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RESEARCH ARTICLE

APPLICATION OF ALGEBRAIC STRUCTURE GROUP THEORY TO INDIAN MUSICAL (*Carnatic* as well as *Hindustani*)-NOTES

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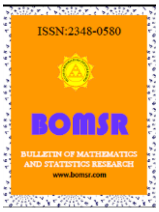

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ABSTRACT

This research paper explains the behavior and relationship that exist between Indian musical notes and group theory. The Indian musical notes construct abelian additive group *modulo*(12).We have also discussed upon some propositions as well as theorems one of important theorem is Dido's Theorem. It is seen that the musical notes behaviors satisfied all group axioms and are connected to group theory.
Keywords: Group theory, abelian group *modulo*(12), Indian musical notes and inverse, Dido's theorem, transposition, invertion, musical clock, Sylow's theorem, cyclic group.

1. INTRODUCTION

Music theory can be explained easily with the help of algebraic structure group. Musical notes are very important in the composition of musical sound. We can't isolate musical notes from musical sound. As music itself is not complete without musical notes. We have Indian music(*Carnatic*) notes as in set { Ri1, Ri2, Ga2, Ga3, Ma1, Ma2, Pa, Da1, Da2, Ni2, Ni3,Sa} and (*Hindustani*) notes as {Re , Re, Ga, Ga, Ma, Ma, Pa, Dha, Dha, Ni, Ni, Sa} . When we logically combine these notes, we hear pleasant sound. The first note Sa is termed as rood note. Thoas M. Flore reffered Western notes{ C, $C^{\#}$, D, $D^{\#}$, E, F, $F^{\#}$, G,$G^{\#}$,A,$A^{\#}$, B} as the $\mathbf{Z}_{12}$ model of pitch class. Similarly we have made the *Carnatic* as well as *Hindustani* music notes {Ri1, Ri2, Ga2, Ga3, Ma1, Ma2, Pa, Da1, Da2, Ni2, Ni3, Sa} and { Re, Re, Ga, Ga, Ma, Ma, Pa, Dha, Dha, Ni, Ni, Sa} as the Z_{12} model of pitch class. He made a musical clock. He suggested, there is a bijective mapping between the set of pitch class and Z_{12} .

2. Music: We can work in field of music with the help of mathematical science. It is a combination of notes. This combination may be done with the help of Group theory. Here we regard musical notes as number. There is Pythagorean School (ca. 580-500 B.C.) whose motto is "*All is number*." In this school pupil used to learn mathematics, philosophy, astronomy, and music. Thus it is clear that, they linked music to number theory, as well as group theory for the development of new researches in the area of music theory.

3. Group Theory:

There is a non-empty set G under binary operation ($\bullet$) as (G, $\bullet$) which satisfied the law of closure, associative, identity, inverse then this algebraic structure (G, $\bullet$) is termed as group. But when this algebraic structure also holds good the commutative law, then it is known as abelian group**.**

4. Definitions with related topics:

We will present here some definitions that will be helpful to us to be familiar with the concepts in music and abstract algebra.

Indian musical notes: Indian musical notes are the following notes,When these notes are combined logically, give out pleasant and sweet sound to the ear. The first note Ri1(Re) is called root note.

Ri2(Re) is called 2nd note

Ga2(Ga) is called 3rd note

Ga3(Ga) is called 4th note

Ma1(Ma) is called 5th note

Ma2(Ma) is called 6th note

Pa is called 7th note

Da1(Dha) is called 8th note

Da2(Dha) is called 9th note

Ni2(Ni) is called 10th note

Ni3(Ni) is called 11th note

Sa is called 12th note

Indian musical Flat b: We can define musical flats as movement of sound from one pitch to the one lower, and it is denoted by b. For example, the movement from Da1(Dha) to any other note to the left.

Musical Sharp: In Indian musical notes Ri2(Re), Ga3(Ga), Ma2(Ma), Da2(Dha), Ni3(Ni) are sharp notes. For example the movement from Da1(Dha) to any other note to the right on the musical notes.

Tone: This means any movement from a musical note to the next note two steps forward or backward on the musical notes. For example, movement from Pa to Da2(Dha) or Pa to Ma1(Ma).

Semitone: This is defined as any movement from a musical note to next note a step forward or backward on the musical notes (scale). For example, the movement from Pa to Da1(Dha) or Pa to Ma2(Ma).

Chord: When two, three, or more notes are sounded together is called a chord. **Transposition**: When playing or writing a given melody at a different pitch higher or lower other than the original is termed as transposition.

5. Group Algebra over addition operation:

Let G be a non empty set with operation (+) as we have taken in this paper, then set (G, +) is called group, when this set satisfies the following axioms:

$G1$ $\forall\, x, y \in G,\ \ x+y \in G$
Closure
$G2$ $\forall x, y \in G\ \ (x+y)+z \in G$
Associative
$G3$ $\forall x, y \in G, \exists\, e \in G\ x+e = e+x = x$ Identity
$G4$ $\forall\, x \in G\ \exists\, x^{-1} \in G\ \ x+x^{-1} = x^{-1}+x$ Inverse
When group is said to be an abelian group then,
$G5$ $\forall\, x, y \in G,\ \ x+y = y+x \in G$ Commutative

Additive Group of the residue class modulo(m): It is also known as a group of additive class of integers modulo m.

P-Group: A finite group is said to be a p-group if its order is power of p.

P-Subgroup: Let us suppose that G be a group. If there be a subgroup $T \leq G$ and $|T| = p^r$ for some $r \geq 0$ then T is called p-subgroup of G.

6. Literary and theoretical understanding:

Several applications of group theory have been observed to many fields such as sciences, games. But very few applications of group theory have been seen in the field of music. Pythagoras used to say that "all is number". In the same way musical notes/numbers are not exceptional, that is Ri1(<u>Re</u>), Ri2(Re), Ga2(<u>Ga</u>), Ga3, Ma1(<u>Ma</u>), Ma2(Ma), Pa, Da1(<u>Dha</u>), Da2(Dha), Ni2(<u>Ni</u>), Ni3(Ni), Sa. The Pythagoreans related certain meanings and characters to numbers. They considered odd numbers as male and even numbers as females. Pythagoreans had said the number as the following manner. One as the number of reason, two as the number of opinion, three as the number of harmony, four as the number of justice, five as the number of marriage, six as the number of creation, seven as the number of awe, and ten is the number of the universe. The reason of these numeral relation which were given by Pythagorean was supposed in many ways. The first one was that Pythagoras might have travelled Egypt and Babylon. Because the number and mystical relations were very common in these two regions. While the second one was due to the contemporary belief in Greek related to these things. At that time, it was believed that earth, air, fire, and water are the four basics principles of things. Here we have studied the Carnatic music as well as Hindustani music notes in this research paper. Carnatic music system is associated with south Indian music system. This musical notes are used in composing music commonly in Karnataka, Andhra Pradesh, Telangana, Kerala, Tamil Nadu, as well as in Sri Lanka. But Hindustani music notes are taken from Persion or Islamic influences from Northern India. Pythagoras had found that the music notes produced from a vibrating string of some length could be characterized by numbers. When we divide a vibrating string by some movable object into two different lengths then it produces different types of musical notes. These notes are then explained as the ratios (numbers) of the lengths of the parts of the vibrating string. It is said that early Indian tried to to show the correlation between the mathematical laws of harmonics as well as rhythms and its relationship towards human well-being. Both music, vocal and instrumental are primarily made up of beats. Musical beats are pulses in which time is marked. This is then played as a series of notes in accordance to a pattern. Musical notes can be combined in an endless variety of grouprings, but the specific number of notes that exist are finite. In mathematics, the result always remains finite despite the various ways in which we can add, multiply, subtract, and divide the numbers. In 1993 M. Flore had construct a musical clock on the notes of western music and Z_{12} model of pitch class. But, Now we suppose Ri1(Ri), Ri2(Ri), Ga2(Ga), Ga3(Ga), Ma1(Ma), Ma2(Ma), Pa, Da1(Dha), Da2(Dha), Ni2(Ni), Ni3(Ni), Sa as $\boldsymbol{Z_{12}}$ model of pitch class on *Carnatic/Hindustani* music and construct a musical clock as below:

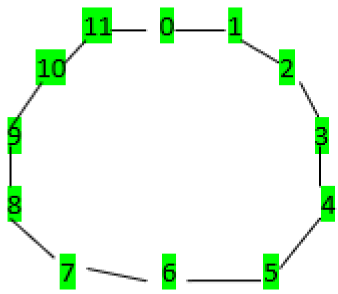

Figure 1

Musial Clock

We have a bijective mapping between the set of pitch classes and $\boldsymbol{Z_{12}}$. So we have defined mapping as, $\boldsymbol{T_n : Z_{12} \to Z_{12}}$ that is $\boldsymbol{T_n (X)}$: $\boldsymbol{\to X+n}$ as well as the inversion mapping as $\boldsymbol{I_n : Z_{12} \to Z_{12}}$ that will be $\boldsymbol{I_n(X): \to -x+n}$ here n is in *mod* 12. We have constructed possibly musical notes with corresponding integers as,

Ri1(Re)	Ri2(Re)	Ga2(Ga)	Ga3(Ga)	Ma1(Ma)	Ma2(Ma)	Pa	Da1(Dha)	Da2(Dha)	Ni2Ni)	Ni3(Ni)	Sa
0	1	2	3	4	5	6	7	8	9	10	11

Here the transposition mapping T_n as that which moves a pitch-class or pitch-class set up by n (mod12) as well as the inversion mapping has been defined as T_nI as the pitch(Ni2/Ni) about Sa(0) and then transposes it by n, that is $T_nI(a) = -a+n(mod12)$. Adam, (2011), had defined the transposition and invertion as in following ways, T_n:$Z_{12} \to Z_{12}$ this means, $T_n(X)$:$x+n$ mod12 and I_n (X):-$x+n$ mod12. Alisa(2009). Asserted that the musical actions of of dihedral group, here he considered two ways in which the dihedral groups act on the set of major and minor traids.Emma,(2011), reffered the musical notes with corresponding integers. Ada Zhang, (2009) told that the Mathieu group M_{12} can be generated by only just two permutations groups. These can be expressed as both two line notation and cyclic notation by generating permutations P_1 and P_0.

$$P_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 5 & 6 & 4 & 4 & 7 & 8 & 2 & 9 & 1 & 10 & 0 & 11 \end{pmatrix} = (0\ 5\ 8\ 1\ 6\ 2\ 4\ 3\ 7\ 9\ 10)(11)$$

$P_0 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 6 & 5 & 7 & 4 & 8 & 3 & 9 & 2 & 10 & 1 & 11 & 0 \end{bmatrix} = (0\ 6\ 9\ 1\ 5\ 3\ 4\ 8\ 10\ 11)\ (2\ 7)$

7. Methodology:

7.(a) Lagngrange's Theorem: If G is a finite group, and H is a its subgroup, then $o(H)\,|\,o(G)$.

7.(b) Every subgroup of a cyclic group is cyclic.

7.(c) Let us suppose that H be a subgroup G, and $H \le G$, then (i) $g \in gH$ (ii) two left cosets of H in G are either identical or disjoint. (iii) the number of elements in gH will be $|H|$.

7.(d) Sylow's first theorem: Let us suppose that G be a finite group and $|G|=p^n m$, where $n\ge 1$ as well as p does not divide m then we have, (i) G contains a subgroup of order p^i for each i here, $1\le i\le n$ (ii) Every subgroup H of G of order p^i is normal subgroup of a subgroup of order p^{i+1} for $1\le i\le n$.

8.RESULTS AND DISCUSSION:

We number the musical notes as,

Ri1(Re)	Ri2(Re)	Ga2(Ga)	Ga3(Ga)	Ma1(Ma)	Ma2(Ma)	Pa	Da1(Dha)	Da2(Dha)	Ni2(Ni)	Ni3(Ni)	Sa
0	1	2	3	4	5	6	7	8	9	10	11

It is clearly showing that the musical notes form a group of integers of *modulo*12 under addition operation. This means Z_{12}= {Ri1(Re), Ri2(Re), Ga2(Ga), Ga3(Ga), Ma1(Ma), Ma2(Ma), Pa, Da1(Dha), Da2(Dha), Ni2(Ni), Ni3(Ni), Sa}.

Indian musical notes and its relation with group theory,

(i) Since Ma2(Ma) and Pa are elements of Z_{12}, so Ma2(Ma)+Pa=Sa $\in Z_{12}$ (Closure law)

(ii) Since Ma2(Ma), Pa, Da1(Da) $\in Z_{12}$,so (Ma2(Ma)+Pa)+Da1(Da) = Ma2(Ma)+(Pa+Da1(Dha))

(Ma2(Ma)+Pa)+Da1(Dha) = Sa+Da1(Dha)=Pa

Ma2(Ma)+(Pa+Da1(Dha)) = Ma2(Ma)+Ri2(Re) =Pa (Associative law)

(iii) Since Pa$\in Z_{12}$ $\exists$ Ri1(Re) $\in Z_{12}$ hence Pa+Ri1(Re) = Ri1(Re)+Pa = Pa (Identity)

(iv) Since Ma1(Ma) $\in Z_{12}$ $\exists$ Da2(Dha) such that Ma1(Ma)+Da2(Dha) = Ri1(Re) $\in Z_{12}$ (Inverse)

(v) Since Ga2(Ga),Ma1(Ma)$\in Z_{12}$ Ga2(Ga)+Ma1(Ma) = Ma1(Ma)+Ga2(Ga) = Pa (Commtative)

Thus, we have seen that Z_{12} is an abelian group and Ri1(Re) (the root note) as the identity element or identity note.

Table 1: List of Musical Notes and their inverse

Notes	Inverse
Sa	Sa
Ri1 (Re)	Ni3 (Ni)
Ri2 (Re)	Ni2 (Ni)
Ga2 (Ga)	Da2 (Dha)
Ga3 (Ga)	Da1 (Dha)
Ma1 (Ma)	Pa
Ma2 (Ma)	Ma2 (Ma)

Thus, we will relate the musical note to Table 1 and formulate a proposition, which we call **Dido's theorem**.

8(a). Proposition: If G is a cyclic group, then there is at least an element which is unique with its inverse.

Proof: Let us suppose that G be a cyclic group. $\forall x \in G$, so each element of G that is x can be written in the form $x = g^{m}$ for some $g \in G$ here m is element of Z, and g is generator of group G. We have some $y \in G$ such that $x \bullet y = e \in G \Rightarrow x = y$ here e is identity element $y = x^{-1} \Rightarrow x = x^{-1}$.

Now, we will use the result of theorem 7.(c) in Indian musical notes.

Z_{12} = {Ri1(<u>Re</u>), Ri2(Re), Ga2(<u>Ga</u>), Ga3(Ga), Ma1(<u>Ma</u>), Ma2(Ma), Pa, Da1(<u>Dha</u>), Da2(Dha), Ni2(<u>Ne</u>), Ni3(Ne), Sa}. H = {Ri1(<u>Re</u>), Ri2(Re), Sa} $\Rightarrow H \leq Z_{12}$. Now we can write, Ga2(<u>Ga</u>)H = {Ga2(<u>Ga</u>)•Ri1(<u>Re</u>), Ga2(<u>Ga</u>)•Ri2(Re), Ga2(Ga)•Sa}

Further we have Ga2(<u>Ga</u>)H = {Ga2(<u>Ga</u>), Ga3(Ga), Ri2(Re)}. Thus it is clear that Ga2(Ga) $\in$ Ga2(<u>Ga</u>)H and also $|H|$ = |Ga2(<u>Ga</u>)H| = 3. Furthermore, we have for some Ni2(<u>Ni</u>), Ma2(Ma) $\in Z_{12}$ there will be (Ni2(<u>Ni</u>)•Ma2(Ma))H = Ga2(<u>Ga</u>)H $\Rightarrow$ we have observed that two left cosets are identical in this case for some Ni2(<u>Ni</u>), Da1(<u>Dha</u>) $\in Z_{12}$. Here we have found that Ni2(<u>Ni</u>)•Da1(<u>Dha</u>) $\in Z_{12}$ but we have Ni2(<u>Ni</u>)•Da1(<u>Dha</u>) = Ma1(<u>Ma</u>) $\neq$ Ga2(<u>Ga</u>). Here we get that two left cosets are disjoint Ma1(<u>Ma</u>)H $\neq$ Ga2(<u>Ga</u>)H.

Now we will use the result of theorem 7(a). in Indian musical notes.

Since we have, $|Z_{12}| = 12$ and $|H| = 3 \Rightarrow |Z_{12}|/|H| = 12/3 = 4$ thus it is clear that the order of a subgroup divides the order of a group.

Now we will use the theorem 7(b). in Indian musical notes.

As we have Z_{12} = {Ri1(<u>Re</u>), Ri2(Re), Ga2(<u>Ga</u>), Ga3(Ga), Ma1(<u>Ma</u>), Ma2(Ma), Pa, Da1(<u>Dha</u>), Da2(Dha), Ni2(Ni), Ni3(Ni), Sa}, and for Ri1(Ni) $\in Z_{12}$.

Ri1^{0} =Ri1(Re) Ri1^{1} = Ri2(Re) Ri1^{3} = Ga3(Ga)

Ri1^{2} = Ga2(<u>Ga</u>) Ri1^{5} = Ma2(Ma)

Ri1^{4} = Ma1(<u>Ma</u>) Ri1^{7} = Da1(<u>Dha</u>)

$Ri1^6$ = Pa $Ri1^9$ = Ni2(Ni)

$Ri1^8$ = Da2(Dha) $Ri1^{11}$ = Sa

Ri^{10} = Ni3(Ni)

$Ri1^{12}$ = Ri1(Re)

Thus we seen that Musical notes are cyclic. This can be represented as $Z_{12} = (C)$. Let us consider a subgroup H = {Ri1(Re), Ri2(Re), Sa} and Sa $\in H \leq Z_{12}$, so we have

$Ri1^0$ = Ri1(Re)

$Ri1^1$ = Ri2(Re)

$Ri1^2$ = Ga2(Ga)

It is clear that H is cyclic. We have also observed that by using Indian musical notes we are satisfied with the theorem which states that **every cyclic group has a subgroup which is also cyclic**.

Proposition: Every Indian musical note is a generator of Z_{12}.

Proof: $Ri1^0$ = Ri1(Re) $Ri1^1$ = Ri2(Re)

$Ri1^2$ = Ga2(Ga) $Ri1^3$ = Ga3(Ga)

$Ri1^4$ = Ma1(Ma) $Ri1^5$ = Ma2(Ma)

$Ri1^6$ = Pa $Ri1^7$ = Da1(Dha)

$Ri1^8$ = Da2(Dha) $Ri1^9$ = Ni2(Ni)

$Ri1^{10}$ = Ni3(Ni) $Ri1^{11}$ = Sa

$Ri1^{12}$ = Ri1(Re)

Thus we have found that a musical note Ri1 has generated the Z_{12}. Similarly, every other note can be considered as generator of Z_{12}.

Since, $|Z_{12}| = 12$, as $12 = 2\times2\times3 = 2^2\times3$, there exist a $H \leq Z_{12}$ such that $|H| = 2^2$. It is a Sylow-2subgroup of Z_{12}. Hence it is clear that every finite group has a Sylow-P subgroup which is known as Sylow's first theorem.

9. Group theory as a structure of atonal music theory:

The numbering of the pitch classes relates their isomorphism to $\boldsymbol{Z_{12}}$**.** We easily found that the group of transposition and inversions, which is denoted by T_n/T_nI is isomorphic to the dihedral group D_{12} **.**

10. Conclusions:

We have found in this paper that the Indian musical notes behaviours satisfied all group axioms and are related to group theory. Since, a fine composition of music affects the soul and mind of a human positively, so it is our suggestion that a deep understanding of group theory to a musician can help in composing the best musical composition and that will give satisfaction to Audience as well as the well beings to their minds.

References:

[1]. Zhang, A. (2009). *The framework of music theory as represented with Groups 9-11*.

[2]. Crans, A. S. (2009). *Musical Actions of Dihedral groups*, 479-482.

[3]. Rose, E. (2011). *Mathematics and Music*, 5-6.

[4]. Fiore, T. M. (1993). *Music and Mathematics*, 7–15. fioret@umich.edu.

[5]. Health, T. (1965). *A History of Greek Mathematics, Vol. 1*. Oxford University Press, Oxford.

[6]. Wussing, H. (2007). *The Genesis of the Abstract Group Concept: A Contribution to the History of the Origin of Abstract Group Theory*. Dover Publications. http://ucl.Wikipedia.org/wiki/Group(Mathematics)

[7]. Townsend, A. (2011). *Mathematics and Music Theory: A Paper Presentation of Undergraduate Mathematics Colloquium at the University College London*. http://ucl.sneffel.com, 5-8.

[8]. Tsok, S. H., & Mshelia, I. B. (2013). *Application of Group Theory to a Local Game Called "Tsorry Checkerboard" (A Case of Klein Four-Group)*. International Organization of Scientific Research Journal of Mathematics (IOSR-JM), 7(3), 04-06. www.iosrjournals.org

[9]. Wright, D. (2009). *Mathematics and Music*, Mathematical World, 28, American Mathematical Society.

[10]. Mao, L. (2011). *Automorphism Group of Maps, Surfaces, and Smarandache Geometries (Second Edition)*. The Education Publisher Inc., USA.