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The Generalization of Divisible Subgroup

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Abstract
In this paper, we introduced a new concept, which is a semi-divisible subgroup, which is considered a generalization of the divisible subgroup. This concept will be adopted as a principal condition for proving the Fuchs problem to solve the Q1: intersection between divisible subgroups, in addition to presenting many of important theorems specific to this concept.

Keyword: divisible, subgroup, direct sum, direct product.

1.Introduction

The subgroups L of abelian group G is called semi divisible in G if $\forall\, x \in G\backslash L$, there exist a divisible subgroup D of the group G and contains L and x not belong to D.

This concept is generalization of divisible group G (G is called divisible if for every x ∈ G and every positive integer n, there is y ∈ G so that ny = x).

This concept semi-divisible will be adopted as a principal condition for proving the Fuchs problem Q1:Is the intersection of family of divisible subgroup is also divisible subgroup? in addition to presenting many of important theorems specific to this concept. From previous studies, we found that both researchers Samir and Charles presented a condition to solve the problem above and we presented a theorem that shows the equivalence of the two conditions of the previous researchers. In addition to the above, we presented a very principal and important theorem that links the conditions of the previous researchers with our new condition (semi-divisible). For a group G and an integer n > 0, let nG= {na | a∈

G} and G[n] = {a | a∈ G, na =0}. Thus g∈nG if and only if the equation nx = g has a solution x in G and g ∈ G[n] if and only if o(g) | n. Clearly, nG and G[n] are subgroups of G.

Definition 1.1[1]:

A group G is called divisible groups if n I a for all a ∈ G and all positive integers n. Thus G is divisible if and only if nG = G for every positive n.

Example 1.2:

The groups 0, Q/Z and Z(p^∞) are divisible groups.

We introduce the new concept semi-divisible as follows

Definition 1.3:

The subgroups L of abelian group G is called semi divisible in G if $\forall\, x \in G\setminus L$, there exist a divisible subgroup D of the group G and contains L and x not belong to D .

i.e. ∀ x , x ∈G\L ,∃ a divisible subgroup D such that L ⊆ D and x ∉ D.

Example 1.4 on semi divisible: let G=Q and L=Z

we claim that L = Z is semi-divisible in G = Q, let $x \in Q\setminus Z\ say\ x = \frac{a}{b}\ with\ b > 1$.

Now define $D = Z\left[\frac{1}{p}\right]$ for some prime p that does not divide b, this is a divisible subgroup containing Z because it includes element fractions with denominators powers of p, since $p \dagger b, x \notin D$, so D is divisible contains Z. Therefore Z is semi divisible in G.

Example 1.5: Let $G = Q\ and\ L = 0$.

Let $L = \{0\}$, for any $0 \neq x \in Q$, we can find a divisible subgroup $D \subseteq Q$ such that $0 \in D$

And $x \notin D$. Now we take a proper divisible subgroup of Q such as $D = Z[\frac{1}{p}] \subseteq Q$.

let $x = \frac{1}{3}$ and let $D = Z[\frac{1}{2}]$, so $x = \frac{1}{3} \notin D$ because 3 is not a power of 2 and D is divisible and contain 0. So {0} is semi-divisible in Q.

Remark 1.6:

It is clear that each divisible subgroup is semi divisible in a group G .

Theorem 1.7:

Let $\{G_i\}$ be a family of abelian groups and for each i, let L_i be a semi-divisible subgroups of G_i.Then the product subgroup $L = \prod L_i$, is semi-divisible in the product group $G = \prod G_i$

Proof:

$Let\ x = (x_i) \in G\setminus L$. Then there exists some index j such that $x_j \notin L_i$. Since L_i is semi-divisible inG_i, there exists a divisible subgroup $D_i \leq G_i$ such that: $L_i \subseteq D_i$ and $x_i \notin D_i$.

Now define a subgroup D of G as follows: $D = \prod D_i$, where

$$D_i = \begin{cases} a\ divisible\ subgroup\ of\ G_i\ containing\ L_i\ and\ not\ containing\ x_i if\ x_i \notin L_i \\ G_i\ \ otherwise \end{cases}$$

Since each D_i, is divisible and the product of divisible abelian groups is divisible, D is a divisible subgroup of G.

Furthermore: $L = \prod L_i \subseteq \prod D_i = D$ and $x = (x_i) \notin D$, because for at least one index $j, x_j \notin D_i$. Thus, for every $x \in G \setminus L$, there exists a divisible subgroup D of G such that $L \subseteq D$ and $x \notin D$.Therefore, L is semi-divisible in G.

Theorem 1.8:

Every quotient of a semi-divisible in abelian group is semi-divisible.

Proof:

Let G be an abelian group and let L be a semi-divisible subgroup of G. Let N be a normal subgroup of G. (Since G is abelian, every subgroup is normal.) We consider the quotient group G/N, and we aim to show that the image of L in G/N, namely L/N, is semi- divisible in G/N. Take any element $x + N$ in G/N that does not belong to L/N. This means that x is in G but not in L, since if x were in L, then x + N would belong to L/N. Since L is semi-divisible in G, there exists a divisible subgroup D of G such that: L is $contained$ in D $(i.e., L \subseteq D)$ and x is not in D.

Now, consider the natural projection map $\prod$ from G to G / N, defined by $\prod(g) = g + N$, $define$ $D' = \prod(D)$, which is a subgroup of G/N.

We claim that: 1. L/N is contained in D', since L is contained in D

2. $x + N$ is not in D' because x is not in D.

Next, we show that D' is a divisible subgroup of G/N. Let y + N be any element in D' and let n be any nonzero integer. Since y + N is in D', there exists y in D such that $\prod(y) = y + N$, because D is divisible, there exists z in D such that $n * z = y$

Then: $n^* (z + N) = n^* z + N = y + N$.

So z +N is in D' and it satisfies $n^* (z + N) = y + N$. This proves that D' is divisible.

We have now found a divisible subgroup D' of G/N that contains L/N but does not contain x + N. Therefore L/N is semi-divisible in G/N and so the quotient of a semi-divisible group is also semi- divisible.

Theorem 1.9:

Every direct sum of semi-divisible abelian groups is semi-divisible.

Proof :

Let $\{G_i\}_{i \in I}$ of abelian groups . For each $i \in I$, let L_i be a semi-divisible subgroup of G_i.

Define the group $G = \oplus_{i \in I} G_i$ and define the subgroup L as the direct sum of all L_i.

We want to prove that L is semi- divisible in G.

1. Let $x = (x_i) \in G \backslash L$ $x \notin L = \oplus L_i$ there must at least one index $i \in I$ such that $x_j, x_j \notin L_j$.

Let sup(x) be the support of x and $supp(x) = \{i \in I, x_i \neq 0\}$, this is a finite set because $x \in \oplus G_i$.

2. Since $x_j \notin L_i$ and L_i is semi – divisible in G_j, there exist a divisible subgroup

$\triangle_i \subseteq G_j$such that $L_j \subseteq D_j$ and $x_j \notin D_j$.

Now we define a subgroup D of G $\oplus G_i$ as fallows :

Let $\triangle_i = \begin{cases} \triangle_i & if\ i = j \\ L_i & if\ i \neq j \end{cases}$

Then define :$D = \oplus_{i\in I} D_i$.

We see that each that $D_j \subseteq G_i$, $D \subseteq G$ because it has finite support and $L_i \subseteq D_j \forall i, so\ L = \oplus L_i \subseteq \oplus \triangle_i = D$ and $x_i \notin \triangle_j \Longrightarrow x \notin D$

Now to show that D is divisible: Each $\triangle_j$is divisible by construction and Fore i not equal to j, $D_i = L_i$ anb L_i may not be divisible , it i's semi – divisible and can by embedded in a divisible group .But here we only need to show that the overall sum is divisible group ,D is itself divisible.

Therefore $L = \oplus_{i\in I} L_i$ is a semi- divisible subgroup of $G = \oplus G_i$.

Remark 1.10:

The intersection of any family of semi-divisible subgroups of an abelian group is not necessarily semi-divisible in general.

Example 1.11:

Let G=Q, the additive group of rational numbers. This group is divisible and hence semi-divisible.

Let L_1 and L_2 be two subgroups of Q define: $L_1 = Z$ be set of all the integers and $L_2 =$ (1/2)Z, be the set of all national numbers of the form n/2, where $n \in z$.

Both L_1 and L_2 are semi- divisible in Q.

For example, given any x not in L_1, we can take D=Q, which is divisible, and it contains L_1 but not x.

Similarly for L_2

Now the intersection is: $L_1 \cap L_2 = Z \cap \left(\frac{1}{2}\right) Z = Z$

So the intersection is Z again. But Z in not semi-divisible in Q.

For example, 1/2 is in Q but not in Z, and there is no divisible subgroup of Q that contains Z and excludes ½.

$Therefore$ the intersection of semi-divisible subgroups may fail to be semi-divisible.

Remark 1.12:

The union of semi- divisible subgroups is not necessarily be semi-divisible in general.

Example 1.13:

Let G=Q and the following chain of subgroups:

$$l \in t\ \ L_n = \frac{1}{n} Z \subseteq Q\ for\ n \in N$$

Each L_n is semi -divisible in Q, because for any $x \in Q \setminus L_n$ one can always find a divisible

Subgroup (such as Q itself) containing L_n but not containing x.

Now the union : $L = \bigcup_{n=1}^{\infty} L_n = \{\frac{a}{b}\ Q |\ \gcd(a, b) = 1, b \in N\}$ $L = \cup\ (n \geq 1)\ L(n)$.

This is the set of rational numbers with finite denominators that is, the set of all rational numbers a/b where $a \in Z$ and $b \in N$.

This set L is known as Q fin clear $L \neq Q$ and is not divisible. To show that L is not semi-divisible in Q .

Let $x = \sqrt{2} \notin Q$, Take x=1/p where p is a prime number not dividing any denominators appearing in a finite part of L. Any divisible subgroup of Q that contains all of L mustcontain all of Q, because for every $b \in N, 1/b \in L$ and the divisible closure of such a set is Q. So, there is no divisible group D such that $L \subseteq D$ and $x \notin D$ .That contradicts semi-divisibility.

More if $L = \cup L_n$, then any divisible $D \subseteq Q$ that contains L will necessarily contain element of Q. So there is no divisible group strictly contain L but containing x.

Therefore The union of semi-divisible subgroups may fail to be semi-divisible

Theorem 1.14 :

Every epimorphic image of a semi - divisible group is semi- divisible .

Proof:

Let G be a semi –divisible abelian group and let $f: G \rightarrow H$ be a epimorphic group homormphism. Let $L \subseteq G$ be a semi-divisible subgroup . Now define $L' = f(l) \subseteq H$.

We want to show that L' semi-divisible in H.

Let $x' \in H \setminus L'$, since f is surjective, there exists $x \in G$ such $that\ f(x) = x'$. Then $f(x) = x'$, $so\ \ x \notin L'$ (otherwise $x' = f(x) \in x' = f(x) \in f(L) = L'$, contradiction)

Since L is semi – divisible in G $\exists$ a divisible subgroup $D \subseteq G$ such that $L \subseteq D, x \in D$.

Let $D' = f(D) \subseteq H$. Then $L' = f(L) \subseteq f(D) = D'$ and $x = f(x) \notin D'$, because homomorphic images of divisible abelian groups are divisible.

Therefore L' is semi-divisible in H.

Theorem 1.15:

Every group can be embedded as a semi- divisible subgroup in abelian group

Proof:

Let G be abelian group. we can embed G in to or a divisible group using the injective or divisible hull D(G), which is a minimal divisible group containing G.

Now, G is trivially a subgroup of D(G). For any $x \in D(G)\Gamma$. We can choose a divisible subgroup $D' = G + Zx \subseteq D(G)$ that contains G but not x, ensuring semi-divisibility .

So G can be embedded as a semi-divisible subgroup of D(G).

Theorem 1.16:

Direct limits of semi-divisible groups are semi – divisible.

Proof:

Let $\{G_i\}$ be a directed system of abelian group, each with a semi- divisible subgroup $L_i \subseteq G_i$. Let $G = \varinjlim G_i$ be the direct limit and let $L = \varinjlim L_i \subseteq G_i$, we want to show that L is semi-divisible in G .

Let $x \in G\backslash L$. Then x comes from some $x_i \in G$ such that it is image in G is not in the image of L_i. Since $L_i \subseteq G_i$ is semi-divisible , there exists a divisible group $D_i \subseteq G_i$ such that $L_i \subseteq D_i$ and $x_i \notin D_i$. Then the image of D_i in G is a divisible subgroup containing the image of L_i but not the image of x_i. So semi-dinsibility is preserved in the direct limit.

We will review khabbazs answer to Q_1 according to the following theorem:

Theorem 1.17[5]:

Let G be a divisible group and let L be a subgroup of G and M is a minimal divisible subgroup of G containing L, G can be represented of the following formula: G= M ⊕ E , then L=M ∩ ($\cap_{w\in I}$ M_w) if and only if there exists a homomorphism h_w : M → E for each w∈ τ such that L= $\cap_{w\in I}$ ker h_w , where ($\cap_{w\in I}$ M_w) represents the intersection of a family of divisible subgroups of G and contain L.

Also we will review Charles answer to Q_1 according to the following theorem:

Theorem 1.18[2]:

Let G be a divisible group and L be a subgroup of G and $\{ Gi \}_{i\in\Lambda}$ is the family of all subgroups of G that are divisible and containing L , then $L = \bigcap_{i\in\Lambda} G^i \Leftrightarrow$ $for\ each\ prim\ number\ p$, either

$$L[p] \neq G[\,p] \qquad or\ L\ is\ divisible\ on\ p$$

Through the semi-divisible, we answered Q_1 by setting a new condition that differs from the condition Khabbaz and charles and it is assumed that G is a commutative group instead of a divisible group as in the following theorem:

Theorem 1.19 :

Let G be abelain group and L be a subgroups of G and let $\{ G_i\}$ $i \in I$ be a family of divisible subgroups of G and contain L, then L is semi-divisible subgroup of G $if\ and\ only\ if$ $L = \bigcap_{i\in I} G_I$.

Proof :

Let L be a semi-divisible subgroups of G. To prove that L =$\bigcap_{i\in I} G_i$.

$It\ is\ clear\ that$ $L \subseteq \bigcap_{i\in I} G_i$, we need to prove that $\bigcap_{i\in I} G_i \subseteq L$.

Let $x \in \bigcap_{i\in I} G_i$, $then\ x \in G_i$, $\forall\ i \in I$. Suppose $x \notin L$, $then\ by$ definition of

Divisible, there exist D divisible subgroup of G such that $L \subseteq D$ and $x \notin D$.

This contradiction with $x \in G_i\ \forall\ i$. $Thus\ x \in L,\ therefore$ $l = \bigcap_{i\in I} G_i$.

To prove the converse:

Suppose that L is semi divisible group of G , then there exist $x \in G \setminus L$ such that for each divisible subgroup $D\ and\ L \subseteq D, we\ get\ x \in D, then\ x \in G_i, \forall i$

Thus $x \in \bigcap_{i\in I} G_i,\ but\ we\ have\ L = \bigcap_{i\in I} G_i\ (by\ hypothesis\ )$, thus $x \in L$, this contradiction. Therefore L is semi divisible subgroup.

Proposition 1.20:

Let G be a group and L be a subgroup of G. if M is minimal divisible subgroup of G and contain L , then

i. M[p] = L[p] , for each prime number p.
ii. for each x ∈ M, there exist p prime number such that , $px \in L$.

Proof : let M be a minimal divisible subgroups and contain L

to prove (i):

It is clear that L [p] ⊆ $M[p]$ $\forall p$ prime number , then it's enough to show that

M[p] ⊆ L[p].

Suppose that M[p] ⊈ L[p] , this means there exist an element x , x ≠ 0 such that x∈ $M[p]$ and x ∉ $L[p]$, thus x ∈ M and x ∉ L and also px = 0.

Claim <x> ∩ L = 0 (when <x> is is subgroups generated by x), suppose that < x > ∩ L ≠o. then there exist element y , y ≠ 0, such that y ∈ $< x > \cap L$, then y ∈$< x >\ \ and\ y \in L$, then y = r x where (r, p) = 1

Since (r , p) = 1 then there exist two integers number λ, m such that $\lambda r + mp = 1$.

So we get $\lambda rx + mpx = x$, but $mPx = 0$, then x=λrx then x∈ L this contradiction.

Therefore <x> ∩ L=0, since M is minimal divisible group and contain L by lemma [ ] then x=0 this contradiction. Then M[p]= L[p] for each prime number p.

To Proof (ii) :

Suppose the converse this means that px ∉ L, for each prime number p. Then <x> ∩ $L = 0$. but x ∈M and M is minimal divisible subgroup and contain L. This means

x=0 by lemma [ ] contradiction.

Proposition 1.21:

Let G be abelian group and L be a subgroup of G, for each prime number p and for each element $l \in L \setminus PL$ and $l \notin pL\ and\ for\ each$ integer number such that (r , p) =1, then $rl \notin PL$.

Proof :

Suppose that r$l \in pL$, this mans there exists $l_o \in L such\ that\ rl = pl_o$, since (r, p) =1 , then $mr + \lambda p = 1$ where each of m and λ is integer numbers this request that $mrl + \lambda pl = l.$ Then $pml_O + \lambda pl = l$

So $p\,(ml_o + \lambda l) = l$. But $ml_o + \lambda l = l_1 \in l$, then $\exists\, l_1$, $pl_1 = l \in pL$, this means $l \in pL$ contradiction, therefore $rl \notin pl$.

The following theorem is the principal and very important theorem in this paper, through which we proved that the condition we set to answer Q_1, is equivalent to the condition of a chabbas and charles, which shows how important it is.

Theorem 1.22:

Let L be a subgroup of divisible group G , then the following are equivalent:

1) L is semi divisible subgroup of G
2) if G [P] $\subseteq L \longrightarrow pL = L$, for each prime number p.

Proof (1) $\longrightarrow$ (2)

Let L be semi divisible subgroup of G and G[p] $\subseteq L$, for each prime number p.

To prove that PL=L, it is clear that $pL \subseteq L$.

Thus we must to prove that $L \subseteq PL$

suppose the convers ,so there exists prime number p ,such that .

$$L \nsubseteq pL\,, then\ \exists\ element\ \ x \in L\ such\ that\ x \notin pL$$

Since G is divisible group , so $\exists$ an element g $\in$ G such that , $pg = x$ if g $\in$L , then pg=x $\in PL$, this contradiction with $x \notin PL$. Thus g $\notin$ L, but we have L is semi divisible subgroup in G. So $\exists$ D divisible subgroup such that $L \subseteq D, g \notin D$.

Since x$\in L \subseteq D\,, then\ \exists\ d \in D\ such\ that\ \ pd = x = pg, then\ p(d-g) = 0$, therefore $d - g \in G[P] \subseteq L\ \ but\ \ L \subseteq D, then\ \ d - g \in D,\ so\ g \in D$ contradiction . Therefore $PL = L$

Now we will prove the convers:

First method : suppose that $G[P] \subseteq L \Longrightarrow PL = L\,, \forall\, p$

To prove that L is semi divisible subgroup in G , suppose that L is not semi divisible subgroup in G , then $\exists$ an element u $\notin$ L such that for each divisible subgroup D , $if\ \ L \subseteq D\,, then\ \ u \in D$,this means

$[\ \exists\, u\,,\ u \in G\backslash L\,, \forall D, D\ is\ divisible\ \ such\ that\ \ L \subseteq D \longrightarrow u \in D\,]$

Let M be a minimal divisible subgroup in G and contain L, then

we get $u \in M$, then $\exists$ prime number P_o such that $p_o\, u = l \in L$ by proposition 1.20

Suppose that $p_o | l\ in\ L\,,\ then\ \exists\ an\ element\ \ l_o \in L\ such\ that$

$p_o l_o = l\ but\ we\ have\ \ p_o\, u\ =\ l\,, then\ \ p_o u\ =\ p_o l_o\ = l\,, then\ \ p_o u -\ p_o l_o = 0$.

$So\ \ \ p_o(\,u -\ l_o\,) = 0\,,$ since each $u\ \ and\ \ l_o \in M, then\ \ u -\ l_o \in M[P_O]$.

But we have M is minimal divisible subgroup in G and contain L, then M[p_o] = L[p_o],

by proposition 1.20 , we get $u -\ l_o \in M\,[p_o] = L\,[P_O] \subseteq L$.

Therefore $u \in L$ this contradiction. Thus $p_o x\ l\ \ in\ \ L\,, then\ l \notin p_o L\ \ so\ \ p_o\, L \nsubseteq L.$

This request G $[P_O]$ $\nsubseteq L$ by hypothesis, then there exists an element z such that

$z \in G\,[P_O]\backslash L,$ then $p_o z = 0$.

Now suppose that $w = z + u\,, \bar{L} = L + < w >, then\ p_o\, w =\ p_o z\ +\ p_o u\,,\ then$

$p_o\, w = p_o u = l$. It is clear that $L \subseteq \bar{L}$ *and* $L\, [P_O] \subseteq [P_O]$.

Now to prove that $\bar{L}\, [p_O] \subseteq L\, [p_O]$

suppose the opposite:

There exist element l^- such that $l^- \in \bar{L}[P_O]$ *and* $l^- \notin L\, [P_O]$, so we get

$l^- = l_1 + rw$, *where* $0 \le r \le p_o$, $l_1 \in L$, *then* $p_o l^- = p_0 l_1 + p_o rw = 0$, *thus* $p_o l_1 = p_o rw = rl$, *this mean* $rl \in p_o L$ contradictory by Proposition 1.21, so

$l \notin p_o L$, therefore $L[\, p_O] = \bar{L}[p_O]$.

Now :

Let M^- be minimal divisible subgroup of G and contain L, then

$L \subseteq \bar{L} \subseteq M^-$, *thus* $u \in M^-$. But we have $w \in \bar{L} \subseteq M^-$, so by proposition 1.20 we get $z = w - u \in M^-[p_O] = \bar{L}\, [P_O] = L[p_O] \subseteq L$, then $z \in L$ this contradiction because

$z \notin L$, *then* L is semi divisible subgroup in G.

The second method :

Suppose L is semi divisible subgroup in G, then $\exists\, u \in G \setminus L$ *such that* for each divisible subgroup D. If $L \subseteq D$, *then* $u \in D$, this means (i .e.) $\exists\, u, u \notin L, \forall D$, D divisible such that $L \subseteq D \longrightarrow u \in D$.

Let M be a minimal divisible subgroup of G contain L, then $u \in M$, so $\exists$ prime number p_o such that p_o u = L $\in$ L by proposition 1.19 , suppose that P_o | L in L , then

$\exists\; l_o \in L$ *such that* $p_o l_o = l$, but we have $p_o\, u = l$, *then*

$p_o\, u = p_o l_o = l\;\; p_o\, u - p_o l_o = 0$, so $p_o\,(u - l_o) = 0$, since each of l_o and u belong to M. Thus $u - l_0 \in M\, [P_O\,]$. But M is minimal divisible subgroup of G and contain L, then $M[\, P_O\,] = L[\, P_O]$ *by proposition* 1.20, *then* $u - l_o \in M\, [P_O] = L[\, P_O] \subseteq L$.

Thus $u \in L$ contradiction, *so* $p_o \dagger l$ *in* L , *then* $l \notin p_o L$. Therefore $p_o l \nsubseteq L$.

This request $G[\, P_O] \nsubseteq L$ by hypothesis, then $\exists$ *an element* z *such that* $z \in G\, [P_O]$ *and* $z \nsubseteq L$. Thus $p_o z = O$

Now we define $w = z + u$, $l^- = L + \langle w \rangle$, *so*

$$p_o w = p_o z + p_o u \;\; then \;\; p_o w = p_o u = l$$

Now, let M^- minimal divisible subgroup of G and contain l^- *then* $L \subseteq L^- \subseteq M^-$.

Claim that M^- is minimal divisible subgroup of G contian L this means to prove that L is essential to L^-, suppose that L is not essential to L^-, *so* $\exists\; 0 \neq A$ subset of L^- such that A $\cap L = 0$.

Let a $\in A \cap L$, *then* $a \in A \subseteq L^-$, *then* $a = l_1 + rw$ *when* $l_1 \in L$ *and* $0 \le r \le p_o$ this request $l_1 + rw = 0$ *then* $- l_1 = rw$ *so* $- p_o\, l_o = p_o rw = rl$.

This means $rl \in p_o\, l$ contradiction by proposition 1.21 because $l \notin p_o L$, we get L is essential to L^-. Therefore M^- is minimal divisible subgroup of G contain L and proposition 1.18, we get $z = w - u \in \bar{M}\, [P] = L[P] \subseteq L$, then $z \in L$ this contradiction because $z \notin l$.

Therefore L is semi divisible subgroup in G .

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