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# Some study of effect of Michell's function of thick circular plate with internal heat generation

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**Abstract**

The present paper deals with the determination of temperature, displacement and thermal stresses in a thick ($M \neq 0$) circular plates. A thick circular plate is subjected to arbitrary known interior temperature under steady state, the fixed circular edge of a circular plate are thermally insulated and lower surface of thick circular plate is at zero temperature. Here we compute the effects of Michell's function on the thickness of circular plate by considering the comparative study of thick and thin circular plate in terms of stresses along radial direction and modify Kulkarni (2008). The governing heat conduction equation has been solved by the method of integral transform technique. The results are obtained in a series form in terms of Bessel's functions. The results for stresses have been computed numerically and illustrated graphically.

**Keywords:** Thermal stresses, inverse problem, thick($M \neq 0$) circular plate, thin($M = 0$) circular plate.

## INTRODUCTION

The inverse thermoelastic problem consists of determination of the temperature of the heating medium, the heat flux on the boundary surfaces of the limiting thick circular plate when the conditions of the displacement and stresses are known at the some points of the limiting thick circular plate under consideration. Noda *et al.* (1989) discussed an analytical method for an inverse problem of three dimensional transient thermoelasticity in a transversely isotropic solid by integral transform technique with newly designed potential function and illustrated practical applicability of the method in engineering problem. Deshmukh and Wankhede (1998) studied an inverse transient problem of quasi static thermal deflection of a thin clamped circular plate.

Bhongade and Durge (2013) considered thick circular plate and discuss, effect of Michell function on steady state behavior of thick circular plate, now here we consider a thick circular plate with internal heat generation subjected to arbitrary known interior temperature. Under steady state, the fixed circular edge of a limiting thick circular plate are thermally insulated and lower surface thick circular plate is kept at zero temperature. Here we compute the effects of Michell's function on the thickness of circular plate by considering the comparative study of thick and thin circular plates in terms of stresses along radial direction and modify Kulkarni (2008). The governing heat conduction equation has been solved by the method of integral transform technique. The results are obtained in a series form in terms of Bessel's functions. A mathematical model has been constructed for thick $(M \neq 0)$ and thin $(M = 0)$ circular plates with the help of numerical illustration by considering aluminum (pure) circular plate. No one previously studied such type of problem. This is new contribution to the field.

The inverse problem is very important in view of its relevance to various industrial mechanics subjected to heating such as the main shaft of lathe, turbines, the role of rolling mill and in the study of aerodynamic heating.

**Formulation of the problem**

Consider a thick $(M \neq 0)$circular plate of thickness *2h* defined by $0 \leq r \leq a, -h \leq z \leq h$. Let the plate be subjected to arbitrary known interior temperature $f(r)$ within region$-h < z < h$. With circular surface $r = a$ are thermally insulated and lower surface $z = -h$ is at zero temperature. Assume the boundary of thick circular plate is free from traction. Under these prescribed conditions, the thermal steady state temperature, displacement and stresses in a thick circular plate with internal heat generation are required to be determined.

The differential equation governing the displacement potential function $\phi(r,z)$ is given in Noda *et al.* (2003)as

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = K\boldsymbol{\tau} \tag{1}$$

where $K$ is the restraint coefficient and temperature change $\tau = T - T_{i,}$ $T_i$ is ambient temperature. Displacement function $\phi$ is known as Goodier's thermoelastic displacement potential.

The steady state temperature $T(r,z)$ of the plate satisfying heat conduction equation as follows,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{q}{k} = 0 \tag{2}$$

with the conditions

$$\frac{\partial T}{\partial r} = 0 \ \ at \ \ r = a, \ -h \leq z \leq h \tag{3}$$

$$T = 0 \ at\ z = -h,\ 0 \le r \le a \tag{4}$$

$$T = f(r)\ (known)\ at\ \ z = \xi\ ,\ -h < \xi < h,\ 0 \le r \le a \tag{5}$$

and

$$T = g(r)\ \text{(unknown) at}\ \ z = h\ ,\ 0 \le r \le a \tag{6}$$

where $k$ is the thermal conductivity of the material of the plate, $q$ is internal heat generation.

The Michell's function $M$ must satisfy

$$\nabla^2\nabla^2 M = 0 \tag{7}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

The components of the stresses are represented by the thermoelastic displacement potential $\phi$ and Michell's function $M$ as

$$\sigma_{rr} = 2G\left\{\frac{\partial^2\phi}{\partial r^2} - K\tau + \frac{\partial}{\partial z}\left[v\nabla^2 M - \frac{\partial^2 M}{\partial r^2}\right]\right\} \tag{8}$$

$$\sigma_{\theta\theta} = 2G\left\{\frac{1}{r}\frac{\partial\phi}{\partial r} - K\tau + \frac{\partial}{\partial z}\left[v\nabla^2 M - \frac{1}{r}\frac{\partial M}{\partial r}\right]\right\} \tag{9}$$

$$\sigma_{zz} = 2G\left\{\frac{\partial^2\phi}{\partial z^2} - K\tau + \frac{\partial}{\partial z}\left[(2-v)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right]\right\} \tag{10}$$

and

$$\sigma_{rz} = 2G\left\{\frac{\partial^2\phi}{\partial r\partial z} + \frac{\partial}{\partial r}\left[(1-v)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right]\right\} \tag{11}$$

where $G$ and $v$ are the shear modulus and Poisson's ratio respectively.

For traction free surface stress functions

$$\sigma_{rr} = \sigma_{rz} = 0\ at\ \ r = a,\ -h \le z \le h \tag{12}$$

Equations (1) to (12) constitute mathematical formulation of the problem.

**Solution**

To obtain the expression for temperature $T$ $(r, z)$, we introduce the finite Hankel transform over the variable $r$ and its inverse transform defined by Ozisik (1968) as

$$\bar{T}(\beta_m, z) = \int_0^a r\, K_0(\beta_m, r)\ T(r,z)\, dr \tag{13}$$

$$T(r,z) = \sum_{m=1}^{\infty} K_0(\beta_m, r)\ \bar{T}(\beta_m, z) \tag{14}$$

$$where\ K_0(\beta_m, r) = \frac{\sqrt{2}}{a}\frac{J_0(\beta_m r)}{J_0(\beta_m a)}, \tag{15}$$

$\beta_1, \beta_2 \ldots\ldots$ are roots of transcendental equation

$$J_1(\beta_m a) = 0 \tag{16}$$

where $J_n(x)$ is Bessel function of the first kind of order n.

On applying the finite Hankel transform defined in the Eq. (13) and its inverse transform defined in Eq. (14) to the Eq. (2), one obtains the expression for temperature as

$$T(r,z) = \sum_{m=1}^{\infty} \frac{\sqrt{2}}{a} \frac{J_0(\beta_m r)}{J_0(\beta_m a)} \begin{Bmatrix} A(\beta_m, -h) \frac{\sin h[\beta_m(z-\xi)]}{\sin h[(\beta_m+\xi)\mathrm{h}]} \\ -[A(\beta_m, \xi) - F(\beta_m)] \frac{\sin h[\beta_m(z+h)]}{\sin h[(\beta_m+\xi)\mathrm{h}]} + A(\beta_m, z) \end{Bmatrix} \tag{17}$$

where $\mathrm{F}(\beta_m)$ is the Hankel transform of $f(r)$ and $A(\beta_m, z)$ is particular integral of differential Eq.(2).

Michells function $M$

Now suitable form of $M$ which satisfy Eq. (8) is given by

$$M = K \sum_{m=1}^{\infty} F(\beta_m) \frac{\sqrt{2}}{a} \frac{J_0(\beta_m r)}{J_0(\beta_m a)} \begin{Bmatrix} B_m \sinh[\beta_m(z+\mathrm{h})] \\ + C_m \beta_m(z+\mathrm{h}) \cosh[\beta_m(z+\mathrm{h}) \end{Bmatrix} \tag{18}$$

where $B_m$ *and* $C_m$ are arbitrary functions.

Goodiers Thermoelastic Displacement Potential $\phi(\mathrm{r}, \mathrm{z})$

Assuming the displacement function $\phi(r,z)$ which satisfies Eq. (1) as

$$\phi(r,z) = \mathrm{K} \sum_{\mathrm{m}=1}^{\infty} \frac{\sqrt{2}}{a} \frac{J_0(\beta_m r)}{J_0(\beta_m a)}$$

$$\times \begin{Bmatrix} A(\beta_m, -h) \frac{\sin h[\beta_m(z-\xi)]}{\sin h[(\beta_m+\xi)\mathrm{h}]} \\ -[A(\beta_m, \xi) - F(\beta_m)] \frac{\sin h[\beta_m(z+h)]}{\sin h[(\beta_m+\xi)\mathrm{h}]} + A(\beta_m, -h)\, e^{\beta_m(\mathrm{z+h})} \end{Bmatrix} \tag{19}$$

Now using Eqs. (17), (18) and (19) in Eqs. (8), (9), (10) and (11), one obtains the expressions for stresses respectively as

$$\frac{\sigma_{rr}}{K} = 2G \sum_{m=1}^{\infty} \frac{\sqrt{2}}{a\, J_0(\beta_m a)} \Big\{ - \left[ {\beta_m}^2 J_1{}' (\beta_m r) + J_0(\beta_m r) \right]$$

$$\times \left[ A(\beta_m, -h) \frac{\sin h[\beta_m(z-\xi)]}{\sin h[(\beta_m+\xi)\mathrm{h}]} + [F(\beta_m) - A(\beta_m, \xi)] \frac{\sin h[\beta_m(z+h)]}{\sin h[(\beta_m+\xi)\mathrm{h}]} \right]$$

$$- \left[ {\beta_m}^2 J_1'(\beta_m r)\, A(\beta_m, -h) e^{\beta_m(\mathrm{z+h})} + J_0(\beta_m r)\, A(\beta_m, z) \right]$$

$$+ {\beta_m}^2 F(\beta_m) C_m \begin{bmatrix} {\beta_m}^2 J_1'(\beta_m r)(z+\mathrm{h}) \sinh[\beta_m(z+\mathrm{h}) \\ [2v\, J_0(\beta_m r) + J_1{}'(\beta_m r)] \beta_m \cosh[\beta_m(z+\mathrm{h})] \end{bmatrix}$$

$$+ {\beta_m}^3 F(\beta_m)\, B_m\ J_1{}'(\beta_m r) \cosh[\beta_m(z+\mathrm{h})] \Big\} \tag{20}$$

$$\frac{\sigma_{\theta\theta}}{K} = 2G \sum_{m=1}^{\infty} \frac{\sqrt{2}}{a\, J_0(\beta_m a)} \Big\{ - \left[ \beta_m \frac{J_1(\beta_m r)}{r} + J_0(\beta_m r) \right]$$

$$\times \left[ A(\beta_m, -h) \frac{\sin h[\beta_m(z-\xi)]}{\sin h[(\beta_m+\xi)\mathrm{h}]} + [F(\beta_m) - A(\beta_m, \xi)] \frac{\sin h[\beta_m(z+h)]}{\sin h[(\beta_m+\xi)\mathrm{h}]} \right]$$

$$- \left[ \beta_m \frac{J_1(\beta_m r)}{r} A(\beta_m, -h) e^{\beta_m(\mathrm{z+h})} + J_0(\beta_m r) A(\beta_m, z) \right]$$

$$+ {\beta_m}^2 F(\beta_m) C_m \begin{bmatrix} {\beta_m}^2 \frac{J_1(\beta_m r)}{r} (z+\mathrm{h}) \sinh[\beta_m(z+\mathrm{h}) \\ [2v\, \beta_m J_0(\beta_m r) + \frac{J_1(\beta_m r)}{r}] \beta_m \cosh[\beta_m(z+\mathrm{h})] \end{bmatrix}$$

$$+ {\beta_m}^2 F(\beta_m)\, B_m\, \frac{J_1(\beta_m r)}{r} \cosh[\beta_m(z+\text{h})]\Big\} \tag{21}$$

$$\frac{\sigma_{zz}}{K} = 2G \sum_{m=1}^{\infty} \frac{\sqrt{2}\, J_0(\beta_m r)}{a\, J_0(\beta_m a)} \left({\beta_m}^2 - 1\right)$$

$$\times \Big\{ \Big[ A(\beta_m, -h) \frac{\sin h[\beta_m(z-\xi)]}{\sin h[(\beta_m+\xi)\text{h}]} + [F(\beta_m) - A(\beta_m, \xi)] \frac{\sin h[\beta_m(z+h)]}{\sin h[(\beta_m+\xi)\text{h}]} \Big]$$

$$+ {\beta_m}^2 A(\beta_m, -h) e^{\beta_m(\text{z+h})} - A(\beta_m, z) - {\beta_m}^3 F(\beta_m)\, B_m \cosh[\beta_m(z+\text{h})]$$

$$+ {\beta_m}^3\, F(\beta_m) C_m \langle\, (1-2v)\cosh[\beta_m(z+\text{h})] - \beta_m(z+\text{h})\sinh[\beta_m(z+\text{h})\,\rangle \Big\} \tag{22}$$

$$\frac{\sigma_{rz}}{K} = 2G \sum_{m=1}^{\infty} \left(-{\beta_m}^2\right) \frac{\sqrt{2}\, J_1(\beta_m r)}{a\, J_0(\beta_m a)}$$

$$\times \Big\{ \Big[ A(\beta_m, -h) \frac{\cos h[\beta_m(z-\xi)]}{\sin h[(\beta_m+\xi)\text{h}]} + [F(\beta_m) - A(\beta_m, \xi)] \frac{\cos h[\beta_m(z+h)]}{\sin h[(\beta_m+\xi)\text{h}]} \Big]$$

$$+ A(\beta_m, -h) e^{\beta_m(\text{z+h})} - {\beta_m}^3 F(\beta_m)\, B_m \sinh[\beta_m(z+\text{h})]$$

$$+ {\beta_m}^4\, F(\beta_m) C_m \langle\, (-2v)\sinh[\beta_m(z+\text{h})] - \beta_m(z+\text{h})\cosh[\beta_m(z+\text{h})\,\rangle \Big\} \tag{23}$$

In order to satisfy condition (12), solving Eqs. (20) and (23) for $B_m$ and $C_m$ one obtain,

$$B_m = 0, \tag{24}$$

$$C_m = \Big\{ \left[{\beta_m}^2 {J_1}'(\beta_m a) + J_0(\beta_m a)\right]$$

$$\times \Big[ A(\beta_m, -h) \frac{\sin h[\beta_m(z-\xi)]}{\sin h[(\beta_m+\xi)\text{h}]} + [F(\beta_m) - A(\beta_m, \xi)] \frac{\sin h[\beta_m(z+h)]}{\sin h[(\beta_m+\xi)\text{h}]} \Big]$$

$$+ {\beta_m}^2 J_1'(\beta_m a) A(\beta_m, -h) e^{2\beta_m \text{h}} + J_0(\beta_m a) A(\beta_m, h) \Big\} \times \frac{1}{R} \tag{25}$$

where $R = {\beta_m}^3 F(\beta_m) \begin{bmatrix} 2\beta_m \text{h}\ {J_1}'(\beta_m a)\sinh(2\beta_m \text{h}) \\ \left(2\text{v}\, J_0(\beta_m a) + {J_1}'(\beta_m a)\right)\cosh(2\beta_m \text{h}) \end{bmatrix}$

## SPECIAL CASE AND NUMERICAL CALCULATIONS

Setting

(1) $\quad \text{f(r)} = r^2$

$$F(\beta_m) = \frac{a\sqrt{2}}{J_0(\beta_m a)} \left[a J_1(\beta_m a) - 2 J_2(\beta_m a)\right]$$

(2) $\quad q(r,z) = \delta(r-r_0)\delta(z-z_0)$

$$\bar{q}(\beta_m, z) = \int_{r'=0}^{a} r' \frac{\sqrt{2}}{a} \frac{J_0(\beta_m r)}{J_0(\beta_m a)} \delta(r-r_0)\delta(z-z_0)\, dr'$$

$$= \frac{\sqrt{2}}{a} \frac{\delta(z-z_0)}{J_0(\beta_m a)} r_0\, J_0(\beta_m r_0)$$

where $\delta(r)$ is well known diract delta function of argument $r$.

$a = 1m$, for thick plate $h = 0.100000000015m$ and for thin plate $h = 0.099999999995m$.

$r_0 = 0.5m, z_0 = 0.05m$

Material Properties

The numerical calculation has been carried out for aluminum (pure) circular plate with the material properties defined as

Thermal diffusivity $\alpha$ = 84.18$\times 10^{-6}$ $\mathrm{m^2 s^{-1}}$,

Specific heat $c_\rho = 896$ J/kg,

Thermal conductivity $k$ = 204.2 W/m K,

Shear modulus $G = 25.5\ G$ pa,

Poisson ratio $\vartheta = 0.281$.

*Roots of Transcendental Equation*

The $\beta_1 = 3.8317$, $\beta_2 = 7.0156$, $\beta_3 = 10.1735$, $\beta_4 = 13.3237$, $\beta_5 = 16.4706$, $\beta_6 = 19.6159$ are the roots of transcendental equation $J_1(\beta_m a) = 0$. The numerical calculation and the graph has been carried out with the help of mathematical software Mat lab.

**DISCUSSION**

In this problem, a thick $M \neq 0$ circular plate is considered which is subjected to arbitrary known interior temperature and determined the expressions for unknown temperature, displacement and stresses. Here we compute the effects of Michell's function on the thickness of circular plate by considering the comparative study of thick $M \neq 0$ and thin $M = 0$ circular plate by substituting $M = 0$ in Eqs. (8), (9), (10) and (11) and plotted the graphs for stresses along radial direction. A**s** a special case mathematical model is constructed for $f(r) = r^2$ and performed numerical calculation by considering aluminum (pure) circular plate with the material properties specified above.

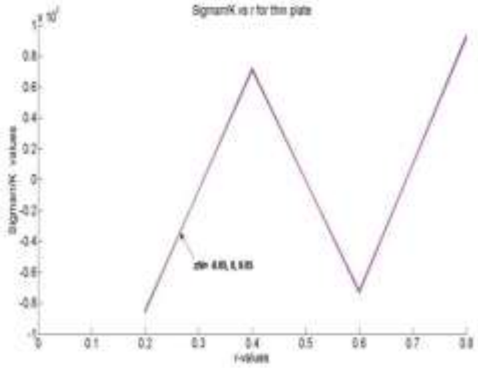


**Figure 1** Radial stresses $\frac{\sigma_{rr}}{K}$ for thick plate.

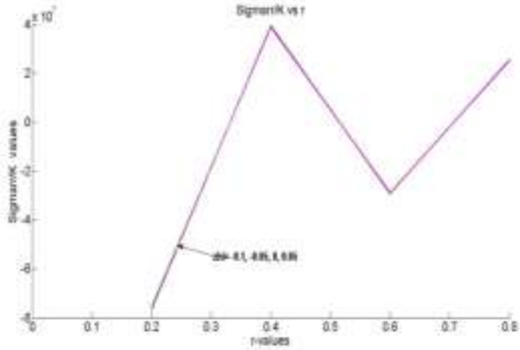


**Figure 2** Radial stresses $\frac{\sigma_{rr}}{K}$ for thin plate.

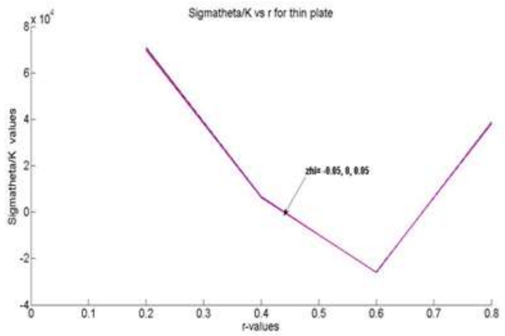


**Figure 3** Angular stresses $\frac{\sigma_{\theta\theta}}{K}$ for thin plate.

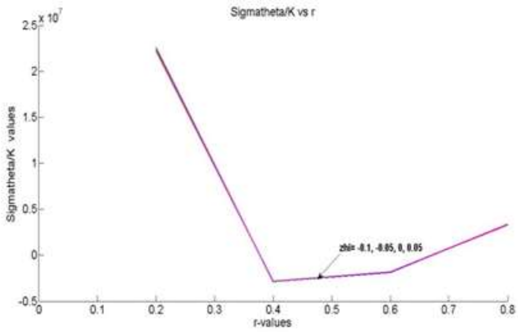


**Figure 4** Angular stresses $\frac{\sigma_{\theta\theta}}{K}$ for thick plate.

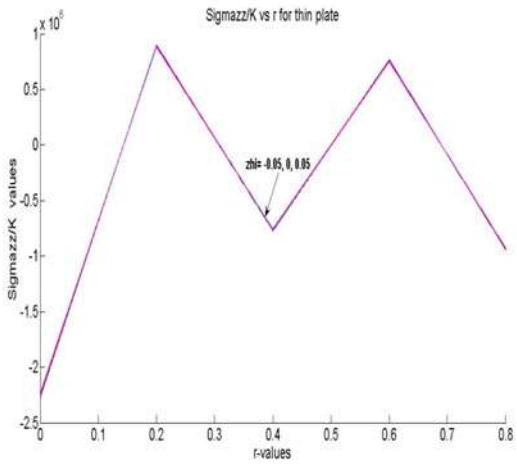


**Figure 5** Axial stresses $\frac{\sigma_{zz}}{K}$ for thin plate.

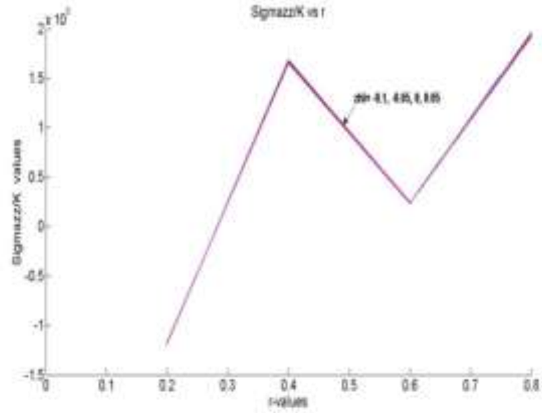


**Figure 6** Axial stresses$\frac{\sigma_{zz}}{K}$ for thick plate.

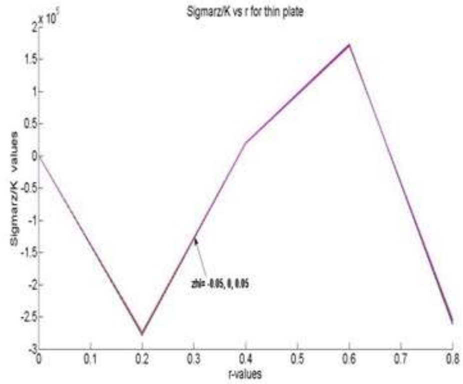


**Figure 7** Stresses $\frac{\sigma_{rz}}{K}$ for thin plate

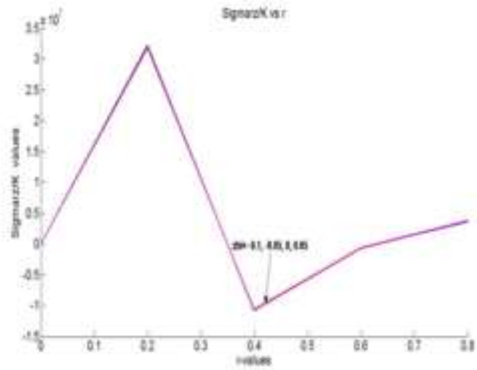


**Figure 8** Stresses$\frac{\sigma_{rz}}{K}$ for thick plate

From figure 1and 2, it is observed that the radial stresses $\frac{\sigma_{rr}}{K}$ for thin $(M = 0)$ and thick $(M \neq 0)$ plate are increasing for $0.2 \leq r \leq 0.4, 0.6 \leq r \leq 0.8$ and decreasing for $0.4 \leq r \leq 0.6$ along radial direction. Due to Michell's function the radial stresses $\frac{\sigma_{rr}}{K}$ are increasing and its nature is tensile along radial direction.

From figure 3, it is observed that the angular stresses $\frac{\sigma_{\theta\theta}}{K}$ for thin $(M = 0)$ plate are decreasing for $0.2 \leq r \leq 0.6$ and it increases rapidly for $0.6 \leq r \leq 0.8$ along radial direction. From figure 4, it is observed that the angular stresses $\frac{\sigma_{\theta\theta}}{K}$ for thick $(M \neq 0)$ plate are decreasing for $0.2 \leq r \leq 0.6$ and increasing for $0.6 \leq r \leq 0.8$ along radial direction. Due to

Michell's function the angular stresses $\frac{\sigma_{\theta\theta}}{K}$ are increasing and its nature is compressive along radial direction.

From figure 5 it is observed that the axial stresses $\frac{\sigma_{zz}}{K}$ for thin $(M = 0)$ plate are increasing for $0 \le r \le 0.2$, $0.4 \le r \le 0.6$ and decreasing for $0.2 \le r \le 0.4, 0.6 \le r \le 0.8$ along radial direction. From figure 6, it is observed that the angular stresses $\frac{\sigma_{\theta\theta}}{K}$ for thick $(M \ne 0)$ plate are increasing for $0.2 \le r \le 0.4, 0.6 \le r \le 0.8$ $a$nd decreasing for $0.4 \le r \le 0.6$ along radial direction. Due to Michell's function the axial stresses $\frac{\sigma_{zz}}{K}$ are increasing and its nature tensile along radial direction. From figure 7, it is observed that the stresses $\frac{\sigma_{rz}}{K}$ for thin $(M = 0)$ plate are decreasing for $0 \le r \le 0.2, 0.6 \le r \le 0.8$ and increasing for $0.2 \le r \le 0.6$ along radial direction. From figure 8, it is observed that the stress $\frac{\sigma_{rz}}{K}$ for thick $(M \ne 0\,)$plate are increasing for $0 \le r \le 0.2$, $0.4 \le r \le 0.8$ and decreasing for $0.2 \le r \le 0.4$ along radial direction. Due to Michell's function the stresses $\frac{\sigma_{rz}}{K}$ are increasing and its nature is compressive along radial direction.

**Conclusion**

Due to Michell's function the radial stresses $\frac{\sigma_{rr}}{K}$ and the axial stresses $\frac{\sigma_{zz}}{K}$ are increased and it is tensile in nature along radial direction, the angular stresses $\frac{\sigma_{\theta\theta}}{K}$ and the stresses $\frac{\sigma_{rz}}{K}$ are decreasing and it is compressive in nature along radial direction. Even though the difference between the thickness of thin and thick plate is very small, but all the stresses are extremely larged due to Michell's function along radial direction.

The results obtained here are useful in engineering problems particularly in the determination of state of stress in a thin clamped circular plate, base of furnace of boiler of a thermal power plant, gas power plant and the measurement of aerodynamic heating.

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