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RESEARCH ARTICLE

DETERMINANTAL IDENTITIES SATISFIED BY $r_k(n)$

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ABSTRACT

We show determinantal identities verified by $r_k(n)$,the number of ways that a positive integer n can be written as sum of k squares, which are implied by the polynomial structure of $r_k(n)$.

Keywords: Sum of divisors, Integer Sequences, Vandermonde's determinant, Sum of square numbers,

Bell polynomials, Lagrangian interpolation.

1. Introduction

In [1-6] we find the following iterated relation for$r_k(n)$,the number of representations of n as ways that a positive integer n can be written as sum of k squares [7-9]:

$$n\, r_k(n) = 2k \sum_{j=1}^{n} A(j)\, r_k(n-j), \tag{1}$$

The term $A(j)$ in (1), can be expressed as:

$$A(j) = (-1)^{j-1}\, j \sum_{odd\ d|j} \frac{1}{d}, \tag{2}$$

From (2), we notice that $A(1) = 1,\ \ A(2) = -2,\ \ A(3) = 4,\ \ A(4) = -4,\ \ etc.$, thus it appears the sequence was studied in [10]: In particular, the first few terms are given by

1, -2, 4, -4, 6, -8, 8, -8, 13, -12, 12, -16, 14, -16, 24, -16, 18, -26, 20, -24, 32, -24, 24, -32, 31, -28, ..., (3)

From [11, 12], we in fact obtain closed expression for $A(n)$:

$$A(n) = \begin{cases} -\left(\sigma(n) - \sigma\left(\frac{n}{2}\right)\right), & n \text{ is even}, \\ \sigma(n), & n \text{ is odd}, \end{cases} \tag{4}$$

which explicitly contains sum of divisors function.

From (1) it is clear that $r_k(n)$ is a polynomial in k of degree n [16-18]:

$$r_k(n) = a(n,n)\, k^n + a(n,n-1)k^{n-1} + \cdots + a(n,2)k^2 + a(n,1)\, k, \tag{5}$$

and it is possible to deduce nice expressions for the coefficients of $r_k(n)$. In view of [17]:

$$a(n,n) = \frac{2^n}{n!}, \quad n \geq 1, \qquad a(n,1) = \frac{2}{n}\, A(n), \quad n \geq 1,$$

$$a(n,n-1) = -\frac{2^{n-1}}{(n-2)!}, \quad n \geq 2, \qquad a(n,n-2) = \frac{2^{n-3}\,(3\,n-1)}{3\,(n-3)!}, \quad n \geq 3, \tag{6}$$

$$a(n,n-3) = \frac{2^{n-4}(n+2)\,(3-n)}{3\,(n-4)!}, \quad n \geq 4, \quad a(n,n-4) = \frac{2^{n-7}}{45}\left[\frac{8(85\,n-371)}{(n-5)!} + \frac{15(n+9)}{(n-7)!}\right], \quad n \geq 5, etc.$$

which enables us to reproduce several polynomials type (5) reported in the literature, for example [2, 13]:

$$r_k(1) = 2k, \qquad r_k(2) = 2k(k-1), \qquad r_k(3) = \frac{4}{3}\, k(k-1)(k-2),$$

$$r_k(4) = \frac{2}{3}\, k[3(2k-1) + k(k-1)(k-5)], \quad r_k(5) = \frac{4}{15}\, k(k-1)[3(2k-3) + k(k-4)(k-5)], \tag{7}$$

$$r_k(6) = \frac{4}{45}\, k(k-1)(k-2)[45 + (k-3)(k-4)(k-5)],$$

$$r_k(7) = \frac{8}{315}\, k(k-1)(k-2)(k-3)(k^3 - 15k^2 + 74k - 15),$$

In [16], it was shown that the $a(n,m)$ can be written in terms of partial exponential Bell polynomials [19-21].The solution of (1) is given by [4]:

$$r_k(n) = \frac{1}{n!} B_n\big(2k \cdot 0!\, A(1), 2k \cdot 1!\, A(2), 2k \cdot 2!\, A(3), \ldots, 2k \cdot (n-1)!\, A(n)\big), \tag{8}$$

which involves complete Bell polynomials.

In Sec. 2 we show that (5) and (6) imply determinantal identities verified by $r_k(n)$.

2. Linear system generated by (5) for $k = 1, 2, \ldots, n+1$.

We write (5) in the form:

$$r_k(n) = a(n,0) + a(n,1)\, k + a(n,2)k^2 + \cdots + a(n,n-1)k^{n-1} + a(n,n)k^n, \tag{9}$$

where $a(n,0) = 0$. From (9) we can obtain a linear system if k takes the values $1, 2, \ldots, n+1$, whose determinant is given by [22]:

$$\begin{vmatrix} 1 & 1 & 1^2 & \cdots & 1^{n-1} & 1^n \\ 1 & 2 & 2^2 & \cdots & 2^{n-1} & 2^n \\ 1 & 3 & 3^2 & \cdots & 3^{n-1} & 3^n \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & n & n^2 & \cdots & n^{n-1} & n^n \\ 1 & n+1 & (n+1)^2 & \cdots & (n+1)^{n-1} & (n+1)^n \end{vmatrix} = \prod_{t=1}^{n+1} (t-1)!, \quad n \geq 1, \tag{10}$$

then (6), (10) and this linear system imply the following identities:

$$\begin{vmatrix} r_1(n) & 1 & 1^2 & \cdots & 1^{n-1} & 1^n \\ r_2(n) & 2 & 2^2 & \cdots & 2^{n-1} & 2^n \\ r_3(n) & 3 & 3^2 & \cdots & 3^{n-1} & 3^n \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ r_n(n) & n & n^2 & \cdots & n^{n-1} & n^n \\ r_{n+1}(n) & n+1 & (n+1)^2 & \cdots & (n+1)^{n-1} & (n+1)^n \end{vmatrix} = 0\,, \qquad n \geq 1, \tag{11}$$

$$\begin{vmatrix} 1 & 1 & 1^2 & \cdots & 1^{n-1} & r_1(n) \\ 1 & 2 & 2^2 & \cdots & 2^{n-1} & r_2(n) \\ 1 & 3 & 3^2 & \cdots & 3^{n-1} & r_3(n) \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & n & n^2 & \cdots & n^{n-1} & r_n(n) \\ 1 & n+1 & (n+1)^2 & \cdots & (n+1)^{n-1} & r_{n+1}(n) \end{vmatrix} = 2^n \prod_{t=1}^{n} (t-1)!\,, \qquad n \geq 1, \tag{12}$$

$$\begin{vmatrix} 1 & 1 & 1^2 & \cdots & r_1(n) & 1^n \\ 1 & 2 & 2^2 & \cdots & r_2(n) & 2^n \\ 1 & 3 & 3^2 & \cdots & r_3(n) & 3^n \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & n & n^2 & \cdots & r_n(n) & n^n \\ 1 & n+1 & (n+1)^2 & \cdots & r_{n+1}(n) & (n+1)^n \end{vmatrix} = 2^{n-1}\, n\,(1-n) \prod_{t=1}^{n}(t-1)!\,,\; n \geq 2, \tag{13}$$

$$\begin{vmatrix} 1 & r_1(n) & 1^2 & \cdots & 1^{n-1} & 1^n \\ 1 & r_2(n) & 2^2 & \cdots & 2^{n-1} & 2^n \\ 1 & r_3(n) & 3^2 & \cdots & 3^{n-1} & 3^n \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & r_n(n) & n^2 & \cdots & n^{n-1} & n^n \\ 1 & r_{n+1}(n) & (n+1)^2 & \cdots & (n+1)^{n-1} & (n+1)^n \end{vmatrix} = \frac{2}{n}\, A(n) \prod_{t=1}^{n+1} (t-1)!\,,\; n \geq 1, etc. \tag{14}$$

The results (9) to (14) indicate that $r_k(n)$ is the Lagrange's interpolating polynomial [23-27] for the data points $\left(j, r_j(n)\right)$, $j = 1, 2, \ldots, n+1$, thus $r_k(n)$ is a polynomial in k of degree n ; then with this approach we can to construct, for example, the polynomials in (7).

Remark 1.- It is possible to prove some expressions for the sequence defined in (2), for example [3, 10]:

$$A(j) = \frac{j}{2} \sum_{k=1}^{j} \frac{(-1)^{k-1}}{k} \binom{j}{k} r_k(j), \sum_{j=1}^{\infty} A(j)\, q^j = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}\, j\, q^j}{1-q^{2j}}\,. \tag{15}$$

Remark 2.-In [2] the following recurrence relations were obtained:

$$r_k(n) = \sum_{t=1}^{n} (-1)^t\, h(t) r_{k+1}(n-t), n\, h(n) = -\sum_{l=1}^{n} A(2l)\, h(n-l), \tag{16}$$

$$h(n) = \frac{1}{n!} B_n\big(-0!\,A(2), -1!\,A(4), -2!\,A(6), \ldots, -(n-1)!\,A(2n)\big),$$

then the sequence A015128 (m) = 2 A014968 (m) results [11]:

$$h(m) = 2, 4, 8, 14, 24, 40, 64, 100, 154, 232, 344, 504, 728, \ldots \tag{17}$$

References

[1]. H. Singh Bal, G. Bhatnagar, *The partition-frequency enumeration matrix,* Ramanujan J. **59** (2022) 51-86.

[2]. A. Hernández-Galeana, M. Muniru Iddrisu, J. López-Bonilla,*A recurrence relation for the number of representations of a positive integer as a sum of squares,* Advances in Maths: Scientific Journal **11**,No. 11 (2022) 1055-1060.

[3]. G. E. Andrews, S. Kumar Jha, J. López-Bonilla, *Sums of squares, triangular numbers, and divisor sums,* J. of Integer Sequences **26** (2023) Article 23.2.5

[4]. R. Sivaraman, J. D. Bulnes, J. López-Bonilla, *Complete Bell polynomials and recurrence relations for arithmetic functions,* European J. of Theoretical and Appl. Sci. **1**, No. 3 (2023) 167-170.

[5]. G Bhatnagar, H. Singh Bal, *Glaisher's divisors and infinite products,* J. of Integer Sequences **27** (2024) Article 24.1.6

[6]. R. Jakimczuk, *A general formula in composition theory,* J. Mathematica Slovaca **74**, No. 3 (2024) 1-14.

[7]. E. Grosswald, *Representations of integers as sums of squares,* Springer-Verlag, New York (1985).

[8]. C. J. Moreno, S. S. Wagstaff Jr, *Sums of squares of integers,* Chapman & Hall / CRC, Boca Raton, Fl, USA (2006).

[9]. M. Aka, M. Einsiedler, T. Ward, *A journey through the realm of numbers,* Springer, Switzerland (2020).

[10]. J. López-Bonilla, J. Yaljá Montiel-Pérez, H. Sherzad Taher, *Recurrence relations and Jacobi theta functions,* J. de Ciencia e Ingeniería **15**, No. 1 (2023) 14-16.

[11]. https://oeis.org/

[12]. Prof. Michael Somos, Private communication, 1th July 2023.

[13]. R. Sivaramakrishnan, *Classical theory of arithmetic functions,* Marcel Dekker, New York (1989).

[14]. K. Ireland, M. Rosen, *A classical introduction to modern number theory,* Springer, New York (1990).

[15]. R. Sivaraman, J. D. Bulnes, J. López-Bonilla, *Sum of divisors function,* Int. J. of Maths. and Computer Res. **11**, No. 7 (2023) 3540-3542.

[16]. M. A. Pathan, H. Kumar, M. Muniru Iddrisu, J. López-Bonilla, *Polynomial expressions for certain arithmetic functions,* J. Mountain Res. **18**, No. 1 (2023) 1-10.

[17]. R. Sivaraman, J. López-Bonilla, S. Vidal-Beltrán, *On the polynomial structure of* $r_k(n)$,Indian J. of Advanced Maths. **3**, No. 2 (2023) 4-5.

[18]. H. Kumar, R. C. Chandel, J. López-Bonilla, *On arithmetic functions, their extended coefficients: various results and relations,* Jnanabha, **53**, No. 2 (2023) 217-223.

[19]. D. Cvijovic, *New identities for the partial Bell polynomials,* Appl. Math. Lett. **24** (2011) 1544-1547.

[20]. M. Shattuck, *Some combinatorial formulas for the partial r-Bell polynomials,* Notes on Number Theory and Discrete Maths. **23**, No. 1 (2017) 63-76.

[21]. M. A. Pathan, H. Kumar, J. D. Bulnes, J. López-Bonilla, *Connection between partial Bell polynomials and* $(q;q)_k$*; partition function, and certain q-hypergeometric series,* J. Ramanujan Soc. Maths. &Math. Sci. **10**, No. 1 (2022) 1-12.

[22]. M. Bahrami-Taghanaki, A. R. Moghaddamfar, *Some identities involving Stirling numbers arising from matrix decompositions,* J. of Integer Sequences **27** (2024) Article 24.5.3

[23]. C. Lanczos, *Applied analysis,* Dover, New York (1988).

[24]. C. Lanczos, *Linear differential operators,* SIAM, Philadelphia, USA (1996).
[25]. G. M. Phillips, *Interpolation and approximation by polynomials,* Springer, New York (2003).
[26]. P. Dechaumphai, N. Wansophark, *Numerical methods in engineering,* Alpha Sci. Int., Oxford, UK (2011).
[27]. I. Guerrero-Moreno, J. López-Bonilla, M. Morales-García, *Lagrangian interpolation formula and Sylvester's matrix function,* Studies in Nonlinear Sci. **6**, No. 1 (2021) 10-16.