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RESEARCH ARTICLE

Analysis of Markovian Finite buffer queue with Threshold dependent impatience

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ABSTRACT

We present a comprehensive stochastic model designed for a single-server finite capacity Markovian queuing system, which captures several intricate features commonly observed in real-world queuing scenarios. Specifically, the model incorporates encouraged arrivals, threshold limits, balking and position-dependent reneging. These characteristics reflect the complexities of real-world queuing systems. Our model is unique in integrating all these elements into a single framework, a novel approach not seen in prior research. This novel approach provides valuable insights into the interplay of these factors and lays the groundwork for further research and practical applications in the design and analysis of queueing systems. Using Markov process methods and probability generating functions (p.g.f's), we derive steady-state probabilities that describe the long-term behaviour of the system. These steady-state probabilities form the backbone of our analysis, providing insights into the distribution of customers within the queue and the server under various operational conditions. Building upon this foundation, we further develop closed form expressions for several key performance measures, which are critical for assessing the functionality and efficiency of the queuing system. Additionally, we introduce some freshly designed measures to evaluate the system's effectiveness better. A comprehensive sensitivity analysis explores how varying parameters impact performance, offering insights into the model's robustness under different conditions. To demonstrate its practical utility, we provide a numerical example that showcases the analytical results and highlights the system's flexibility and applicability to real-life scenarios. This work contributes significantly to understanding and optimizing complex queuing systems.

Keywords. Markovian, threshold limit, encouraged arrival, position-dependent reneging.
Mathematics Subject Classification (2020). 60K25, 68M20, 90B22.

1 INTRODUCTION

Managing the flow of customers is essential for businesses to run smoothly and deliver quality service. Queueing systems play a key role in this process, helping organizations reduce wait times, improve efficiency and enhance the overall customer experience. By studying how queues function, businesses can make smarter decisions about allocating resources, streamlining services and keeping customers engaged, ultimately staying competitive and delivering better performance.

When customers arrive at a service facility, they may encounter busy servers and need to wait in line. However, not everyone is willing to wait. Some customers may choose to leave right away if the queue looks too long—a behavior known as balking. Others may initially join the queue but get impatient and leave before being served, which is called reneging. Take a hospital emergency room as an example. Patients often evaluate the waiting area, considering how many people are ahead of them and how long they think it will take to be seen. If the room is too crowded or the expected wait is too long, some might decide not to join the queue at all (balking). Others who start waiting might leave partway through (reneging), especially if they have alternatives like going to a different hospital or clinic.

Understanding these behaviours has been a topic of interest for researchers for decades. Early studies by Barrer(1957a, 1957b), Haight(1957,1959), and others laid the foundation for analyzing balking and reneging. More recently, researchers have explored how these behaviours interact with complex queueing systems. For example, Bouchentouf et al. (2022) studied queues with features like feedback, breaks for servers, and customer impatience. Other studies, like those by Choudhury and Medhi (2011a, 2011b, 2011c) and Medhi and Choudhury (2023), have focused on specific behaviours like state-dependent balking and retaining customers who initially leave the queue.

An emerging idea in queueing theory is encouraged arrivals. This happens when businesses use incentives like promotions or discounts to attract more customers. While these strategies can increase customer numbers and boost revenue, there's a limit to how effective they can be. If the system becomes overwhelmed, customers might lose patience, introducing what researchers call a threshold. Beyond this point, customers get impatient and display reneging as well as balking. For example, shopping malls often use marketing campaigns to draw crowds during sales or special events. Many malls also have waiting lounges with limited seating, creating a threshold where most customers are comfortable and less likely to leave. However, during peak times, those who can't find a spot to wait comfortably may choose to leave instead of sticking around. This scenario illustrates how encouraged arrivals and thresholds interact in real-world settings. Previous studies, including those by Som and Seth (2017), examined M/M/1/N models with encouraged arrivals and their steady-state characteristics. Later research expanded on these concepts to include multi-server models and investigated reverse reneging phenomena. Rao, Kumar, and Kumar (2020) incorporated thresholds into finite-capacity Markovian systems featuring both encouraged and discouraged arrivals, whereas Ahmad and Jayalalitha (2021) focused on analysing steady-state properties in related models.

This research builds on these ideas by introducing a detailed model for managing queues in multi-server finite capacity settings. The model incorporates features like encouraged arrivals, thresholds, state-independent balking and position-dependent reneging to give businesses better tools for managing customer flow. It offers valuable insights for improving resource allocation, customer engagement, and service efficiency. The concepts are applicable to a range of scenarios, from restaurants to hospitals and shopping malls.

The paper is organized as follows: Section 2 introduces the model and its assumptions, while Section 3 examines its steady-state behavior. Section 4 dives into performance metrics, followed by a practical example in Section 5 and sensitivity analysis in Section 6. The study concludes in Section 7, summarizing the key findings and future research opportunities. Detailed technical derivations are provided in the appendix.

2 ASSUMPTIONS AND DESCRIPTION OF THE MODEL

The suggested model operates under the following assumptions:

1. The arrival process follows a Poisson distribution with a rate parameter λ. The encouraged arrival is taken as λ(1 + η) up to a specified threshold, where η represents the proportional increase in arrivals due to an incentive. The threshold system size is denoted by k. This threshold signifies that the increased arrival rate under the "encouraged arrival strategy" is only valid up to k. Beyond this point, the arrival rate reverts to λ as it is assumed that customers may lose patience and either abandon or decide not to join the system. Essentially, the encouraged strategy is effective only up to the threshold since prolonged waiting can lead to customer impatience, rendering the strategy ineffective.
2. Service rate follows Poisson law with parameter μ.
3. The reneging rate follows a Poisson distribution with a rate parameter α. Reneging occurs only when the system size exceeds the threshold k, meaning customers within the threshold do not renege. The reneging rate depends on the system's state and is a function of the state of the system, i.e., position-dependent reneging.

   A customer who is at a position 'n' in the system will be assumed to have the reneging rate,

$$r(n) = \begin{cases} 0 & for\ \ 0 \le n \le k \\ (n-k)\alpha & for\ \ n \ge k+1 \end{cases} \tag{2.1}$$

4. Customers have a probability '(1- ξ)' of balking after the threshold value k. There is no balking before the threshold.
5. There is only one server.
6. System capacity is finite and restricted to N.
7. Service discipline is of first come first serve (FCFS) type.
8. Calling population is infinite.

The application of our designated model is evident in hospital OPDs, especially for renowned doctors. In such OPDs, patients are typically willing to wait longer than usual. While a seating area (often equipped with chairs) is provided for waiting patients, its limited capacity means some patients have to stand nearby when seats are full. We define the number of seats as the threshold limit. Within this threshold, patients rarely leave before seeing the doctor, so our model assumes no reneging up to this limit. Due to the doctor's reputation, patients are generally willing to wait even beyond the threshold, standing if necessary. However, patients who are standing are less comfortable and thus more likely to experience impatience of both types, balking (refusing to wait) as well as reneging (leaving before being seen). The doctor's efficiency draws even more patients to the OPD, increasing the arrival rate, so our model also incorporates the concept of encouraged arrival.

3 SYSTEM STATE PROBABILITIES

In this section, steady state probabilities of the system are derived using Markov process method. The state transition diagram for the suggested model is given below:

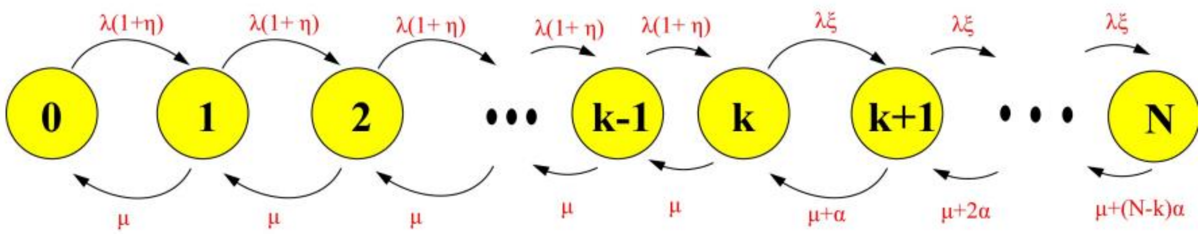

Figure 1: State transition diagram

Using the "Rate in=Rate out" principle, from the transition diagram (Fig. 1) we get the following steady state equations.

$$\mu p_1 = \lambda(1+\eta)p_0\ ; \qquad for\ n = 0 \tag{3.1}$$

$$\lambda(1+\eta)p_{n-1} + \mu p_{n+1} = [\mu + \lambda(1+\eta)]p_n\ ; \qquad for\ n = 1,2,\dots,k-1 \tag{3.2}$$

$$\lambda(1+\eta)p_{n-1} + (\mu+\alpha)p_{n+1} = (\mu+\lambda\xi)p_n\ ; \qquad for\ n = k \tag{3.3}$$

$$\lambda\xi p_{n-1} + \{\mu+(n-k1)\alpha\}p_{n+1} = [\{\mu+(n-k)\alpha\}+\lambda\xi]p_n\ ; \quad for\ n=k+1,k+2,\dots,N-1 \tag{3.4}$$

$$\lambda\xi p_{n-1} = \{\mu+(n-k)\alpha\}p_n; \qquad for\ n = N \tag{3.5}$$

Where, p_n denotes the probability that there are n number of customers in the system, n = 0,1,2,....N

Solving the above equations, we get

$$p_n = \begin{cases} \left[\frac{\lambda(1+\eta)}{\mu}\right]^n p_0 & for\ n = 0,1,2,\dots,k \\ \left(\frac{1+\eta}{\mu\xi}\right)^k \frac{(\lambda\xi)^n}{\prod_{i=1}^{n-k}(\mu+i\alpha)} p_0 & for\ n = k+1, k+2, \dots, N \end{cases} \tag{3.6}$$

We have,

$$\sum_{n=0}^{N} p_n = 1$$

Using this condition, we obtain

$$p_0 = \left[1 + \sum_{n=1}^{k} \left\{\frac{\lambda(1+\eta)}{\mu}\right\}^n + \left(\frac{1+\eta}{\mu\xi}\right)^k \sum_{n=k+1}^{N} \frac{(\lambda\xi)^n}{\prod_{i=1}^{n-k}(\mu+i\alpha)}\right]^{-1} \tag{3.7}$$

4 PERFORMANCE MEASURES

1. Expected system size (L): Let, $P(s)$ be the p.g.f of the steady-state probabilities. Subsequently, we find that,

$$L = \sum_{n=0}^{N} np_n$$
$$= P'(1)$$
$$= \frac{d}{ds}P(s)|_{s=1}$$

The derivation of $P'(1)$ is given in the Appendix section. From the Appendix, the expected system size is,

$$L = \frac{\lambda}{\alpha}\left[(1+\eta)\sum_{n=0}^{k-1} p_n + \xi\left(\sum_{n=k}^{N} p_n - p_N\right)\right] - \frac{\mu}{\alpha}\sum_{n=1}^{N} p_n + \sum_{n=1}^{k} np_n + k\sum_{n=k+1}^{N} p_n \tag{4.1}$$

2. Expected queue size (Lq): The expected queue size is given by the formula,

$$L_q = \sum_{n=2}^{N} (n-1)p_n$$
$$= L + p_0 + 1$$

$$= \frac{\lambda}{\alpha}\left[(1+\eta)\sum_{n=0}^{k-1} p_n + \xi\left(\sum_{n=k}^{N} p_n - p_N\right)\right] - (1-p_0)\left(\frac{\mu}{\alpha}+1\right) + \sum_{n=1}^{k} np_n + k\sum_{n=k+1}^{N} p_n \tag{4.2}$$

Using Little's formula ($L = \lambda W$ or $L_Q = \lambda W_Q$) one can obtain the expected waiting time in system, W and expected waiting time in queue W_Q.

3. **Effective arrival rate (λ_e):** Customers enter the system at an increased arrival rate of $\lambda(1+\eta)$ until the system size reaches the threshold k. Beyond this threshold, the arrival rate decreases to λ. However, not all arriving customers choose to join the system due to balking. As a result, the effective arrival rate of customers entering the system differs from the overall arrival rate and is expressed as:

$$\lambda_e = \sum_{n=0}^{k-1} \lambda(1+\eta)p_n + \sum_{n=k}^{N-1} \lambda\xi p_n$$

$$= \lambda\xi(1 - p_N) + \lambda(1 + \eta - \xi)\sum_{n=0}^{k-1} p_n \tag{4.3}$$

4. **Average reneging rate (α^A):** Since a position dependent reneging rate is assumed, the reneging rate of the system will vary based on its current state. The average reneging rate is given by,

$$\alpha^A = \sum_{n=k+1}^{N} (n-k)\alpha p_n$$

$$= \alpha \sum_{n=k+1}^{N} n p_n - k\alpha \sum_{n=k+1}^{N} p_n$$

$$= \alpha\left(L - \sum_{n=1}^{k} p_n\right) - k\alpha\left(1 - \sum_{n=0}^{k} p_n\right)$$

$$= \lambda\left[(1+\eta)\sum_{n=0}^{k-1} p_n + \xi\left(\sum_{n=k}^{N} p_n - p_N\right)\right] - \mu\sum_{n=1}^{N} p_n \tag{4.4}$$

5. **Probability that a customer will join the system within the threshold given that he/she is not turned away due to finite buffer restriction (γ):** It is assumed that customers are encouraged up to the threshold limit k. Any encouraging strategy developed by the business organization will work until the system size reaches the threshold value k, after that customer impatience can be observed. The probability that a customer will join the system within the threshold, provided they are not rejected due to finite buffer limitations can be expressed as,

$$\gamma = \frac{\sum_{n=0}^{k-1} p_n}{1 - p_N} \tag{4.5}$$

6. **Proportion of customers lost due to reneging (P^α):** In real-world situations, when customers leave the queue (reneging) due to impatience, it leads to a loss of business. Businesses are keen to understand how many customers are being lost, as this gives a clear picture of the overall impact on revenue. With this information, management can take steps to address the issue and minimize these losses.

The proportion of customers lost as a result of reneging is given by,

$$P^\alpha = \frac{\alpha^A}{\lambda_e}$$

$$= \frac{\lambda\left[(1+\eta)\sum_{n=0}^{k-1} p_n + \xi\left(\sum_{n=k}^{N} p_n - p_N\right)\right] - \mu\sum_{n=1}^{N} p_n}{\lambda_e} \tag{4.6}$$

7. **Average arrival rate (post-threshold) adjusted for balking (λ_{PT}):**

$$= \lambda\xi \sum_{n=k}^{N=1} p_n \tag{4.7}$$

8. **Proportion of customers lost due to balking and finite buffer restriction (P^{B+F}):** The proportion of customers lost due to balking and limited space shows how many customers either decide not to join or are turned away because the system is full. The proportion of customers lost as a result of balking and finite buffer restriction is given by,

$$P^{\xi} = \frac{\sum_{n=k}^{N} \lambda p_n - \sum_{n=k}^{N} \lambda \xi p_n}{\lambda_e} \tag{4.8}$$

9 **Proportion of total customers lost (P^T):** In a finite buffer queue, some customers are lost because they leave the queue (reneging), decide not to join (balking) or are turned away when the system is full. So, the proportion of total customers lost can be obtained by combining the proportion of customers lost due to reneging, balking and finite buffer restriction and is expressed as,

$$P^{T} = \frac{\sum_{n=k}^{N} \lambda p_n - \sum_{n=k}^{N} \lambda \xi p_n + \alpha^A}{\lambda_e} \tag{4.9}$$

10 **Actual server load or Arrival rate of customers reaching service station (λ^s):** Customers who leave the queue before being served don't contribute to the server's workload. So, the server's workload depends only on the customers who stay and get served. This helps in measuring the server's actual workload.

λ^s=λ^e(1-proportion of customers lost due to reneging out of those joining the system)

$$\begin{aligned} \lambda^s &= \lambda^e \left(1 - \frac{\alpha^A}{\lambda^e}\right) \\ &= \mu(1-p_0) + \lambda \xi p_N \end{aligned} \tag{4.10}$$

5. NUMERICAL EXAMPLE

To illustrate the usefulness of our suggested model, we consider the following example from Taha (2013).

"Patients arrive at a 1-doctor clinic according to a Poisson distribution at the rate of 20 patients per hour. The waiting room doesn't accommodate more than 14 patients. Examination time per patient is exponential, with a mean of 8 minutes. (a) What is the probability that an arriving patient will not wait? (b) What is the probability that an arriving patient will find a seat in the room? (c) What is the expected total time a patient spends in the clinic?"

The features of the above queuing system fit the features of our assuming queuing model. We recall that our queuing model is a single server Markovian queuing system with encouraged arrival, threshold limit, balking and position dependent reneging. The above example describes a 1-doctor clinic. The Markovian assumptions are clearly stated. The existence of a threshold is also specific as the waiting room does not accommodate more than 14 patients. Including the one being served, the threshold will be 15. We shall additionally assume that because of the skill of the doctor, patients are willing to wait outside the waiting room. However, the space outside can hold a maximum of 11 patients.

We assume that due to the doctor's expertise, patients are consistently motivated to schedule appointments at the clinic. This aligns with our theoretical assumption of encouraged arrivals. Although it is mentioned that some patients may wait outside the main waiting room if needed, there is no denying that this is considered a second preferenced choice. As a result, it is not surprising that patients in this waiting area may get impatient and could display reneging or balking.

To make the given queuing system more realistic, we have assumed existence of a threshold limit, encouraged arrival, balking and position dependent reneging. To model different scenarios, we assume two sets of parameters.

Table 1: performance measures when $\lambda = 20/hr, \mu = 8/hr, a = 0.5/hr, \eta = 0.3, \xi = 0.8, k = 15, N = 26$

Performance Measures	
Probability that the server remains idle	$7.1 * 10^{-11}$
Mean system size	23.5195
Effective arrival rate	12.2608
Actual server load	11.7543
Average reneging rate	4.2608
Proportion of customers lost due to reneging	0.3475
Average arrival rate (post-threshold) adjusted for balking	15.9759
Proportion of customers lost due to balking and finite buffer restriction	0.3257
Proportion of total customers lost	0.6733
Probability that a customer will join the system with threshold incentive	0.002

Table 2: performance measures when $\lambda = 20/hr, \mu = 8/hr, a = 1/hr, \eta = 0.3, \xi = 0.6, k = 15, N = 26$

Performance Measures	
Probability that the server remains idle	$1.5 * 10^{-9}$
Mean system size	19.1873
Effective arrival rate	12.2331
Actual server load	8.2109
Average reneging rate	4.2331
Proportion of customers lost due to reneging	0.346
Average arrival rate (post-threshold) adjusted for balking	11.4086
Proportion of customers lost due to balking and finite buffer restriction	0.6322
Proportion of total customers lost	0.9792
Probability that a customer will join the system with threshold incentive	0.004

Some interesting observations can be made from the above two scenarios.

1. Probability that an arriving customer receives instant service goes up as the balking probability and reneging rate increases. This is expected. As more customers leave the system because of impatience (balking and reneging), the probability that the system is idle goes up.

2. Mean system size decreases as balking and reneging increases. We reiterate that balking and reneging occurs only after the threshold. The expected size of the system has been computed to be approximately 24 and 19 respectively for the 2 scenarios. It is note worthy that both the system sizes are in excess of the threshold and therefore the customers beyond

the threshold experience balking as well as reneging. The average system size in second scenario is the lower because of the higher balking probability (0.4) and higher reneging rate (1/hr).

3. Effective arrival rate slightly goes down as the balking probability goes up.
4. Actual server load decreases as more customers leave the system because of higher impatience (balking and reneging).
5. Average reneging rate has decreased in the second scenario even though a higher reneging rate has been assumed. This might appear contradictory however, we recall that reneging is observed only among those customers who join the system beyond the threshold. We note that the balking prob. in the second scenario is twice the balking probability of the first scenario. Consequently, more customers balk in the second scenario compared to the first. This in turn means that reneging is observed on fewer customers and therefore it is not unexpected that the average reneging rate has decreased. The same logic applies to reduction of proportion of customers lost due to reneging.
6. The proportion of customers lost due to balking and finite buffer restriction has increased in the second scenario even though the size of the buffer is the same(N=26). This increase is due to increase in the balking probability. The balking probability in the second scenario is twice that of the first and consequently the proportion of customers lost due to balking and finite buffer restriction is increases from 32% to 63% in the second scenario.
7. Proportion of total customer lost is higher in the second scenario. This is because more customers leave the system due to higher reneging rate and balking probability.
8. From the perspective of a customer, given that he has not been turned away due to finite buffer restriction the probability of joining the system inside the threshold is of particular interest. We note that in both our scenarios the buffer size has been kept constant at twenty-six so as to aid comparison. We also note that the average system size in the first scenario is around 24 and in the second scenario around 19. In other words, there are around nine customers beyond the threshold in the first scenario and around four customers only in the second. Consequently, it is expected that an arriving customer will have a higher probability of joining inside the threshold in the second scenario. Our numerical results confirm the same. The only issue that could arise is with regard to low probability of joining inside the threshold (0.002 in the first scenario and 0.004 in the second scenario). This is because of the fact that there would be around nine customers on the average in the system beyond the threshold in the first scenario and around four customers in the second scenario. The probability of joining has gone up in the second scenario because the average number of customers beyond the threshold is lower compared to the first.

Answer to the specific questions posed by the hospital planners are as follows (Considering the first scenario).

1. The probability that an arriving patient will not wait = $p_0 = 7.1 * 10^{-11}$
2. The probability that an arriving patient will find a seat in the room $=\sum_{n=0}^{k-1} p_n = 0.002$
3. The expected total time a patient spends in the clinic $=W = \frac{L}{\lambda} = 1.18hr$

6 SENSITIVITY ANALYSIS

To study the variations in the different performance measures with variations in the system parameters, a sensitivity analysis is carried out in this section.

I. Let $\lambda_1 > \lambda_0$, then

$$\frac{p_o(\lambda_1,\mu,\alpha)}{p_o(\lambda_0,\mu,\alpha)} < 1$$

$$\Rightarrow \frac{\left[1+\sum_{n=1}^{k}\left\{\frac{\lambda_1(1+\eta)}{\mu}\right\}^n+\sum_{n=k+1}^{N}\left(\frac{1+\eta}{\mu\xi}\right)^k\frac{(\lambda_1\xi)^n}{\prod_{i=1}^{n-k}(\mu+i\alpha)}\right]^{-1}}{\left[1+\sum_{n=1}^{k}\left\{\frac{\lambda_0(1+\eta)}{\mu}\right\}^n+\sum_{n=k+1}^{N}\left(\frac{1+\eta}{\mu\xi}\right)^k\frac{(\lambda_0\xi)^n}{\prod_{i=1}^{n-k}(\mu+i\alpha)}\right]^{-1}} < 1$$

$$\Rightarrow 1+\sum_{n=1}^{k}\left\{\frac{\lambda_0(1+\eta)}{\mu}\right\}^n+\sum_{n=k+1}^{N}\left(\frac{1+\eta}{\mu\xi}\right)^k\frac{(\lambda_0\xi)^n}{\prod_{i=1}^{n-k}(\mu+i\alpha)} < 1+$$

$$\sum_{n=1}^{k}\left\{\frac{\lambda_1(1+\eta)}{\mu}\right\}^n+\sum_{n=k+1}^{N}\left(\frac{1+\eta}{\mu\xi}\right)^k\frac{(\lambda_1\xi)^n}{\prod_{i=1}^{n-k}(\mu+i\alpha)}$$

$$\Rightarrow (\lambda_0-\lambda_1)\left(\frac{1+\eta}{\mu}\right)+(\lambda_0^2-\lambda_1^2)\left(\frac{1+\eta}{\mu}\right)^2+\cdots+(\lambda_0^k-\lambda_1^k)\left(\frac{1+\eta}{\mu}\right)^k+(\lambda_0^{k+1}-\lambda_1^{k+1})$$

$$\left(\frac{1+\eta}{\mu\xi}\right)^k\frac{\xi^{k+1}}{(\mu+\alpha)}+(\lambda_0^{k+2}-\lambda_1^{k+2})\left(\frac{1+\eta}{\mu\xi}\right)^k\frac{\xi^{k+2}}{(\mu+\alpha)(\mu+2\alpha)}+\cdots+(\lambda_0^N-\lambda_1^N)$$

$$\left(\frac{1+\eta}{\mu\xi}\right)^k\frac{\xi^N}{(\mu+\alpha)(\mu+2\alpha)\ldots(\mu+(N-k)\alpha)} < 0$$

which is true. Hence, p_0 decreases as λ increases

II. Let $\mu_1 > \mu_0$, then

$$\frac{p_o(\lambda,\mu_1,\alpha)}{p_o(\lambda,\mu_0,\alpha)} > 1$$

$$\Rightarrow \frac{\left[1+\sum_{n=1}^{k}\left\{\frac{\lambda(1+\eta)}{\mu_1}\right\}^n+\sum_{n=k+1}^{N}\left(\frac{1+\eta}{\mu_1\xi}\right)^k\frac{(\lambda\xi)^n}{\prod_{i=1}^{n-k}(\mu_1+i\alpha)}\right]^{-1}}{\left[1+\sum_{n=1}^{k}\left\{\frac{\lambda(1+\eta)}{\mu_0}\right\}^n+\sum_{n=k+1}^{N}\left(\frac{1+\eta}{\mu_0\xi}\right)^k\frac{(\lambda\xi)^n}{\prod_{i=1}^{n-k}(\mu_0+i\alpha)}\right]^{-1}} > 1$$

$$\Rightarrow 1+\sum_{n=1}^{k}\left\{\frac{\lambda(1+\eta)}{\mu_0}\right\}^n+\sum_{n=k+1}^{N}\left(\frac{1+\eta}{\mu_0\xi}\right)^k\frac{(\lambda\xi)^n}{\prod_{i=1}^{n-k}(\mu_0+i\alpha)} > 1+$$

$$\sum_{n=1}^{k}\left\{\frac{\lambda(1+\eta)}{\mu_1}\right\}^n+\sum_{n=k+1}^{N}\left(\frac{1+\eta}{\mu_1\xi}\right)^k\frac{(\lambda\xi)^n}{\prod_{i=1}^{n-k}(\mu_1+i\alpha)}$$

$$\Rightarrow \lambda(1+\eta)\left(\frac{1}{\mu_0}-\frac{1}{\mu_1}\right)+\{\lambda(1+\eta)\}^2\left(\frac{1}{\mu_0^2}-\frac{1}{\mu_1^2}\right)+\ldots+\{\lambda(1+\eta)\}^k\left(\frac{1}{\mu_0^k}-\frac{1}{\mu_1^k}\right)+(1+\eta)^k\lambda^{k+1}\xi\left\{\frac{1}{\mu_0^k(\mu_0+\alpha)}-\frac{1}{\mu_1^k(\mu_1+\alpha)}\right\}+(1+\eta)^k\lambda^{k+2}\xi^2\left\{\frac{1}{\mu_0^k(\mu_0+\alpha)(\mu_0+2\alpha)}-\frac{1}{\mu_1^k(\mu_1+\alpha)(\mu_1+2\alpha)}\right\}+\ldots+(1+\eta)^k\lambda^N\xi^{N-k}\left\{\frac{1}{\mu_0^k(\mu_0+\alpha)(\mu_0+2\alpha)\ldots(\mu_0+(N-k)\alpha)}-\frac{1}{\mu_1^k(\mu_1+\alpha)(\mu_1+(N-k)\alpha)}\right\} > 0$$

which is true. Hence, p_0 increases as μ increases.

III. Let $a_1 > a_0$, then

$$\frac{p_o(\lambda,\mu,\alpha_1)}{p_o(\lambda,\mu,\alpha_0)} > 1$$

$$\Rightarrow \frac{\left[1+\sum_{n=1}^{k}\left\{\frac{\lambda(1+\eta)}{\mu}\right\}^n+\sum_{n=k+1}^{N}\left(\frac{1+\eta}{\mu\xi}\right)^k\frac{(\lambda\xi)^n}{\prod_{i=1}^{n-k}(\mu+i\alpha_1)}\right]^{-1}}{\left[1+\sum_{n=1}^{k}\left\{\frac{\lambda(1+\eta)}{\mu}\right\}^n+\sum_{n=k+1}^{N}\left(\frac{1+\eta}{\mu\xi}\right)^k\frac{(\lambda\xi)^n}{\prod_{i=1}^{n-k}(\mu+i\alpha_0)}\right]^{-1}} > 1$$

$$\Rightarrow 1+\sum_{n=1}^{k}\left\{\frac{\lambda(1+\eta)}{\mu}\right\}^n+\sum_{n=k+1}^{N}\left(\frac{1+\eta}{\mu\xi}\right)^k\frac{(\lambda\xi)^n}{\prod_{i=1}^{n-k}(\mu+i\alpha_0)} > 1+\sum_{n=1}^{k}\left\{\frac{\lambda(1+\eta)}{\mu}\right\}^n+\sum_{n=k+1}^{N}\left(\frac{1+\eta}{\mu\xi}\right)^k\frac{(\lambda\xi)^n}{\prod_{i=1}^{n-k}(\mu+i\alpha_1)}$$

$$\Rightarrow (\lambda\xi)^{k+1}\left\{\frac{1}{\mu+\alpha_0}-\frac{1}{\mu+\alpha_1}\right\}+(\lambda\xi)^{k+2}\left\{\frac{1}{(\mu+\alpha_0)(\mu+2\alpha_0)}-\frac{1}{(\mu+\alpha_1)(\mu+2\alpha_1)}\right\}$$
$$+\ldots+(\lambda\xi)^N\left\{\frac{1}{(\mu+\alpha_0)(\mu+(N-k)\alpha_0)}-\frac{1}{(\mu+\alpha_1)(\mu+(N-k)\alpha_1)}\right\}>0$$

which is true. Hence, p_0 increases as a increases.

Table 3: Variations in performance measures w.r.t. mean arrival rate λ considering $\mu = 8/hr, a = 0.5/hr, k = 15, \eta = 0.3, \xi = 0.8, N = 26$

λ	p_0	L	λe	λ^s	α^A	P^α	P^{B+F}	P^I	γ
20	7.1×10^{-11}	23.5194	12.2608	11.7543	4.2608	0.3475	0.3257	0.6733	0.002
21	2.2×10^{-11}	23.7767	12.3889	12.4206	4.3889	0.3543	0.3386	0.6929	0.001
22	7.3×10^{-12}	23.9926	12.4967	13.1096	4.4966	0.3598	0.3518	0.7117	0.0008
23	2.5×10^{-12}	24.1752	12.5878	13.8163	4.5878	0.3645	0.3653	0.7298	0.0005
24	9.0×10^{-13}	24.3307	12.6655	14.5372	4.6655	0.3684	0.3789	0.7472	0.0003
25	3.3×10^{-13}	24.4642	12.7322	15.2696	4.7322	0.3717	0.3926	0.7643	0.0002

It can be seen from Table 3 that as the arrival rate λ increases, the effective arrival rate, system size and the proportion of total customers lost all rise. In contrast, the probability of the server remaining idle decreases. This is expected, as a higher number of arriving customers naturally reduces the likelihood of idle time for the server. The increase in other performance metrics is due to the fact that a higher customer arrival rate leads to a longer queue, which, in turn, increases the system size. As a result, customers experience longer waiting times, leading to a higher average reneging rate and a greater proportion of customers lost due to impatience.

Table 4: Variations in performance measures w.r.t. mean service rate μ considering $\lambda = 20/hr, a = 0.5/hr, k = 15, \eta = 0.3, \xi = 0.8, N = 26$

μ	p_0	L	λe	λ^s	α^A	P^α	P^{B+F}	P^I	γ
8	7.1×10^{-11}	23.5194	12.2608	11.7543	4.2608	0.3475	0.3257	0.6733	0.002
9	8.9×10^{-10}	22.9966	13.0013	12.0372	4.0013	0.3078	0.3065	0.6142	0.0047
10	8.3×10^{-9}	22.3846	13.6994	12.3881	3.6994	0.27	0.2894	0.5595	0.0102
11	6.1×10^{-8}	21.6874	14.3592	12.8196	3.3592	0.2339	0.2736	0.5075	0.0202
12	3.5×10^{-7}	20.9202	14.9909	13.3412	2.9909	0.1995	0.2579	0.4575	0.0362
13	1.7×10^{-6}	20.1067	15.6099	13.956	2.6099	0.1672	0.2417	0.4089	0.0602

As shown in Table 4, an increase in the service rate μ leads to a reduction in the system size. Consequently, both the average reneging rate and the proportion of total customers lost also decrease. With a higher service rate, customers experience shorter wait times, making it less likely for them to renege, which results in fewer customers leaving the queue due to impatience. Additionally, an increase in the service rate raises the probability of the system being idle, which is clearly observed. The effective arrival rate increases as the service rate rises, because a higher service rate reduces wait times, allowing the system to handle more customers in a given period. This results in a higher rate of arrivals being processed, effectively increasing the arrival rate.

Table 5: Variations in performance measures w.r.t. reneging rate α considering $\lambda = 20/hr$, $\mu = 8/hr, k = 15, \eta = 0.3, \xi = 0.8, N = 26$

α	p_0	L	λe	λ^s	α^A	P^α	P^{B+F}	P^I	γ
0.5	7.1×10^{-11}	23.5194	12.2608	11.7543	4.2608	0.3475	0.3257	0.6733	0.002
0.6	1.02×10^{-10}	23.1553	12.8951	11.1266	4.8951	0.3796	0.3095	0.6891	0.003
0.7	1.4×10^{-10}	22.7819	13.4503	10.5798	5.4503	0.4052	0.2964	0.7017	0.004
0.8	1.9×10^{-10}	22.4059	13.9294	10.1109	5.9294	0.4257	0.286	0.7116	0.005
0.9	2.5×10^{-10}	22.034	14.3374	9.7149	6.3374	0.442	0.2775	0.7195	0.006
1	3.1×10^{-10}	21.6716	14.6811	9.3848	6.6811	0.4551	0.2707	0.7257	0.007

As observed in Table 5, when the reneging rate α increases, the proportion of total customers lost also rises. This is because a higher reneging rate results in more customers abandoning the queue before receiving service. Consequently, the system size decreases as fewer customers remain in the queue. As a result, when an arriving customer observes a shorter queue, they are more likely to join the system, increasing the effective arrival rate.

7 CONCLUSION

This study examines a multi-server queuing model featuring encouraged arrivals, threshold limits, balking and position-dependent reneging. The analysis focuses on steady-state conditions and provides explicit formulas for both traditional and innovative performance metrics. A numerical example demonstrates the model's practical relevance and application in real-world scenarios. Sensitivity analysis evaluates how parameter changes influence performance measures, offering insights into system behaviour. The findings highlight opportunities for extending the model by incorporating encouraged arrivals and threshold limits. Future research could address non-Markovian settings, transient state conditions, alternative arrival patterns such as batch arrivals and customer behaviour under varied service disciplines.

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Appendix

Derivation of P ′ (1):

From equation (3.2), we have

we have $\lambda(1+\eta)p_{n-1} + \mu p_{n+1} = [\mu + \lambda(1+\eta)]p_n$; *for* $n = 1,2,\ldots,k-1$

Multiplying both sides of the equation by S^n and summing over n,

$$\sum_{n=1}^{k-1}\left[\lambda(1+\eta)p_{n-1} + \mu p_{n+1}\right]S^n = \sum_{n=1}^{k-1}\left[\{\mu + \lambda(1+\eta)p_n\}\right]S^n$$

$$\Rightarrow \lambda S(1+\eta)\sum_{n=1}^{k-1} p_{n-1}S^{n-1} + \frac{\mu}{S}\sum_{n=1}^{k-1} p_{n+1}S^{n+1} = \mu\sum_{n=1}^{k-1} p_n S^n + \lambda(1+\eta)\sum_{n=1}^{k-1} p_n S^n \quad (8.1)$$

From equation (3.3), we have

$\lambda(1+\eta)p_{k-1} + (\mu + a)p_{k+1} = (\mu + \lambda\xi)p_k$; *for* $n = k$

Multiplying both sides of the equation by S^k,

$$\lambda S(1+\eta)p_{k-1}S^{k-1} + \frac{1}{S}(\mu+\alpha)p_{k+1}S^{k+1} = (\mu+\lambda\xi)p_k S^k \quad (8.2)$$

From equation (3.4), we have

$\lambda\xi p_{n-1}+\{\mu + (n - k + 1)a\}p_{n+1} = [\{\mu + (n - k)a\} + \lambda\xi]p_n$; *for* $n = k+1,k+2,\ldots,N-1$

Multiplying both sides of the equation by S^n and summing over n,

$$\sum_{n=k+1}^{N-1}\left[\lambda\xi p_{n-1} + \{\mu + (n-k+1)\alpha\}p_{n+1}\right]S^n = \sum_{n=k+1}^{N-1}\left[\{\mu + (n-k)\alpha\} + \lambda\xi\right]p_n S^n$$

$$\Rightarrow \lambda\xi S\sum_{n=k+1}^{N-1} p_{n-1}S^{n-1} + \frac{1}{S}\sum_{n=k+1}^{N-1}\{\mu + (n-k+1)\alpha\}\,p_{n+1}S^{n+1}$$

$$= \sum_{n=k+1}^{N-1}\{\mu + (n-k)\alpha\}\,p_n S^n + \lambda\xi\sum_{n=k+1}^{N-1} p_n S^n \qquad (8.3)$$

From equation (3.5), we have

$\lambda\xi p_{n-1} = \{\mu + (n - k)a\}p_n$;   *for* $n = N$

Multiplying both sides of the equation by S^n and summing over n,

$\lambda\xi S p_{N-1}S^{N-1} = \{\mu + (N - k)a\}p_N S^N$ (8.4)

Adding (8.1), (8.2), (8.3) and (8.4),

$$\lambda S(1+\eta)\sum_{n=1}^{k-1} p_{n-1}S^{n-1} + \frac{\mu}{S}\sum_{n=1}^{k-1} p_{n+1}S^{n+1} + \lambda S(1+\eta)p_{k-1}S^{k-1} + \frac{1}{S}(\mu+\alpha)p_{k+1}S^{k+1}$$

$$+\lambda\xi S\sum_{n=k+1}^{N-1} p_{n-1}S^{n-1} + \frac{1}{S}\sum_{n=k+1}^{N-1}\{\mu + (n-k+1)\alpha\}\,p_{n+1}S^{n+1} + \lambda\xi S p_{N-1}S^{N-1}$$

$$= \mu\sum_{n=1}^{k-1} p_n S^n + \lambda(1+\eta)\sum_{n=1}^{k-1} p_n S^n + \mu p_k S^k + \lambda\xi p_k S^k + \sum_{n=k+1}^{N-1}\{\mu + (n-k)\alpha\}\,p_n S^n + \lambda\xi\sum_{n=k+1}^{N-1} p_n S^n + \{\mu + (N-k)\alpha\}p_N S^N$$

$$-\frac{1}{S}\left[\mu\sum_{n=1}^{k-1} p_{n+1}S^{n+1} + \mu p_{k+1}S^{k+1} + \alpha p_{k+1}S^{k+1} + \mu\sum_{n=k+1}^{N-1} p_{n+1}S^{n+1} + \alpha\sum_{n=k+1}^{N-1} n p_{n+1}S^{n+1} - k\alpha\sum_{n=k+1}^{N-1} p_{n+1}S^{n+1} + \alpha\sum_{n=k+1}^{N-1} p_{n+1}S^{n+1}\right]$$

$$\Rightarrow \lambda S\left[(1+\eta)\left\{P(S) - \sum_{n=k}^{N} p_n S^n\right\} + \xi\left\{P(S) - \sum_{n=0}^{k-1} p_n S^n - p_N S^N\right\}\right]$$

$$-\lambda\left[(1+\eta)\left\{P(S) - p_0 - \sum_{n=k}^{N} p_n S^n\right\} + \xi\left\{P(S) - \sum_{n=0}^{k-1} p_n S^n - p_N S^N\right\}\right]$$

$$= \mu\sum_{n=1}^{N} p_n S^n + \alpha S\sum_{n=k+1}^{N} n p_n S^{n-1} - k\alpha\sum_{n=k+1}^{N} p_n S^n -$$

$$\Rightarrow \lambda S\left[(1+\eta)\sum_{n=1}^{k} p_{n-1}S^{n-1}+\xi\sum_{n=k+1}^{N} p_{n-1}S^{n-1}\right]-\lambda\left[(1+\eta)\sum_{n=1}^{k-1} p_nS^n+\xi\sum_{n=k}^{N-1} p_nS^n\right]$$

$$=\mu\sum_{n=1}^{k} p_nS^n+\sum_{n=k+1}^{N}\{\mu+(n-k)\alpha\}p_nS^n-$$

$$\frac{1}{S}\left[\mu\sum_{n=1}^{k-1} p_{n+1}S^{n+1}+(\mu+\alpha)p_{k+1}S^{k+1}+\sum_{n=k+1}^{N-1}\{\mu+(n-k+1)\alpha\}p_{n+1}S^{n+1}\right]$$

$$\Rightarrow \lambda S\left[(1+\eta)\left\{P(S)-\sum_{n=k}^{N} p_nS^n\right\}+\xi\left\{P(S)-\sum_{n=0}^{k-1} p_nS^n-p_NS^N\right\}\right]$$

$$-\lambda\left[(1+\eta)\left\{P(S)-p_0-\sum_{n=k}^{N} p_nS^n\right\}+\xi\left\{P(S)-\sum_{n=0}^{k-1} p_nS^n-p_NS^N\right\}\right]$$

$$=\mu\sum_{n=1}^{k} p_nS^n+\mu\sum_{n=k+1}^{N} p_nS^n+\alpha\sum_{n=k+1}^{N} np_nS^n-\alpha k\sum_{n=k+1}^{N} p_nS^n-$$

$$-\frac{1}{S}\left[\mu\sum_{n=1}^{N-1} p_{n+1}S^{n+1}+\alpha\sum_{n=k}^{N-1} p_{n+1}S^{n+1}+\alpha S\sum_{n=k+1}^{N-1} np_{n+1}S^n-k\alpha\sum_{n=k+1}^{N-1} p_{n+1}S^{n+1}\right]$$

$$\Rightarrow \lambda S\left[(1+\eta)\left\{P(S)-\sum_{n=k}^{N} p_nS^n\right\}+\xi\left\{P(S)-\sum_{n=0}^{k-1} p_nS^n-p_NS^N\right\}\right]$$

$$-\lambda\left[(1+\eta)\left\{P(S)-p_0-\sum_{n=k}^{N} p_nS^n\right\}+\xi\left\{P(S)-\sum_{n=0}^{k-1} p_nS^n-p_NS^N\right\}\right]$$

$$=\mu\sum_{n=1}^{N} p_nS^n+\alpha S\sum_{n=k+1}^{N} np_nS^{n-1}-k\alpha\sum_{n=k+1}^{N} p_nS^n-$$

$$-\frac{1}{S}\left[\mu\sum_{n=1}^{N-1} p_{n+1}S^{n+1}+\alpha\sum_{n=k}^{N-1} p_{n+1}S^{n+1}+\alpha S\sum_{n=k+1}^{N-1} np_{n+1}S^n-k\alpha\sum_{n=k+1}^{N-1} p_{n+1}S^{n+1}\right]$$

$$\Rightarrow \lambda S\left[(1+\eta)\left\{P(S)-\sum_{n=k}^{N} p_nS^n\right\}+\xi\left\{P(S)-\sum_{n=0}^{k-1} p_nS^n-p_NS^N\right\}\right]$$

$$-\lambda\left[(1+\eta)\left\{P(S)-p_0-\sum_{n=k}^{N} p_nS^n\right\}+\xi\left\{P(S)-\sum_{n=0}^{k-1} p_nS^n-p_NS^N\right\}\right]$$

$$=\mu\{P(S)-p_0\}+\alpha S\left\{P'(S)-\sum_{n=1}^{k} np_nS^{n-1}\right\}-k\alpha\left\{P(S)-\sum_{n=0}^{k} p_nS^n\right\}-$$

$$-\frac{1}{S}\left[\mu\{P(S)-p_0-p_1S\}+\alpha\left\{P(S)-\sum_{n=0}^{k} p_nS^n\right\}-k\alpha\left\{P(S)-\sum_{n=0}^{k+1} p_nS^n\right\}+\left\{\alpha S\sum_{n=k+1}^{N-1}(n+1)p_{n+1}S^n-\alpha S\sum_{n=k+1}^{N-1} p_{n+1}S^n\right\}\right]$$

$$\Rightarrow \lambda S\left[(1+\eta)\left\{P(S)-\sum_{n=k}^{N} p_nS^n\right\}+\xi\left\{P(S)-\sum_{n=0}^{k-1} p_nS^n-p_NS^N\right\}\right]$$

$$-\lambda\left[(1+\eta)\left\{P(S)-p_0-\sum_{n=k}^{N}p_nS^n\right\}+\xi\left\{P(S)-\sum_{n=0}^{k-1}p_nS^n-p_NS^N\right\}\right]$$

$$=\mu\{P(S)-p_0\}+\alpha S\left\{P'(S)-\sum_{n=1}^{k}np_nS^{n-1}\right\}-k\alpha\left\{P(S)-\sum_{n=0}^{k}p_nS^n\right\}-$$

$$-\frac{1}{S}\left[\mu\{P(S)-p_0-p_1S\}+\alpha\left\{P(S)-\sum_{n=0}^{k}p_nS^n\right\}-k\alpha\left\{P(S)-\sum_{n=0}^{k+1}p_nS^n\right\}+\alpha S\left\{P'(S)-\sum_{n=1}^{k+1}np_nS^{n-1}\right\}-\alpha\left\{P(S)-\sum_{n=o}^{k+1}p_nS^n\right\}\right]$$

$$\Rightarrow \lambda(1+\eta)p_0+\lambda S\left(1-\frac{1}{S}\right)\left[(1+\eta)\left\{P(S)-\sum_{n=k}^{N}p_nS^n\right\}+\xi\left\{P(S)-\sum_{n=0}^{k-1}p_nS^n-p_NS^N\right\}\right]$$

$$=\mu p_1+\left(1-\frac{1}{S}\right)\left[\mu P(S)-\mu p_0+\alpha SP'(S)-\alpha S\sum_{n=1}^{k}np_nS^{n-1}-k\alpha P(S)+k\alpha\sum_{n=0}^{k}p_nS^n\right]$$

$$\Rightarrow \lambda S\left[(1+\eta)\sum_{n=0}^{k-1}p_nS^n+\xi P(S)-\xi\sum_{n=0}^{k-1}p_nS^n-\xi p_NS^N\right]=\mu P(S)-\mu p_0+\alpha SP'(S)-\alpha S\sum_{n=1}^{k}np_nS^{n-1}-$$

$$-k\alpha P(S)+k\alpha\sum_{n=0}^{k}p_nS^n$$

$$\Rightarrow \alpha SP'(S)=\lambda(1+\eta)\sum_{n=0}^{k-1}p_nS^{n+1}+\lambda S\xi P(S)-\lambda S\xi\sum_{n=0}^{k-1}p_nS^n-\lambda S\xi p_NS^N-\mu P(S)+\mu p_0+\alpha S\sum_{n=1}^{k}np_nS^{n-1}+$$

$$+k\alpha P(S)-k\alpha\sum_{n=0}^{k}p_nS^n$$

$$\Rightarrow P'(S)=\frac{\lambda}{\alpha}(1+\eta)\sum_{n=0}^{k-1}p_nS^n+\frac{\lambda}{\alpha}\xi P(S)-\frac{\lambda}{\alpha}\xi\sum_{n=0}^{k-1}p_nS^n-\frac{\lambda}{\alpha}\xi p_NS^N-\frac{\mu}{\alpha}\frac{P(S)}{S}+\frac{\mu}{\alpha}\frac{p_0}{S}+$$

$$+\sum_{n=1}^{k}np_nS^{n-1}+k\frac{P(S)}{S}-k\sum_{n=0}^{k}p_nS^{n-1}$$

Taking limit as S→1, we get

$$P'(1)=\frac{\lambda}{\alpha}(1+\eta)\sum_{n=0}^{k-1}p_n+\frac{\lambda}{\alpha}\xi-\frac{\lambda}{\alpha}\xi\sum_{n=0}^{k-1}p_n-\frac{\lambda}{\alpha}\xi p_N-\frac{\mu}{\alpha}+\frac{\mu}{\alpha}p_0+\sum_{n=1}^{k}np_n+k-k\sum_{n=0}^{k}p_n$$

$$\Rightarrow L=\frac{\lambda}{\alpha}\left[(1+\eta)\sum_{n=0}^{k-1}p_n+\xi\left(1-p_N-\sum_{n=0}^{k-1}p_n\right)\right]-\frac{\mu}{\alpha}(1-p_0)+\sum_{n=1}^{k}np_n+k\left(1-\sum_{n=0}^{k}p_n\right.$$

$$\Rightarrow L=\frac{\lambda}{\alpha}\left[(1+\eta)\sum_{n=0}^{k-1}p_n+\xi\left(\sum_{n=k}^{N}p_n-p_N\right)\right]-\frac{\mu}{\alpha}\sum_{n=1}^{N}p_n+\sum_{n=1}^{k}np_n+k\sum_{n=k+1}^{N}p_n$$