Vol.12.Issue.1.2024 (Jan-March)

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com

Email:editorbomsr@gmail.com

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

RESEARCH ARTICLE

SOME COMMON FIXED POINT THEOREMS USING GENERALIZED CONE METRIC SPACES WITH BANACH ALGEBRA

ANIL KUMAR MISHRA[1*], PADMAWATI[2]

[1]Department of Mathematics, Govt.V.Y.T. P.G.Auto.College, Durg, C.G., India

[2]Department of Mathematics, Govt.V.Y.T.P.G.Auto.College, Durg, C.G., India

Email:mshranil@gmail.com[1*]& padmavati.sudha62@gmail.com[2]

*Corresponding author

DOI:10.33329/bomsr.12.1.1

ABSTRACT

In this paper, we have proved some common fixed point results using generalized contraction mapping in generalized cone metric space over Banach algebras. Our results are generalized results of Naval [18] and Öztürk, [20].

Keywords: Generalized cone metric spaces, Banach algebras, Weakly compatible maps, generalized Lipschitz conditions, Common fixed point.

MSC(2020) : 47H10, 54H25

1. INTRODUCTION

Fixed point theory has great significance in the field of analysis. Fixed point theory is used to find solution of different mathematical problems. Several mathematicians studied fixed point theorems over different spaces as metric space, D- metric space, G-metric space, cone metric space and many more. One of the most important and fruitful result in metric space for contractive type mapping was given by Banach [14] called Banach Contraction principal. This principal was generalized and its several variants were studied over different spaces. Dhage's theory for D-metric space [7], Mustafa and Sims ([8],[9]) introduced a more appropriate generalization of metric space, which are called G- metric space or Generalized metric space. Motivated by the idea of Huang and Zhang [3] about cone metric space in 2010, Ismat Beg et.al [13] introduce the concept of G-cone metric space by replacing the set of real numbers by an ordered Banach space, proved convergence properties of sequence and some fixed point theorems in this space. The concept of a G-cone metric space is more general than that G- metric space and cone metric space.

The study of unique common fixed point for map satisfying certain contraction has been center of rigorous research activity. Jungck [21, 22], proved a common fixed point theorem for commuting maps, generalized the Banach Contraction principal and also defined a pair of self mapping to be weakly compatible if they commute at their coincidence points. Later Sing and Jain proved a common fixed point theorem for two self mappings satisfying the concept of compatibility in G-cone metric space which generalizes the result of Ismat Beg et.al [13].

In this paper, we have proved some common fixed point theorem on G-cone metric space over Banach algebra. Many fixed point theorems have been proved in normal or non normal cone metric space by some authors [1,2,13,16,17,22].

In this paper, we assume that P is a cone in A with int$P \neq \varphi$(θ, the additive identity element of A) and $\preceq$ is the partial ordering with respect to P where A is a real Banach algebra. That is, A is a real Banach space in which an operation of multiplication is defined, satisfying the following properties[4] (for all x, y, z $\in$ A, $\alpha \in$ R) :

1. $(xy)z = x(yz)$;
2. $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
4. there exists $e \in E$ such that $xe = ex = x$;
5. $||e|| = 1$;
6. $||xy|| \leq ||x||.||y||$;

An element $x \in A$ is called invertible if there exists $x^{-1} \in A$ such that $xx^{-1} = x^{-1}x = e$.

Proposition 2.1. [4] Let $x \in A$ be a Banach algebra with a unit e, then the spectral radius $\rho(u)$ of $u \in A$ holds

$\rho(u) = \lim_{n\to\infty} ||u_n||^{1/n} = \inf||u^n||^{1/n} < 1$

Further, $e - u$ is invertible and $(e - u)^{-1} = \sum_0^\infty u^i$

Consider a Banach algebra A, θ be the null vector, e be the identity element of A and a subset P of A is called a cone if it satisfies the following:

1. $\{\theta, e\} \subset P$ and P is closed;
2. $P^2 = P P \subset P$;
3. $\alpha P + \beta P \subset P$, for all non-negative real numbers α and β;
4. $P \cap (-C P) = \{\theta\}$;

With respect to cone P, a partial ordering $\preceq$ is defined as $u \preceq w$ if and only if $w - u \in P$ and $u \prec w$ if $u \preceq w$ and $u \neq w$ whereas $u \ll w$ means $w - u \in$ int$P$. If A is a Banach space and $P \subset A$, satisfies the conditions 1, 3 and 4 then P is called a cone of A.

Remark 2.2. [4] If $\rho(x) < 1$, then $\|x_n\| \to 0$ as $n \to \infty$

Definition 2.3. [4] Consider X is a non-empty set, A be a Banach algebra and $P \subseteq A$ be a cone. Suppose the mapping $d : X \times X \to A$ satisfies the following for all x, y, z $\in$ X,

1. $d(x, z) = \theta$ if and only if $x = z$, and $\theta \preceq d(x, z)$,

2. $d(x, z) = d(z, x)$, (symmetry).

3. $d(x, z) \preceq d(x, y) + d(y, z)$ for every $x, y, z \in X$, (rectangle inequality)

Here d is called a cone metric and (X, d) is called Cone metric space over a Banach algebra A (In Short CMSBA). Note that $d(x, z) \in P$ for all $x, y \in X$.

Definition 2.4. [6] Let X be a non-empty set, A, a Banach algebra and $G : X^3 \to A$ be a function satisfying the following properties:

1. $G(x, y, z) = \theta$ if and only if $x = y = z$
2. $\theta \prec G(x, y, z)$,for all $x, y \in X$, with $x \neq y$
3. $G(x, x, y) \preceq G(x, y, z)$ for all $x, y, z \in X$, with $\mathbf{z} \neq y$
4. $G(x, y, z) = G(y, z, x) = G(x, z, y) = \ldots$ (symmetry).
5. $G(x, y, z) \preceq G(x, a, a) + G(a, y, z)$ for all $a, x, y, z \in X$ (rectangle inequality)

Then G is called a G-cone metric over Banach algebra A and the pair (X, G) denotes a G-cone metric space over Banach algebra.

Remark 2.5. 1. If A is a Banach space in Definition 2.3, then (X, G) becomes a G-cone metric space and if in addition z = y, then it becomes a cone metric space as in Huang and Zhang [3]

2. If A = R in Definition 2.3, we obtain a G-metric space as in Mustafa and Sims [9] and if in addition, z = y in G(x, y, z), then it becomes a metric space.

Definition 2.6. [4] Let (X, d) be a cone metric space over Banach algebra A and $\{x_n\}$ a sequence in X. We say that

1. $\{x_n\}$ is a convergent sequence if, for every $c \in A$ with $\theta \ll c$, there is an N such that $d(x_n, x) \ll c$ for all $n \geq N$. Ones write it by $x_n \to x (n \to \infty)$;

2. $\{x_n\}$ is a Cauchy sequence if, for every $c \in A$ with $\theta \ll c$, there is an N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$;

3. (X, d) is a complete cone metric space if every Cauchy sequence in X is convergent.

Lemma 2.7. [4] Let A be a Banach algebra and k, a vector in A. If $0 \leq \rho(k) < 1$, then we have $r((e - k)^{-1}) < (1 - \rho(k))^{-1}$.

Lemma 2.8. [2] Let A be a Banach algebra and x, y be vectors in A. If x and y commute, then the following holds:

1. $r(xy) \leq r(x)r(y)$;
2. $r(x + y) \leq r(x) + r(y)$;
3. $|r(x) - r(y)| \leq r(x - y)$.

Lemma 2.9.[2] If A is real Banach algebra with a solid cone P and $\{x_n\}$ is a sequence in A. Suppose $||x_n|| \to 0$ $(n \to \infty)$ for any $\theta \ll c$. Then $x_n \ll c$ for all $n > N^1$, $N^1 \in N$

Lemma 2.10 [6] If E is a real Banach space with a solid cone P and if $||x_n|| \to 0$ as $n \to \infty$, then for any $\theta \ll c$, there exists $N \in N$ such that, for any $n > N$, we have $x_n \ll c$.

Lemma 2.11.[17] Let A be a Banach algebra and $k \in A$. If $\rho(k) < 1$, then $\lim_{n\to\infty} \|k^n\| = 0$.

Lemma 2.12.[16] If X is a symmetric G−cone metric space, then d_G (x, y) = 2G (x, y, y).

Example 2.13. [2] Let A be the Banach space of all continuous real-valued functions C(K) on a compact Hausdorff topological space K, with multiplication defined pointwise. Then A is a Banach algebra, and the constant function f(t) = 1 is the unit of A.

Let P = {f ∈A : f(t) ≥ 0 for all t ∈ K}. Then P ⊂ A is a normal cone with a normal constant M = 1. Let X = C(K) with the metric d : X × X → A defined by

$$d(f, g) = |f(t) - g(t)|$$

where t ∈ K. Then (X, d) is a cone metric space over a Banach algebra A.

Definition 2.14. [6] Let (X, G) be a G-cone metric space over Banach algebra. G is said to be symmetric if:

$$G(x, y, y) = G(x, x, y)$$

for all x, y, z ∈ X.

Definition 2.15. [6] A G-cone metric space over Banach algebra A is said to be G-bounded if for any x, y, z ∈ X, there exists K ≻ θ such that

$$||G(x, y, z)|| \preceq K.$$

Definition 2.16. [6] Let (X, G) be a G-cone metric space over Banach algebra and $\{x_n\}$ a sequence in X, $c \gg \theta$ with c ∈ A. Then

1. $\{x_n\}$ converges to x ∈ X if and only if $G(x_m, x_n, x) \ll c$ for all n, m > N^1 , N^1∈ N.
2. $\{x_n\}$ is Cauchy sequence if and only if $G(x_n, x_m, x_p) \ll c$ for all n, m > p > N^1 , N^1 ∈ N.
3. (X, G) is complete G-cone metric space over Banach algebra if every Cauchy sequence converges.

Definition 2.17.[18] Let f and g be self maps of a set X. If u = fx = gx for some x in X, then x is called a coincidence point of f and g , and u is called a point of coincidence of f and g.

Lemma 2.18.[20] Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence w = fx = gx, then w is the unique common fixed point of f and g.

2. MAIN RESULTS

Theorem 3.1. Let (X, G) be a complete symmetric G-cone metric space over a Banach algebra A and P be a non normal cone in A. Suppose that the mapping f, g : X → X satisfy the following contractive condition

$$G(fx, fy, fz) \preceq \mu\, G(gx, gy, gz) \tag{3.1}$$

for all x,y,z ∈ X ,where μ ∈ P is a generalized Lipschitz constant with ρ(μ) ∈ [0, 1). If f(X)⊂ g(X) and g(X) is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if (f, g) is weakly compatible, then f and g have a unique common fixed point.

Proof. Assuming that f satisfies the inequality (3.1), then for all x, y ∈ X

$$G(fx, fy, fy) \preceq \mu\, G(gx, gy, gy) \tag{3.2}$$

And

$$G(fy, fx, fx) \preceq \mu\, G(gy, gx, gx) \tag{3.3}$$

Since X is a symmetric G−cone metric space, by adding (3.2) and (3.3) we have

$$d_G(fx, fy) \preceq \mu\, d_G(gx, gy) \tag{3.4}$$

for all x,y ∈ X

Let $x_0 \in X$ and choose a point x_1 in X such that $fx_0 = gx_1$. This can be done in view of f(X)⊂ g(X). Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that $fx_n = gx_{n+1}$ for all n∈ N. Then we have

$$\begin{aligned} d_G(gx_{n+1}, gx_n) &= d_G(fx_n, fx_{n-1}) \preceq \mu\, d_G(gx_n, gx_{n-1}) \\ &\preceq \mu^2 d_G(gx_{n-1}, gx_{n-2}) \\ &\preceq \ldots\ldots\ldots\ldots \mu^n d_G(gx_1, gx_0) \end{aligned} \tag{3.5}$$

Then, for n > m, we have

$$\begin{aligned} d_G(gx_n, gx_m) &\preceq d_G(gx_n, gx_{n-1}) + d_G(gx_{n-1}, gx_{n-2}) + \ldots. + d_G(gx_{m+1}, gx_m) \\ &\preceq (\mu^{n-1} + \mu^{n-2} + \ldots\ldots.. + \mu^m)\, d_G(gx_1, gx_0) \\ &\preceq \left(\textstyle\sum_{n=0}^{\infty} \mu^n\right)\mu^m\, d_G(gx_1, gx_0) \\ &\preceq \frac{\mu^m}{e-\mu} d_G(gx_1, gx_0) \\ &\preceq (e-\mu)^{-1}\mu^m\, d_G(gx_1, gx_0) \end{aligned}$$

Then by Lemma 2.11,one has

$$\|(e-\mu)^{-1}\mu^m d_G(gx_1, gx_0)\| \leq \|(e-\mu)^{-1}\|.\|\mu^m\|.\|d_G(gx_1, gx_0)\| \to 0 (m \to \infty)$$

Which implies that $(e-\mu)^{-1}\mu^m d_G(gx_1, gx_0) \to 0 (m \to \infty)$. So by Lemma 2.10 for each $\theta \ll$ c there exists N_1 such that $d_G(gx_n, gx_m) \ll$ c for all n > N_1. Therefore $\{gx_n\}$ is a Cauchy sequence. Since g(X) is a complete subspace of X, then there exists w ∈ g (X) such that $gx_n \to w$ as $n \to \infty$. Hence we can find v in X such that gv = w. Note that $gx_{n-1} \to w(n \to \infty)$, it is easy to see for every real $\varepsilon > 0$ choose c_0 with $\theta \ll c_0$ and $\|\mu\|\|c_0\| < \in$ Then there is N_2 such that $d_G(gx_{n-1}, w) \ll c_0$ for all n > N_2 . Consequently,

$$d_G(gx_n, fv) = d_G(fx_{n-1}, fv) \preceq \mu d_G(gx_{n-1}, gv) = \mu d_G(gx_{n-1}, w) \preceq \mu c_0$$

that is to say $gx_n \to fv$ $(n \to \infty)$. Indeed, owing to

$$\|\mu c_0\| \leq \|\mu\|\|c_0\| < \varepsilon \Rightarrow \mu c_0 \to \theta\ (n \to \infty)$$

Then by Lemma 2.10 for each $\theta \ll$ c there exists N_3 such that $d_G(gx_n, fv) \ll$ for all n > N_3. As a result, $gx_n \to fv$ $(n \to \infty)$. By virtue of of$gx_n \to w = gv$ $(n \to \infty)$, then fv = gv. In the following we shall prove f and g have a unique point of coincidence.

We suppose for absurd that there exists v*≠ v such that f v* = g v*. Then

$$d_G(gv^*, gv) = d_G(fv^*, fv) \preceq \mu d_G(gv^*, gv) \preceq \ldots\ldots\ldots\ldots \preceq \mu^m d_G(gv^*, gv)$$

Making use of Lemma 2.11 and the following fact that

$$\|\mu^m d_G(gv*, gv)\| \leq \|\mu^m\|.\|d_G(gx_1, gx_0)\| \to 0\ (n \to \infty),$$

We speculate $\mu^n d_G(gv^*, gv) \to \theta$ $(n \to \infty)$. So by utilizing Lemma 2.10, one has gv* = gv. Finally if (f, g) is weakly compatible, then by using Lemma 2.17, we claim that f and g have a unique common fixed point.

Corollary 3.2. Let X be a complete symmetric G-cone metric space over a Banach algebra A. Suppose that the mapping T : X → X satisfies the following

$$G(fx, fy, fz) \preceq \mu\, G(x, y, z)$$

for all x, y, z ∈ X, where $\rho(\mu) \in [0, 1)$ is a constant. Then T has unique common fixed point.

Proof. The proof can be obtained from Theorem 3.2 by taking g = I where I is identity map.

Theorem 3.3. Let (X, G) be a complete symmetric G-cone metric space over a Banach algebra A and P be a cone in A. Suppose that the mapping f, g : X → X satisfy one of the following contractive condition

G(fx,fy,fz) ⪯ a G(gx,gy,gz) + b G(fx,fx,gx) + c G(fy,fy,gy) + d G(fz,fz,gz) + e G(fy,fy,gx) + h G(fx,fx,gy).......(1)

Or

G(fx,fy,fz) ⪯ a G(gx,gy,gz) + b G(fx,gx,gx) + c G(fy,gy,gy) + d G(fz,gz,gz) + e G(fy,gx,gx) + hG(fx,gy,gy)........(2)

For all x,y,z ∈ X and a,b,c,d,e,h ∈ [0,1), a + b +c + e + h < 1. Suppose f and g are weakly compatible and f(X) ⊂ g(X) such that (X)or g(X) complete subspace of X. Then the mappings f and g have a unique common fixed point. Moreover, for any x_0 ∈ X the sequence ,$\{x_n\}$ ∈ X defined by gx = fx_{n-1} for all n, converges to the fixed point.

Proof : Suppose that f satisfies condition (1) and (2), then for all x,y ∈ X

G(fx,fy,fy) ⪯ a G(gx,gy,gy) + b G(fx,fx,gx) + (c + d)G(fy,fy,gy) + e G(fy,fy,gx) + h G(fx,fx,gy)......(3)

and

G(fy,fx,fx) ⪯ a G(gy,gx,gx) + b G(fy,fy,gy) + (c + d)G(fx,fx,gx) + e G(fx,fy,gy) + h G(fy,fx,gx)......(4)

Since X is a symmetric G-cone metric space therefore by adding (3) and (4) we have ,

$$d_G(fx,fy) \preceq \ a d_G(gx,gy) + \frac{b+c+d}{2} d_G(fx,gx) + \frac{b+c+d}{2} d_G(fy,fy) + \frac{e+h}{2} d_G(fy, gx) + \frac{e+h}{2} d_G(fx, gy)$$

Let $a = \alpha_1$, $\frac{b+c+d}{2} = \alpha_2$, $\frac{b+c+d}{2} = \alpha_3$,$\frac{e+h}{2} = \alpha_4$, $\frac{e+h}{2} = \alpha_5$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$$

$d_G(fx,fy) \preceq \alpha_1 d_G(gx,gy) + \alpha_2 d_G(fx,gx) + \alpha_3 d_G(fy,fy) + \alpha_4 d_G(fy, gx) + \alpha_5 d_G(fx, gy)$................(5)

If $fx_n = fx_{n-1}$ for all n∈ N then $\{fx_n\}$ is a Cauchy sequence. If $fx_n \neq fx_{n-1}$ for all n ∈ N then put x = x_{n+1} and y = x_n in (5), we get

$$d_G(fx_{n+1}, fx_n) \preceq \alpha_1 d_G(x_{n+1}, gx_n) + \alpha_2 d_G(fx_{n+1}, gx_{n+1}) + \alpha_3 d_G(fx_n , fx_n)$$
$$+ \ \alpha_4 d_G(fx_n, gx_{n+1}) + \alpha_5 d_G(fx_{n+1}, gx_n)$$

Using the fact that $gx_n = fx_{n-1}$ for all n, we have

$$d_G(fx_{n+1}, fx_n) \preceq \alpha_1 d_G(fx_n, f x_{n-1}) + \alpha_2 d_G(fx_{n+1}, fx_n) + \alpha_3 d_G(fx_n, fx_{n-1})$$
$$+ \alpha_4 d_G(fx_n,\ fx_n) + \alpha_5 d_G(fx_{n+1}, fx_{n-1})$$

$$d_G(f x_{n+1}, fx_n) \preceq \alpha_1 d_G(fx_n, f x_{n-1}) + \alpha_2 d_G(fx_{n+1}, fx_n) + \alpha_3 d_G(fx_n , fx_{n-1})$$
$$+ \alpha_4 d_G(fx_n,\ fx_n) + \alpha_5 (d_G(fx_{n+1}, fx_n) + d_G(fx_n, f x_{n-1}))$$

$$\{1-(\alpha_2+\alpha_5)\}\, d_G(f x_{n+1}, fx_n) \preceq \{\alpha_1+\alpha_3+\alpha_5\} d_G(fx_n, f x_{n-1})$$

It further implies that

$$d_G(f x_{n+1}, fx_n) \preceq \mu d_G(fx_n, f x_{n-1})$$

where $\mu = \frac{\alpha_1+\alpha_3+\alpha_5}{1-(\alpha_2+\alpha_5)} < 1.$

Consequently

$$d_G(f x_{n+1}, fx_n) \preceq \mu^n d_G(fx_n, f x_{n-1})$$

Now for all m, n$\in$ N with m > n, we have

$$d_G(fx_m, fx_n) \preceq d_G(fx_m, f x_{m-1}) + d_G(f x_{m-1}, f x_{m-2}) + \dots\dots\dots + d_G(f x_{n+1}, fx_n)$$

$$= (\mu^{m-1}+\mu^{m-2}+\cdots\dots\dots\dots.\mu^n)\, d_G(fx_1, fx_0)$$

$$\preceq (\textstyle\sum_{n=0}^{\infty}\mu^n)\mu^m d_G(fx_1, fx_0)$$

$$\preceq \frac{\mu^n}{e-\mu} d_G(fx_1, fx_0)$$

$$\preceq (e-\mu)^{-1}\mu^m d_G(fx_1, fx_0)$$

Then by Lemma 2.11, one has

$$\|(e-\mu)^{-1}\mu^m d_G(fx_1, fx_0)\| \leq \|(e-\mu)^{-1}\|.\|\mu^m\|.\|d_G(fx_1, fx_0)\| \to 0(m\to\infty)$$

Which implies that $(e-\mu)^{-1}\mu^m d_G(fx_1, fx_0) \to 0(m\to\infty)$. So by Lemma 2.10 for each $\theta \ll$ c there exists N_1 such that $d_G(fx_m, fx_n) \ll$ c for all n > N_1. Therefore $\{fx_n\}$ is a Cauchy sequence. Since f(X) or g(X) is a complete subspace of X, then there exists w $\in$ f (X) such that $fx_n \to$ w and $gx_n \to$ w as n $\to \infty$. Hence we can find v in X such that gv = w. Note that $fx_{n-1} \to w\ (n\to\infty)$, it is easy to see for every real $\varepsilon > 0$ choose c_0 with $\theta \ll c_0$ and $\|\mu\|\|c_0\| < \in$. Then there is N_2 such that$d_G(fx_{n-1}, w) \ll c_0$ for all n > N_2. Consequently,

$$d_G(fx_n, gv) = d_G(gx_{n-1}, gv) \preceq \mu d_G(fx_{n-1}, fv) = \mu d_G(fx_{n-1}, w) \preceq \mu c_0$$

that is to say $fx_n \to$ gv $(n\to\infty)$. Indeed, owing to

$$\|\mu c_0\| \leq \|\mu\|\|c_0\| < \varepsilon \Rightarrow \mu c_0 \to \theta(n\to\infty)$$

Then by Lemma 2.10 for each $\theta \ll$ c there exists N_3 such that $d_G(fx_n, gv) \ll$ for all n > N_3. As a result, $fx_n \to gv(n\to\infty)$. By virtue of of $fx_n \to$ w = fv$(n\to\infty)$, then fv = gv. In the following we shall prove f and g have a unique point of coincidence.

We suppose for absurd that there exists v*$\neq$ v such that fv* = gv*. Then

$$d_G(fv^*, fv) = d_G(gv^*, gv) \preceq \mu d_G(fv^*, fv) \preceq \dots\dots\dots\dots \preceq \mu^m d_G(fv^*, fv)$$

Making use of Lemma 2.11 and the following fact that

$$\|\mu^m d_G(fv*, fv)\| \leq \|\mu^m\|.\|d_G(fx_1, fx_0)\| \to 0(n\to\infty),$$

We speculate $\mu^n d_G(fv^*, fv) \to \theta(n\to\infty)$. So by utilizing Lemma 2.10, one has fv* = fv. Finally if (f, g) is weakly compatible, then by using Lemma 2.17, we claim that f and g have a unique common fixed point.

3. CONCLUTION

We have proved common fixed point theorems for generalize contraction mappings in G-cone metric spaces with Banach algebra, which generalize several existing results in literature.

References

[1]. Bakhtin, I.(1989): The contraction mapping principle in quasimetric spaces, Func. An., Gos. Ped. Inst. Unianowsk, 30, 26-37.

[2]. Ahmed, A. and Salunke, J.N.(2017): Fixed Point Theorem of Expanding Mapping without Continuity in Cone Metric Space over Banach Algebra, In International Conference on Recent Trends in Engineering and Science, vol. 20, 19-22 .

[3]. Huang,L. and Zhang, X. (2007): Cone metric spaces and fixed point theorems of contractive mappings , J. Math. Anal. Appl. 332(2), 1468-1476 .

[4]. Hao,L. and Shaoyuan,X.(2013): Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings, Fixed Point Theory and Applications, 1-10.

[5]. Xu,S. and Radenovic,S.(2014):Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebra without assumption of normality,Fixed Point Theory Appl.,1-12.

[6]. Adewale,O.K. and Osawaru,E.K. (2019): G-cone metric Spaces over Banach Algebras and Some Fixed Point Results, International Journal of Mathematical Analysis and Optimization: Theory and Applications,Vol. No. 2, 546 – 557.

[7]. Dhage,B. (1992): Generalized metric space and mapping with fixed points Bulletin of the Calcutta Mathematical Society , 84, 329-336.

[8]. Mustafa,M. and Sims,B. .(2006): A new approach to generalized metric spaces J., Nonlinear Convex Analysis,7(2),289-297.

[9]. Mustafa,M. and Sims,B.(2009):Fixed point theorems for contractive mappings in complete G-metric spaces, Fixed Point Theory and Applications,1-10.

[10]. Olaleru,J.O.(2010): Some generalizations of fixed point theorems in cone metric spaces, Mathematics, E-Note,11, 41-49.

[11]. Radenovic,S. and Rhoades,B.E.(2009): Fixed point theorem for two non-self mappings in cone metric spaces,Comput. Math. Appl., 57,1701-1707.

[12]. Hao,L. and Shaoyuan,X. ,(2013): Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings,Fixed Point Theory and Applications,1-10.

[13]. Beg, I., Abbas, M. and Nazir,T. Generalized cone metric spaces, J. Nonlinear Sci. Appl, 3(1),(2010),21-31.

[14]. Banach,S.(1922): Sur les operations dans ensembles abstraits et leur application aux equations integrales,FundamentaMathematicae Vol. 3, 133-181.

[15]. Turkoglu, D. and Bilgili, N.(2012):Some fixed point theorem for mapping on complete G-cone metric spaces, Journal of Applied Functional Analysis, Vol.7, No's 1-2,118-137.

[16]. Mishra, A.K. and padmawati, (2022):On Some Fixed Point Theorem in Ordered G-Cone Metric Spaces Over Banach Algebra**,** International Journal of Mathematics And its Applications 10.4 ,29-38.

[17]. Mishra, A.K., Padmawati, Rathour,L.,Mishra,V.N.(2023):Some Fixed Point Theorem using Generalized Cone Metric Spaces with Banach Algebra, High Technology Letters Volume 29, Issue 1, 153-162.

[18]. Singh, Naval and Jain, Reena(2017):Common Fixed Point Theorems in Generalized Cone Metric, European Journal of Business and Management ,Vol 3, No.4,184-190

[19]. HUANG,H.P.(2016):Common fixed point theorems in non-normal cone metric spaces with Banach algebras**,** Chinese Quarterly Journal of Mathematics, *31*(2), 155-161.

[20]. Öztürk, M. A. (2011): On some common fixed point theorems with φ-maps on G-cone metric spaces, Bull. Math. Anal. Appl, *3*(1), 121-133.

[21]. Jungck,G. (1996): Common fixed points for noncontinuous nonself maps on non-metric spaces, Far East J. Math. Sci., 4, 199-215.

[22]. Abbas,M. and Jungck,G.,(2008):Common fixed point results for noncommuing mappings without continuity in cone metric spaces, Journal of Mathematical Analysis and Applications, 341 ,416-420.