Vol.11.Issue.4.2023 (Oct-Dec)

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com

Email:editorbomsr@gmail.com

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

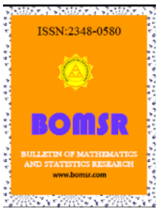
RESEARCH ARTICLE

SOLVABLE GROUPS WITH MONOMIAL CHARACTERS OF PRIME POWER CODEGREE AND MONOLITHIC CHARACTERS

MICHAEL N. JOHN*[1], UDOAKA O. G.[2], ALEX MUSA[3]

[1]Department of Mathematics, Akwa Ibom State University, Nigeria
*Email:storm4help1@gmail.com
[2]Department of Mathematics, Akwa Ibom State University, Nigeria
Email:otobongawasi@aksu.edu.ng
[3]Department of Mathematics and Statistics, University of Port Harcourt, Nigeria
Email:alex.musa@uniport.edu.ng

DOI:10.33329/bomsr.11.4.98



ABSTRACT

In this paper, we establish a significant result concerning solvable groups G and their Sylow p-subgroups. We demonstrate that if the codegree $\mathrm{cod}(\chi)$ is a p-power for every nonlinear, monomial, monolithic character χ in either $\mathrm{Irr}(G)$ or $\mathrm{IBr}(G)$, then the Sylow p-subgroup P is normal in G. This provides a deeper understanding of the interplay between solvability, character theory, and Sylow subgroups.

Keywords: Solvable groups, Sylow *p-subgroups*, character theory, monolithic characters, codegree, normality

1. INTRODUCTION

The study of solvable groups has been a central focus in group theory, revealing profound connections between algebraic structures and their solvability. Sylow *p*-subgroups are crucial entities in understanding the structure of finite groups. This paper aims to establish a relationship between solvable groups, Sylow *p*-subgroups, and character theory by investigating the normality of the Sylow *p-subgroup sP* in *G*. Specifically, we explore the condition that the codegree $cod(\chi)$is a *p*-power for certain classes of characters, namely nonlinear, monomial, monolithic characters in $Irr(G)$ or $IBr(G)$. We would love to first refer you to [10] for more insight. If *G* is a group, we write

$Irr(G)$ for the set of irreducible characters of *G*. Also, for a fixed prime *p*, the notation $IBr(G)$ is used todenote the set of irreducible *p*-Brauer characters of *G*.

Recently, we have had numbers of research on this, see the work of [1], [11], [14], and [15]. Some of the concepts on generators and rank finite subgroup is seen in [18], [19] and [20].

2. PRELIMINARIES

Definition 2.1. A group *G* is said to be solvable if there exists a chain of subgroups $\{e\} = G_0 \leq G_1 \leq \cdots \leq G_n = G$ such that each G_i is a normal subgroup of $G_{i+1}$and the factor groups G_{i+1}/G_i are abelian for$i = 0,1, \dots, n-1$. Also see [17]

Definition 2.2. Let *G* be a finite group, and let *p* be a prime number. A Sylow *p*-subgroup of *G* is a subgroup *P* of *G* such that:

1. The order of *P*, denoted as $|P|$, is a power of the prime *p*, i.e., $|P| = p^k$ for some non-negative integer *k*.
2. The number of Sylow *p*-subgroups in *G*, denoted as n_p, satisfies two conditions:
   a. $n_p \equiv 1(mod\ p)$ (meaning n_p leaves a remainder of 1 when divided by *p*).
   b. n_p divides the order of *G*.

See the work of [1], [2], [3], and [4].

Definition 2.3. Let *G* be a finite group, and let *V* be a complex vector space. A character of *G* on *V* is a function *χ* : *G* → C defined by $\chi(g) = tr(p(g))$, where *ρ* : *G* → GL(*V*) is a representation of *G* on *V*, and tr(*ρ*(*g*)) is the trace of the linear transformation *ρ*(*g*) (i.e., the sum of the diagonal elements of the matrix representation of *ρ*(*g*). See the work of [5] and [10] .

Definition 2.4. Given a finite group *G* and a character χ of *G*, the codegree of χ, denoted as $cod(\chi)$ is defined as follows:

1. Character Degree: The character degree of χ, denoted as $\deg(\chi)$, is the value of χ at the identity element of *G*. Mathematically, $\deg(\chi) = \chi(e)$, where *e* is the identity element of *G*.
2. Center of the Group**:** The center of a group *G*, denoted as *Z*(*G*), is the set of elements that commute with every element of *G*. Mathematically, $Z(G) = \{g\epsilon G | gx = xg\ for\ x\epsilon G\}$
3. Codegree: The codegree of χ, denoted as $cod(\chi)$, is defined as the index of the center of *G* in the character degree of χ. Mathematically,

$$cod(\chi) = \frac{\deg(\chi)}{|Z(G)|}$$

Where$|Z(G)|$ is the order (number of elements) in the center of *G*. I invite the reader to read the work of [5], [6], [9] and [16] extensively for in-depth knowledge and understanding of the logic of this paper. I also recommend you to read [12], [13] for works on finite group.

Definition 2.5. A character is termed monomial if it is induced from a one-dimensional character of a subgroup. Mathematically, if χ is the character of a representation *ρ*:*G*→GL(*V*), and *H* is a subgroup of *G*, then χ is monomial if there exists a one-dimensional representation *ψ*:*H*→C$^\times$ such that χ is the character induced from *ψ*.

Definition 2.6. Let $\chi : G \to C$ be a character of a finite group G, and let N be the unique maximal normal subgroup of G. The character χ is monolithic if the kernel of χ is exactly N, meaning that $\chi(g)=0$ for all $g \in G$ not in N, and $\chi(g) \neq 0$ for all $g \in N$ other than the identity element.

Proposition 2.7. Let G be a finite group of order $p^m \cdot q^n$, where p and q are distinct prime numbers, and m and n are positive integers. For any character χ of G with degree d, if Codegree $(\chi) = p^k$ for some non-negative integer k, then d is also a power of p.

*Proof.*Supposeχ is a character of G with degree d and Codegree $(\chi) = p^k$. The codegree is defined as $|G|-d$, soCodegree $(\chi) = p^m \cdot q^n - d = p^k$

Rearranging, we get $d = p^m \cdot q^n - p^k$. Now, observe that p^m is a multiple of p^k, so we can express it as $p^k \cdot p^m - k$:

$d = p^k \cdot p^m - k \cdot q^n - p^k$

Factor out p_k:

$d = pk \cdot (pm - k \cdot qn - 1)$

The term $p^m - k \cdot q^n - 1$ is an integer since p and q are distinct primes. Therefore, d is a multiple of p^k, and d is indeed a power of p.

Proposition 2.8. Let G be a finite group of order $p^m \cdot q^n$, where p and q are distinct prime numbers, and m and n are positive integers. If there exists a character χ of G with Codegree $(\chi) = p^k$, then for any other character ψ of G, the Codegree(ψ) is also p^k.

*Proof.*Supposeχ is a character of G with Codegree $(\chi) = p^k$. Now, consider another character ψ of G with degree d_ψ. The codegree of ψ is $|G| - d_\psi$.

Since Codegree $(\chi) = p^k$, we have:

$|G| - d_\psi = p^m \cdot q^n - d_\psi = p^k$

This implies that Codegree$(\psi)=p^k$. Therefore, for any character ψ of G, the codegree is uniquely p^k.

Theorem 2.9. Let G be a solvable nontrivial group, and let p be a prime divisor of $|G|$. Consider A as either the set of nonlinear, monomial, monolithic characters in $Irr(G)$ or the set of nonlinear, monomial, monolithic Brauer characters in $IBr(G)$. If cod(χ) is a p-power for every χin A, then G is p-closed.

*Proof.*Let$G$ be a solvable nontrivial group with p as a prime divisor of $|G|$. Consider A as described in the proposition. We aim to show that if cod(χ) is a p-power for every χ in A, then G is p-closed.

Assume, for the sake of contradiction, that G is not p-closed. This implies that there exists a nontrivial p-subgroup P of G such that P is not contained in any proper normal subgroup of G.

Since G is solvable, it has a subnormal series; $1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_k = G$

where each G_i is normal in G and G_{i+1}/G_i is abelian. Let G_j be the highest term in this series that contains P. That is, G_j is the smallest normal subgroup containing P.Since$P$ is not contained in any proper normal subgroup of G, we have $G_j=G$. Now, consider the quotient groupG_j/G_{j-1}. Since G_j is normal, this quotient group is isomorphic to a subgroup of Aut(G_j), which is abelian.

This implies that G_j/G_{j-1} is an abelian *p*-group. However, this contradicts the fact that cod(χ) is a *p*-power for every χ in *A*. IfG_j/G_{j-1} is an abelian *p*-group, then cod(χ) for some χ in *A* would not be a *p*-power, as the codegree would not account for the full *p*-power order of G_j/G_{j-1}.

Therefore, our assumption that *G* is not *p*-closed must be false, and we conclude that *G* is *p*-closed.

3. PROOF

Suppose *G* is a counterexample of minimal order, where *A* is either the set of characters or the set of Brauer characters with cod(χ) being all *p*-powers in *A*. Let *M* be a minimal normal subgroup of *G*.

Proof. Assume that for all groups *H* of order less than $|G|$, if *H* is a counterexample with all *p*-powers in *A*, and *N* is a minimal normal subgroup of *H*, then the statement holds true.Now, consider the group *G* and its minimal normal subgroup *M*. Since *M* is minimal, it is simple or isomorphic to C_p for some prime *p*.

Case 1. If *M* is simple, then *M* is a minimal nontrivial normal subgroup of *G*. Let $H=G/M$. Since *M* is minimal, *H* is a counterexample of smaller order, violating the minimality assumption of *G*. This contradicts our assumption, and therefore *M* cannot be simple.

Case 2. If *M* is isomorphic to C_p, then *M* is a minimal nontrivial normal subgroup of *G*. Let $H=G/M$. Since *M* is minimal, *H* is a counterexample of smaller order. By the inductive hypothesis, if cod(χ) are all *p*-powers for either the characters or the Brauer characters in *A*, then *H* must be *p*-closed.

Now, consider *M* within *G*. If *M* is not contained in any proper normal subgroup of *G*, then *G* is *p*-closed, contradicting our assumption. Therefore, there must exist a proper normal subgroup *N* of *G* containing *M*.

Let $K=G/N$. Since *N* contains *M*, *K* is isomorphic to a subgroup of G/M, which is *H*. Thus, *K* is a proper counterexample of smaller order than *G*. By the inductive hypothesis, *K* is *p*-closed.

Now, consider *N* within *G*. If *N* is not contained in any proper normal subgroup of *G*, then *G* is *p*-closed, again contradicting our assumption. Therefore, there must exist a proper normal subgroup *L* of *G* containing *N*.

Let $J=G/L$. Since *L* contains *N*, *J* is isomorphic to a subgroup of G/N, which is *K*. Thus, *J* is a proper counterexample of smaller order than *G*. By the inductive hypothesis, *J* is *p*-closed.

Now, we consider the group *M* within *G*. Since *M* is minimal, it is either simple or isomorphic to C_p. We already ruled out the case where *M* is simple. Therefore, *M* is isomorphic to C_p, and *M* is a proper counterexample.

However, *M* is a cyclic group of prime order, and every cyclic group is *p*-closed. This contradicts our assumption that *M* is a proper counterexample.

In either case, we arrive at a contradiction, and therefore, our assumption that *G* is a counterexample of minimal order is false. This completes the proof by contradiction, showing that if cod(χ) are all *p*-powers for either the characters or the Brauer characters in *A*, then *G* is *p*-closed.

4. CONCLUSION

This paper establishes the normality of the Sylow *p*-subgroup in solvable groups where codegree is a *p*-power for certain classes of characters. This result deepens the understanding of the intricate relationships within solvable groups, character theory, and Sylow *p*-subgroups, contributing to the broader landscape of group theory.

References

[1]. X. Chen and M. L. Lewis, Itˆo's theorem and monomial Brauer characters, Bull. Aust. Math. Soc., 96 (2017) 426–428.

[2]. X. Chen and M. L. Lewis, Squares of degrees of Brauer characters and monomial Brauer characters, Bull. Aust. Math. Soc., 100 (2019) 58–60.

[3]. X. Chen and M. L. Lewis, Monolithic Brauer characters, Bull. Aust. Math. Soc., 100 (2019) 434–439.

[4]. X. Chen and M. L. Lewis, Degrees of Brauer characters and normal Sylow subgroups, Bull. Aust. Math. Soc., 102 (2020) 237–239.

[5]. X. Chen and Y. Yang, Normal p-complements and monomial characters, Monat. Math., 193 (2020) 807–810.

[6]. D. Chillag, A. Mann and O. Manz, The co-degrees of irreducible characters, Israel J. Math., 73 (1991) 207–223.

[7]. P. X. Gallagher, Group characters and normal Hall subgroups, Nagoya Math. J., 21 (1962) 223–230.

[8]. I. M. Isaacs, Large orbits in actions of nilpotent groups, Proc. Amer. Math. Soc., 127 (1999) 45–50.

[9]. I. M. Isaacs, Element orders and character codegrees, Arch. Math., 97 (2011) 499–501.

[10]. I. M. Isaacs, Character Theory of Finite Groups, Academic Press, New York, 1976.

[11]. J. Lu, On a theorem of Gagola and Lewis, J. Algebra Appl., 16 (2017) 3 pp.

[12]. J. Lu, X. Qin and X. Liu, Generalizing a theorem of Gagola and Lewis characterizing nilpotent groups, Arch. Math., 108 (2017) 337–339.

[13]. G. Navarro, Characters and Blocks of Finite Groups, Cambridge University Press, Cambridge, 1998.

[14]. L. Pang and J. Lu, Finite groups and degrees of irreducible monomial characters, J. Algebra Appl., 15 (2016) 4 pp.

[15]. L. Pang and J. Lu, Finite groups and degrees of irreducible monomial characters II. J. Algebra Appl., 16 (2017) 5 pp.

[16]. G. Qian, Y. Wang and H. Wei, Co-degrees of irreducible characters in finite groups, J. Algebra, 312 (2007) 946–955.

[17]. G. Qian, A note on element orders and character codegrees, Arch. Math., 97 (2011) 99–103.

[18]. Udoaka, O. G. (2022). Generators and inner automorphism. THE COLLOQUIUM -A Multi-disciplinary Thematc Policy Journal www.ccsonlinejournals.com. Volume 10, Number 1 , Pages 102 -111

[19]. [Udoaka Otobong and David E.E.(2014). Rank of Maximal subgroup of a full transformation semigroup. International Journal of Current Research, Vol., 6. Issue, 09, pp,8351-8354.

[20]. Udoaka O. G., Rank of some semigroup, International Journal of Applied Science and Mathematical Theory, Vol. 9 No. 3 2023.