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RESEARCH ARTICLE

OBSERVATIONS ON THE HOMOGENEOUS CONE $x^2 + y^2 = 17z^2$

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ABSTRACT

Patterns of integer solutions to the homogeneous cone $x^2 + y^2 = 17z^2$ are presented.

Keywords: Homogeneous cone ,Ternary quadratic equation , Integer solutions

Introduction

It is quite obvious that Diophantine equations, one of the areas of number theory, are rich in variety [1-3].In particular, the ternary quadratic Diophantine equations in connection with geometrical figures occupy a pivotal role in the orbit of mathematics and have a wealth of historical significance. In this context, one may refer [4-10] for second degree Diophantine equations with three unknowns representing different geometrical figures.

In this paper, the ternary quadratic Diophantine equation representing cone given by $x^2 + y^2 = 17z^2$ is studied for determining its integer solutions successfully through elementary algebra.

Method of analysis

The ternary quadratic equation representing homogeneous cone under consideration is

$$x^2 + y^2 = 17z^2 \tag{1}$$

The process of obtaining patterns of integer solutions to (1) are illustrated below :

Pattern 1

Assume

$$z = a^2 + b^2 \tag{2}$$

Express the integer 17 on the R.H.S. of (1) as the product of complex conjugates as

$$17 = (4+i)(4-i) \tag{3}$$

Substituting (2) & (3) in (1) and employing the method of factorization , consider

$$\begin{aligned} x + i\,y &= (4+i)(a+i\,b)^2 \\ &= (4+i)[f(a,b) + i\,g(a,b)] \end{aligned} \tag{4}$$

where

$$f(a,b) = a^2 - b^2 , g(a,b) = 2ab$$

Equating the real and imaginary parts in (4) ,we have

$$\begin{aligned} x &= 4f(a,b) - g(a,b), \\ y &= f(a,b) + 4g(a,b). \end{aligned} \tag{5}$$

Thus,(2) & (5) satisfy (1).

Pattern 2

One may also write the integer 17 on the R.H.S. of (1) as the product of complex conjugates as

$$17 = (1+4i)(1-4i) \tag{6}$$

Following the procedure similar to Pattern 1,we get

$$x = f(a,b) - 4g(a,b) \; , y = 4f(a,b) + g(a,b). \tag{7}$$

Thus,(2) & (7) satisfy (1).

Pattern 3

Let

$$\begin{aligned} f(u,v,p,q) &= u(p^2 - q^2) + v(2pq) \; , \\ g(u,v,p,q) &= u(2pq) - v(p^2 - q^2) \end{aligned} \tag{8}$$

where

$p, q\,(p > q > 0)$ represent the generators of a Pythagorean triangle and

$$17 = u^2 + v^2 .$$

Taking $u=4, v=1$ in (8) ,we get

$$f(4,1,p,q)=4(p^2-q^2)+(2pq)\ ,$$
$$g(4,1,p,q)=4(2pq)-(p^2-q^2)$$

Observe that

$$17=\frac{[f(4,1,p,q)+i\,g(4,1,p,q)][f(4,1,p,q)-i\,g(4,1,p,q)]}{(p^2+q^2)^2} \tag{9}$$

Substituting (2) & (9) in (1) and employing the method of factorization ,

consider

$$x+iy=\frac{[f(4,1,p,q)+i\,g(4,1,p,q)]}{(p^2+q^2)}[f(a,b)+i\,g(a,b)] \tag{10}$$

Taking

$$a=(p^2+q^2)A, b=(p^2+q^2)B \tag{11}$$

in (10) and equating the coefficients of corresponding terms ,we have

$$\begin{aligned} x&=(p^2+q^2)[f(4,1,p,q)f(A,B)-g(4,1,p,q)g(A,B)]\ ,\\ y&=(p^2+q^2)[f(4,1,p,q)g(A,B)+g(4,1,p,q)f(A,B)]. \end{aligned} \tag{12}$$

From (11) and (2) ,we obtain

$$z=(p^2+q^2)^2(A^2+B^2) \tag{13}$$

Thus,(12) and (13) satisfy (1).

Example :

Let $A=2, B=1, p=3, q=2$

$$f(4,1,3,2)=32, g(4,1,3,2)=43, f(A,B)=3, g(A,B)=4$$

$$17=\frac{(32+i43)(32-i43)}{13^2}$$

From (12) and (13) ,we have

$$x=-13*76, y=13*257, z=5*169$$

which satisfy (1).

Pattern 4

Write (1) as

$$17z^2-y^2=x^2*1 \tag{14}$$

Assume

$$x=17a^2-b^2 \tag{15}$$

Express the integer 1 on the R.H.S. of (14) as

$$1=(\sqrt{17}+4)(\sqrt{17}-4) \tag{16}$$

Substituting (15) & (16) in (14) and using factorization ,we have

$$\sqrt{17}\,z+y=(\sqrt{17}+4)(\sqrt{17}\,a+b)^2=(\sqrt{17}+4)(17a^2+b^2+\sqrt{17}\,2ab)$$

Equating the rational and irrational parts in the above equation , we have

$$\begin{aligned} y&=4(17a^2+b^2)+34ab, \\ z&=(17a^2+b^2)+8ab \end{aligned} \tag{17}$$

Thus,(15) & (17) satisfy (1).

Pattern 5

Apart from (16) , the integer 1 on the R.H.S. of (14) may be written as

$$1=\frac{(\sqrt{17}+1)(\sqrt{17}-1)}{16} \tag{18}$$

Substituting (15) & (18) in (14) and using factorization , we have

$$\sqrt{17}\,z+y=\frac{(\sqrt{17}+1)}{4}(\sqrt{17}\,a+b)^2=\frac{(\sqrt{17}+1)}{4}(17a^2+b^2+\sqrt{17}\,2ab)$$

Equating the coefficients of corresponding terms ,we get

$$\begin{aligned} y&=\frac{1}{4}(17a^2+b^2+34ab)\ , \\ z&=\frac{1}{4}(17a^2+b^2+2ab)\ . \end{aligned} \tag{19}$$

To obtain the solutions in integers ,we have to choose the values of a,b suitably in (19) .The suitable choices to a,b and the corresponding solutions to (1) are presented below :

Choice 1

Take $a=2A, b=2B$

The corresponding integer solutions to (1) are given by

$$\begin{aligned} x&=4(17A^2-B^2)\ , \\ y&=(17A^2+B^2+34AB)\ , \\ z&=(17A^2+B^2+2AB). \end{aligned}$$

Choice 2

Take $a=2A+1, b=2B+1$

The corresponding integer solutions to (1) are given by

$$x = (68A^2 - 4B^2 + 68A - 4B + 16)\ ,$$
$$y = (17A^2 + B^2 + 34AB + 34A + 18B + 13)\ ,$$
$$z = (17A^2 + B^2 + 2AB + 18A + 2B + 5).$$

Choice 3

Take $a = (4n + 2k + 1)b$

The corresponding integer solutions to (1) are given by

$$x = [17(4n + 2k + 1)^2 - 1]b^2\ ,$$
$$y = (68n^2 + 17k^2 + 68nk + 68n + 34k + 13)b^2\ ,$$
$$z = (68n^2 + 17k^2 + 68nk + 36n + 18k + 5)b^2\ .$$

Note 1

One may also choose the options given by

$$a = (4n - (2k + 1))b, b = (4n \pm (2k + 1))a$$

Thus, we get three more choices of integer solutions to (1).

Pattern 6

The substitution of the transformations

$$z = U + T, y = U + 17T, x = 4X \qquad (20)$$

in (1) leads to the homogeneous ternary quadratic equation

$$U^2 = X^2 + 17T^2 \qquad (21)$$

By scrutiny ,it is seen that (21) is satisfied by

$$U = 9T, X = 8T$$

In view of (20) , the integer solutions to (1) are given by

$$x = 32T, y = 26T, z = 10T$$

Note 2

In addition to (20) ,one may consider the transformations

$$z = U - T, y = U - 17T, x = 4X \qquad (22)$$

In this case , the integer solutions to (1) are given by

$$x = 32T, y = -8T, z = 8T$$

Pattern 7

Write (21) as the pair of equations

$$U + X = T^2\ ,$$
$$U - X = 17.$$

Solving the above system of equations , we get

$$T = 2k + 1,$$
$$U = 2k^2 + 2k + 9,$$
$$X = 2k^2 + 2k - 8.$$

Employing (20) ,the integer solutions to (1) are given by

$$x = 4(2k^2 + 2k - 8),$$
$$y = (2k^2 + 36k + 26),$$
$$z = (2k^2 + 4k + 10).$$

Note 3

Employing (22) ,the integer solutions to (1) are given by

$$x = 4(2k^2 + 2k - 8),$$
$$y = (2k^2 - 32k - 8),$$
$$z = (2k^2 + 8).$$

Pattern 8

Write (21) as the pair of equations

$$U + X = 17T^2,$$
$$U - X = 1.$$

Solving the above system of equations , we get

$$T = 2k + 1,$$
$$U = 34k^2 + 34k + 9,$$
$$X = 34k^2 + 34k + 8.$$

Employing (20) ,the integer solutions to (1) are given by

$$x = 4(34k^2 + 34k + 8),$$
$$y = (34k^2 + 68k + 26),$$
$$z = (34k^2 + 36k + 10).$$

Note 4

Employing (22) ,the integer solutions to (1) are given by

$$x = 4(34k^2 + 34k + 8),$$
$$y = (34k^2 - 8),$$
$$z = (34k^2 + 32k + 8).$$

Pattern 9

Rewrite (21) as

$$X^2 + 17T^2 = U^2 * 1 \tag{23}$$

Assume

$$U = 9(a^2 + 17b^2) \tag{24}$$

Express the integer 1 on the R.H.S. of (23) as

$$1 = \frac{(8 + i\sqrt{17})(8 - i\sqrt{17})}{81} \tag{25}$$

Substituting (24) & (25) in (23) and applying factorization ,

consider

$$X + i\sqrt{17}\,T = (8 + i\sqrt{17})\ (a + i\sqrt{17}\,b)^2$$

Equating the coefficients of corresponding terms in the above

equation ,we have

$$X = 8(a^2 - 17b^2) - 34ab,$$
$$T = (a^2 - 17b^2) + 16ab.$$

In view of (20), the integer solutions to (1) are given by

$$x = 32(a^2 - 17b^2) - 136ab,$$
$$y = 26\ a^2 - 136b^2 + 272ab,$$
$$z = 10\ a^2 + 136b^2 + 16ab.$$

Note 5

Employing (22) ,the integer solutions to (1) are given by

$$x = 32(a^2 - 17b^2) - 136ab,$$
$$y = -8\ a^2 + 442b^2 - 272ab,$$
$$z = 8\ a^2 + 170b^2 - 16ab.$$

Note 6

In addition to (25) , the integer 1 on the R.H.S. of (23) may be represented as follows :

$$1 = \frac{(4 + i3\sqrt{17})(4 - i3\sqrt{17})}{169},$$
$$1 = \frac{(17r^2 - s^2 + i\sqrt{17}\,2rs)(17r^2 - s^2 - i\sqrt{17}\,2rs)}{(17r^2 + s^2)^2}$$

The repetition of the above process gives two more choices of integer solutions to (1).

Conclusion

In this paper, the homogeneous ternary quadratic equation representing homogeneous cone given by $x^2 + y^2 = 17z^2$ is studied for obtaining its integer solutions through substitution technique and factorization method. One may search for other forms of quadratic equations with multiple variables to determine their integer solutions.

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