Vol. 12. Issue.4. 2024 (Oct-Dec)
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RESEARCH ARTICLE

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

INTERNATIONAL STANDARD SERIAL NUMBER 2348-0580

Euler's operator and Sun's binomial inversion formulae

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DOI:10.33329/bomsr.12.4.6

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ABSTRACT

In this paper, we establish the role of Euler's operator in the Sun's binomial inversion formulae and arrive at useful results.

Keywords: Stirling numbers, Euler's operator, Sun's binomial inversion, Bernoulli numbers, Grünert's formula.

1.- Introduction

For the binomial expression:

$$F(n) = \sum_{k=0}^{n} \binom{n}{k} f(k) - \lambda\, f(n), \qquad n \geq 0, \tag{1}$$

Sun [1-4] obtained the corresponding inversion formulae:

$$f(n) = \frac{1}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} F(k)\, B_{n+1-k}\,, \qquad \lambda = 1, \tag{2}$$

$$f(n) = \sum_{m=0}^{n} \binom{n}{m} F(m)\, Q_{n-m}\,, \qquad Q_n = \sum_{j=0}^{n} \frac{(-1)^j\, j!}{(1-\lambda)^{j+1}}\, S_n^{[j]}, \qquad \lambda \neq 1, \tag{3}$$

in terms of Bernoulli and Stirling numbers of the second kind [5-8], respectively.

The Sec. 2 considers the case $\lambda = -1$ and it is proven that the corresponding Q_k can be written in terms of Bernoulli numbers. The Sec. 3, for $\lambda \neq 1$, shows the connection between Q_m and the Euler's operator [9, 10] via the Grünert's operational formula [6, 10, 11].

2.- Q_n for $\lambda = -1$

From (1) and (3) with $\lambda = -1$:

$$F(n) = \sum_{k=0}^{n} \binom{n}{k} f(k) + f(n), \qquad Q_n = \sum_{j=0}^{n} \frac{(-1)^j\, j!}{2^{j+1}}\, S_n^{[j]}, \tag{4}$$

besides, we know the relation [6, 12, 13]:

$$B_{n+1} = \frac{n+1}{1-2^{n+1}} \sum_{j=0}^{n} \frac{(-1)^j\, j!}{2^{j+1}}\, S_n^{[j]}, \tag{5}$$

therefore:

$$Q_n = \frac{1-2^{n+1}}{n+1}\, B_{n+1}, \qquad Q_{2m} = 0, \qquad m \geq 1, \tag{6}$$

$$f(n) = \sum_{j=0}^{n} \binom{n}{j} \frac{1-2^{n+1-j}}{n+1-j}\, F(j)\, B_{n+1-j},$$

which gives the following Z-transform [4]:

$$\mathcal{Z}\left\{ \frac{1-2^{n+1}}{(n+1)!}\, B_{n+1} \right\} = (\, e^{1/z} + 1)^{-1}. \tag{7}$$

3.- Q_n in terms of the Euler operator for $\lambda \neq 1$

Grünert's formula [6, 10, 11]:

$$(x \frac{d}{dx})^n\, g(x) = \sum_{j=0}^{n} x^j\, S_n^{[j]}\, g^{(j)}(x), \tag{8}$$

for the Euler operator $x \frac{d}{dx}$ [9, 10], can be applied to the function:

$$g(x) = \frac{1}{x-\lambda} \qquad \therefore \qquad g^{(j)}(1) = \frac{(-1)^j\, j!}{(1-\lambda)^{j+1}}, \tag{9}$$

then (3), (8) and (9) imply the interesting relationship:

$$Q_n = \sum_{j=0}^{n} S_n^{[j]}\, g^{(j)}(1) = [\, (x \frac{d}{dx})^n \frac{1}{x-\lambda}\,]_{x=1}, \tag{10}$$

hence:

$$Q_0 = \frac{1}{1-\lambda}, \qquad Q_1 = -\frac{1}{(1-\lambda)^2}, \qquad Q_2 = \frac{1+\lambda}{(1-\lambda)^3}, \qquad Q_3 = -\frac{\lambda^2+4\lambda+1}{(1-\lambda)^4}, \ldots \tag{11}$$

Conclusion

As conclusion of this paper, we make the following remarks which would summarize the purpose of whole paper.

Remark 1.- The inversion of (3) is given by:

$$\frac{(-1)^n\, n!}{(1-\lambda)^{n+1}} = \sum_{j=0}^{n} Q_j\, S_n^{(j)}, \tag{12}$$

in terms of Stirling numbers of the first kind [5, 6].

Remark 2.- The inversion of (5) generates the Shirai-Sato identity [13, 14]:

$$\sum_{j=0}^{n} \frac{1-2^{j+1}}{j+1} B_{j+1} S_n^{(j)} = \frac{(-1)^n \, n!}{2^{n+1}}, \qquad n \geq 0. \tag{13}$$

Remark 3.- From (6) and (10):

$$B_{n+1} = \frac{n+1}{1-2^{n+1}} \left[(x\frac{d}{dx})^n \frac{1}{x+1} \right]_{x=1}. \tag{14}$$

Remark 4.- The expression (1) is a binomial transform [15] if $\lambda = 0$, then (10) implies that $Q_n = (-1)^n$ and thus (3) gives the well-known inversion property [6]:

$$F(n) = \sum_{k=0}^{n} \binom{n}{k} f(k) \qquad \Leftrightarrow \qquad f(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} F(k). \tag{15}$$

In [4] the Sun's inversion formulae [1] were studied using the Z-transform, and here the corresponding analysis has been carried out employing the Euler's operator [9, 10].

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