300346525	0.000001498991125	0.999987373976503	0.081877005124409
100	98	0.98	155.512803035381	1.5349345925105	0.000006056849125	0.999948983496128	0.081876239686939
100	97	0.97	154.732408602913	1.5349423029189	0.000013767257525	0.999884040791483	0.081874954545554
100	96	0.96	153.954068977126	1.5349532628070	0.000024727145625	0.999791731748236	0.081873128740673
100	95	0.95	153.177815915124	1.5349675724869	0.000039036825525	0.999671214852201	0.081870744921279
100	94	0.94	152.403681875157	1.5349853357976	0.000056800136225	0.999521620085841	0.081867785923806
100	93	0.93	151.631700037168	1.5350066602563	0.000078124594925	0.999342047771291	0.081864233941659
100	92	0.92	150.861904324107	1.5350316572175	0.000103121556125	0.999131567356817	0.081860070382150
100	91	0.91	150.094329424045	1.5350604420407	0.000131906379325	0.998889216143399	0.081855276014499
100	90	0.90	149.329010813121	1.5350931342674	0.000164598606025	0.998613997947909	0.081849831143354
Rows 89 to 26 deleted							
100	25	0.25	107.230272189460	1.5598793292295	0.024950793568125	0.800000000000000	0.077784961189479
100	24	0.24	106.774014536549	1.5611802276534	0.026251691992025	0.790157981542961	0.077574408434046
100	23	0.23	106.327536868368	1.5625567254105	0.027628189749125	0.779810003790686	0.077351847827221
100	22	0.22	105.891173106713	1.5640151426459	0.029086606984525	0.768920616364497	0.077116285904241
100	21	0.21	105.465276743059	1.5655626093531	0.030634073691725	0.757450528091874	0.076866601240618
100	20	0.20	105.050222698445	1.5672072131955	0.032278677534125	0.745355992499930	0.076601520811938
a	**b**	**b/a**	**S= Q(100, b; Sim.)**	**p: $1+(b/a)^p=(S/a)^p$**	**d = p - ln(2)/ln(π/2)**	**GM/AM**	**k: d = 1.45*(1 - $(GM/AM)^k$)**
100	20	0.20	105.050222698445	1.5672072131955	0.032278677534125	0.745355992499930	0.076601520811938
100	19	0.19	104.646409451062	1.5689581833555	0.034029647694125	0.732588057737928	0.076319590533511
100	18	0.18	104.254261485833	1.5708261216290	0.035897585967625	0.719091641884625	0.076019138141396
100	17	0.17	103.874232134835	1.5728232963232	0.037894760661825	0.704804380447463	0.075698225866419
100	16	0.16	103.506806897050	1.5749640209003	0.040035485238925	0.689655172413793	0.075354589303757
100	15	0.15	103.152507352688	1.5772651488939	0.042336613232525	0.673562321079551	0.074985557295030
100	14	0.14	102.811895824480	1.5797467313451	0.044818195683725	0.656431120486656	0.074587945193235
100	13	0.13	102.485580990886	1.5824329062210	0.047504370559625	0.638150668223715	0.074157910044331
100	12	0.12	102.174224732294	1.5853531269050	0.050424591243625	0.618589574131742	0.073690749913515
100	11	0.11	101.878550604060	1.5885438999996	0.053615364338225	0.597590052316288	0.073180619098178
100	10	0.10	101.599354502522	1.5920513125618	0.057122776900425	0.574959574576069	0.072620112503900
100	9	0.09	101.337518361821	1.5959348289380	0.061006293276625	0.550458715596330	0.071999638907769
100	8	0.08	101.094028165077	1.6002732207721	0.065344685110725	0.523782800878924	0.071306438216062
100	7	0.07	100.869998319400	1.6051742759571	0.070245740295725	0.494532955339176	0.070522965566946
100	6	0.06	100.666705836685	1.6107916545762	0.075863118914825	0.462167875996826	0.069624072852845
100	5	0.05	100.485640478649	1.6173564339297	0.082427898268325	0.425917709999960	0.068571706962883
100	4	0.04	100.328582826687	1.6252423495220	0.090313813860625	0.384615384615385	0.067303881215431

100	3	0.03	100.197736240671	1.6351211438411	0.100192608179725	0.336320545159005	0.065708237581741	
100	2	0.02	100.095979045020	1.6484222730186	0.113493737357225	0.277296776935901	0.063543084218685	
100	1	0.01	100.027463597804	1.6693302026113	0.134401666949925	0.198019801980198	0.060067120141285	

Table 1: p- norm formula for Quarter Perimeter Approximation of the Standard Ellipse: Part (B)

a	b	b/a	S = Q (100, b; Sim.)	k_e = estimate of k (with 0.0046, and 0.084)	d_e = estimate of d = 1.45* (1- $(GM/AM)^{ke}$)	p_e = estimate of p = 1.45* (1- $(GM/AM)^{ke}$) + ln(2)/ln(π/2)	Q_e = Q (a, b; p_e) = estimate of Q (a, b)	Relative Error = (Q_e - S)/S
100	99	0.99	156.295221198759	0.083953768455074	0.000001537012156	1.534930072673530	156.295219450521000	-1.11855E-08
100	98	0.98	155.512803035381	0.083907067546339	0.000006207080641	1.534934742742020	155.512796163150000	-4.41908E-08
100	97	0.97	154.732408602913	0.083859887645570	0.000014101022013	1.534942636683390	154.732393415230000	-9.81545E-08
100	96	0.96	153.954068977126	0.083812218825207	0.000025312780315	1.534953848441690	153.954042471047000	-1.72169E-07
100	94	0.94	152.403681875157	0.083715373142897	0.000058081972574	1.534986617633950	152.403624498095000	-3.76481E-07
100	93	0.93	151.631700037168	0.083666174812960	0.000079844174012	1.535008379835390	151.631623504821000	-5.04725E-07
100	92	0.92	150.861904324107	0.083616444598880	0.000105334032399	1.535033869693770	150.861806427889000	-6.48913E-07
100	91	0.91	150.094329424045	0.083566170874432	0.000134663287340	1.535063198948720	150.094208163422000	-8.07896E-07
100	90	0.90	149.329010813121	0.083515341627974	0.000167947725056	1.535096483386430	149.328864399088000	-9.80479E-07
Rows 89 to 26 deleted								
100	25	0.25	107.230272189460	0.077623045938849	0.024899305119884	1.559827840781260	107.231025560040000	7.02573E-06
100	24	0.24	106.774014536549	0.077435264764055	0.026205032934011	1.561133568595390	106.774666695540000	6.10784E-06
100	23	0.23	106.327536868368	0.077239490537729	0.027588442696576	1.562516978357950	106.328066218586000	4.97849E-06
100	22	0.22	105.891173106713	0.077035012429903	0.029056261589623	1.563984797251000	105.891557123783000	3.62653E-06
100	21	0.21	105.465276743059	0.076821020357983	0.030616101233203	1.565544636894580	105.465492202699000	2.04294E-06
100	20	0.20	105.050222698445	0.076596585602803	0.032276621234101	1.567205156895480	105.050245973533000	2.21562E-07
a	**b**	**b/a**	**S = Q (100, b; Sim.)**	**k_e = estimate of k (with 0.0056 and 0.0856)**	**d_e = estimate of d = 1.45* (1- $(GM/AM)^{ke}$)**	**p_e = estimate of p = 1.45* (1- $(GM/AM)^{ke}$) + ln(2)/ln(π/2)**	**Q_e = Q (a, b; p_e) = estimate of Q (a, b)**	**Relative Error = (Q_e - S)/S**
100	20	0.20	105.050222698445	0.076587147690369	0.032272688833259	1.567201224494630	105.050290484375000	6.45272E-07
100	19	0.19	104.646409451062	0.076299905241799	0.034020974132279	1.568949509793650	104.646501804315000	8.82527E-07
100	18	0.18	104.254261485833	0.075997128802685	0.035887322457108	1.570815858118480	104.254363864005000	9.82005E-07
100	17	0.17	103.874232134835	0.075677041685182	0.037884295556567	1.572812831217940	103.874329478251000	9.37128E-07
100	16	0.16	103.506806897050	0.075337543803010	0.040026555218892	1.574955090880270	103.506883953307000	7.44456E-07
100	15	0.15	103.152507352688	0.074976128084639	0.042331368007290	1.577259903668670	103.152549092362000	4.04640E-07
100	14	0.14	102.811895824480	0.074589768004312	0.044819273864634	1.579747809526010	102.811887965348000	-7.64419E-08
100	13	0.13	102.485580990886	0.074174763360251	0.047514987673686	1.582443523335060	102.485510650475000	-6.86344E-07
100	12	0.12	102.174224732294	0.073726524197280	0.050448639749966	1.585377175411340	102.174081230690000	-1.40448E-06
100	11	0.11	101.878550604060	0.073239260486138	0.053657522698355	1.588586058359730	101.878326440143000	-2.20031E-06
100	10	0.10	101.599354502522	0.072705523479233	0.057188618291310	1.592117153952680	101.599046525554000	-3.03129E-06
100	9	0.09	101.337518361821	0.072115504591550	0.061102369701230	1.596030905362600	101.337129153729000	-3.84071E-06
100	8	0.08	101.094028165077	0.071455919591874	0.065478528209189	1.600407063870560	101.093567630502000	-4.55551E-06
100	7	0.07	100.869998319400	0.070708143793176	0.070425637031083	1.605354172692460	100.869485436861000	-5.08459E-06
100	6	0.06	100.666705836685	0.069844899986144	0.076097307345215	1.611025843006590	100.666170423863000	-5.31867E-06
100	5	0.05	100.485640478649	0.068823899268098	0.082722234340853	1.617650770002230	100.485124591577000	-5.13394E-06
100	4	0.04	100.328582826687	0.067574295380738	0.090665089400877	1.625593625062250	100.328140844157000	-4.40535E-06
100	3	0.03	100.197736240671	0.065963275775008	0.100567684694060	1.635496220355440	100.197431173227000	-3.04465E-06
100	2	0.02	100.095979045020	0.063692671169602	0.113750148532317	1.648678684193690	100.095867928552000	-1.11010E-06
100	1	0.01	100.027463597804	0.059811046958467	0.133855998999840	1.668784534661220	100.027541695350000	7.80761E-07

References

[1]. Erwin Kreiszig, "Advanced Engineering Mathematics", Ch. 17; Wiley, 2010.

[2]. Gerard P Michon, "Perimeter of an ellipse: A Review", Weblink: Numericana.