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RESEARCH ARTICLE

Cubic Diophantine Polynomial: (a+b+c)(ab+bc+ca)

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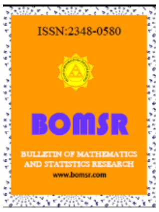

ABSTRACT

There is not much mention about Cubic Diophantine equation-(a+b+c)(ab+bc+ca), in math literature. A question was raised recently on the Mathblog, math stack exchange, about the asymmetrical equation,$(a+b+c)(de+df+ef) = (d+e+f)(ab+bc+ca)$. This got the authors thinking if a parametric solution is possible for the symmetrical equation, $(a+b+c)(ab+bc+ca) = (d+e+f)(de+df+ef)$. Turns out there are multiple parametric solutions. The authors have given three parametric solutions. Also, we have shown that integers of the form, $n = m(6m+1)$ can be represented as, $(a+b+c)(ab+bc+ca)$ in the form of rational numbers in a parametric form.

Consider the below equation:

$$(x+y+z)(xy+yz+zx) = (u+v+w)(uv+vw+wu).. \tag{1}$$

Case (1):

We look for a parametric solution:

Taking: $(x+y+z) = (u+v+w)$

And, $(xy+yz+zx) = (uv+vw+wu)$

Let: $(x, y, z) = [(m + a), (pm - a), (qm)]$ (2)

&

$(u, v, w) = [(m), (pm + a), (qm - a)]$ (3)

We get the condition, $[q = 2p - 1]$

Substituting the value of 'q' in (2) & (3) above we get:

$(x, y, z) = [(m + a), (pm - a), (m(2p - 1))]$

We get :

$(u, v, w) = [(m), (pm + a), (2mp - m - a)]$

$(x + y + z) = (u + v + w) = 3mp$

&

$(xy + yz + zx) = (uv + vw + wu)$

$= (2pm^2 + 2p^2m^2 - ma + pma - a^2 - m^2)$

For, (m,a,p)=(5,3,2) we get:

$(x, y, z, u, v, w) = (8,7,15,5,13,12)$

Case (2):

$(x + y + z)(xy + yz + zx) = (u + v + w)(uv + vw + wu)$ (4)

We take: $(x, y, z) = [(t), (-8t + a), (t + b)]$ (5a)

& $(u, v, w) = [(t), (t - a), (-8t - b)]$ (6a)

Substituting the above in (1) we get:

$$t = \frac{2ab}{9(b - a)}$$

We substitute the value of 't' in equations (5a) & (6a) & we get:

$(x, y, z) = [(2ab), -a(9a + 7b), b(-7a + 9b)]$

& $(u, v, w) = [(2ab), a(9a - 7b), -b(9b + 7a)]$

For (a,b)=(8,9) we get:

$(x, y, z, u, v, w) = (16, -120, 25, 16, 8, -137)$

Case (3):

$(x + y + z)(xy + yz + zx) = (u + v + w)(uv + vw + wu)$ (7)

Substitute,

$x = t, y = pt + a, z = qt + b, u = t - a, v = pt - b, w = qt,$

into above equation (1):

We take, $a = -(p^2 + 5p + 4pq + 5q + q^2 + 2),$

$b = (5pq + p^2 + 4p + 2q^2 + 5q + 1)$

Then, we get after substituting in (1):

$t = (\frac{1}{r})*(p^2 + 5p + 2 + 4pq + 5q + q^2)(5pq + p^2 + 4p + 1 + 2q^2 + 5q)$

Where,

$r = (p^3 + 4p^2q + 4p^2 - pq^2 + pq - q^3 - 3q^2 - p - 3q - 1)$

x=-ab

y=-a$(p^2q + 2p + 4pq + 3pq^2 + 1 + q^3 + 3q^2 + 3q)$

z=b($(p^3 + 4p^2 + 5p^2q - p + 3pq^2 + 6pq - q - 1 + 2q^2)$

u=-a$(p^3 + 5p^2 + 4p^2q + 3p - pq^2 + 6pq + 2q - q^2 - q^3)$ v=b$(p^2 + 2pq^2 + 3p + 4pq + 3q + 1 + q^3 + 3q^2)$

w=-abq

$Where, \quad a = -(p^2 + 5p + 2 + 4pq + 5q + q^2),$

$b = (5pq + p^2 + 4p + 1 + 2q^2 + 5q)$

Since, p, q are any integers.

For (p, q) = (3,2), the numerical solution is:

$[x, y, z, u, v, w]$ = [2240, 3552, 7945, 5408, 3255, 4480]

Case (4)

Next we consider integer representation of the numbers of the form, n=m(6m+1).

Let:

$$n = (x + y + z)(xy + yz + zx) \tag{8}$$

We need to find, postive integers which are represented by the positive rational numbers, x,y,z.

For, $n = (x + y + z)(xy + yz + zx)$

We consider the special case of y=z. After simplification of equation (1) we get:

$$2y^3 + 5xy^2 + 2x^2y = n \tag{9}$$

For the quadratic in (x) equation (2) inorder to have rational solutions, the discriminant must be a rational square, then we get:

$$v^2 = 9y^4 + 8yn \tag{10}$$

An equation (3) can be transformed to the parameterized elliptic curve (4).

$$Y^2 = X^3 + 576n^2, \ with \ \ y = \frac{8n}{X}, v = \frac{8Yn}{X^2} \tag{11}$$

Let, $n = m(6m + 1)$, then we get a parametric solution of (4) as follows.

$(X, Y) = (144m^2 + 24m, 24m(6m + 1)(12m + 1))$.

$(n, x, y, z) = (m\ (6m + 1), (-\frac{1}{6} + 3m), \frac{1}{3}\ , \frac{1}{3})$.

For m=2, we get n=26 &

$$(x, y, z) = (\frac{35}{6}, \frac{2}{6}, \frac{2}{6}) \tag{12}$$

Using the group structure, we get other parametric solution. $n = m(6m + 1)$

$x=(1/6w)(-30233088m^8 - 7558272m^7 - 139968m^6 - 1189728m^5 - 559872m^4 - 86832m^3 - 5508m^2 - 162m + 10077696m^9 - 1)$

$y = 18m(6m + 1)(36m^2 + 6m + 1)^3$

$z = 18m(6m + 1) (36m^2 + 6m + 1)^3$

where:

$w = (36m^2 + 6m + 1)(216m^3 + 216m^2 + 18m - 1)(216m^3 - 108m^2 - 36m - 1)$

We can get the positive solution, with m>=4.

For m=4, n= $m(6m + 1)$,

we get a numerical example:

(n, x, y, z):

$n = 100$

$$x = \frac{534676592951}{747746654406}$$

$$y = \frac{650161800}{207361801}$$

$$z = \frac{650161800}{207361801}$$

Note: A smaller numerical solution from equation (12) above, after clearing the denominators is :

$For\ n = 5616, (x, y, z) = (35, 2, 2)$

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