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Convex Optimization of Havrda-Charvat Distance (Divergence) Metric by Employing Lagrangian Policy in Intuitionistic Fuzzy Setting

Rohit Kumar Verma
Associate Professor, Department of Mathematics,
Bharti Vishwavidyalaya, Durg, C.G., India.

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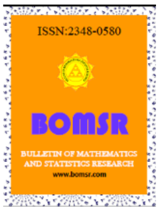

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Abstract

According to the established requirements, the distribution that minimizes the Kullback-Leibler divergence is chosen using the minimal likeliness distance (divergence) metric principle. This fundamental principle generalizes various approaches that have been put forth independently and cover a wide range of distributions. Additionally, we point out that the Lagrangian approach is a particular instance of minimum distance (divergence) metric (MDM) with a uniform posterior distribution. To provide much-needed clarification, this study is done in intuitionistic fuzzy environment thatgives us the analytical solutionand the direction which highlights this link.

Keywords: Aggregation, Lagrangian approach, distance (divergence) metric, Gamma Distribution, Thresholding, Optimization.

1. Introduction

For every uncertain decision, estimating the underlying probability distribution of the decision options is a necessary step [4]. For instance, the distribution of profitability is necessary when making investments, and the probability of failure for each alternative is necessary when constructing an engineered solution. The minimum distance (divergence) metric approach was put forth by Edwin T. Jaynes [5] as a way to establish prior probabilities in decision analysis. The Kullback-Leibler [6] divergence is the objective of entropy methods, which rely on the optimization of an objective function.

The optimization problem, which has also been solved by Verma [11, 12, 13, 14, 15] incorporates the available information as constraints. In decision analysis, both directions of the distance metric approach are frequently employed, especially when combining expert opinion [1]. The (divergence) metric [10] assesses how closely two probability distributions, $P$and Q, are connected. It can be used to determine the distribution P that satisfies a set of requirements and is closest to a target distribution Q using the notion of minimal (divergence) metric (MDM), where the "closeness" is determined by the Kullback-Leibler divergence [6, 7]. To find the solution to the probabilistic problem presented above, Kullback [7] first maximized a measure of directed divergence owing to Chernoff [2] concerning the relevant parameter. The seven optimization problems have been solved by Kapur's [8]and Verma's [11, 13], and the solution is dependent on the Kullback-Leibler divergence's measure of directed divergence [6] for a discrete reference distribution Q calculated with a discrete distribution P is

$$D(P,Q)=\sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i}$$

To find the solution to the probabilistic problem presented below, Kullback [7] first maximized a measure of directed divergence owing to Chernoff [2] concerning the relevant parameter. For a discrete reference distribution Q estimated with a discrete distribution P, the Kullback-Leibler divergence is $D(P,Q)=\sum_{i=1}^{n} p_i\, ln \frac{p_i}{q_i}$, and Kapur's [8] and Verma's [11, 12, 16] solution relies on this measure. We are now given three probability distributions, $P=(p_1,\dots,p_n)$, $Q=(q_1,\dots,q_n)$ and $R=(r_1,\dots,r_n)$, each of which has a $p_i>0, q_i>0$and $r_i>0$ with $\sum_{i=1}^{n} p_i=1$,$\sum_{i=1}^{n} q_i=1$ and $\sum_{i=1}^{n} r_i=1$.Let $P=(p_1,\dots,p_n)$ be a probability distribution. Shannon's [9] provides the measure $S(P)=\sum_{i=1}^{n} p_i \ln p_i$.

We take into account the following optimization issues:

Problem Find the probability distribution that is closest to Q (or R) among all those that are equally distant from Q and R.

Here, the term "distance of P from Q" refers to the directed divergence of Havrda and Charvat [3] of $P$from Q, $i.e.$,

$$D(P,Q)=\frac{1}{1-\alpha}\left(\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha}-1\right), \alpha\neq 1.$$

To find the solution to the probabilistic problem presented above, Kullback [7] first maximized a measure of directed divergence owing to Chernoff [2] concerning the relevant parameter. The seven optimization problems have been solved by Kapur's [8], and the solution is based on the Kullback-Leibler [6] measure of directed divergence of $P$from Q, which is

$$D(P,Q)=\sum_{i=1}^{n} p_i \log_D \frac{p_i}{q_i}.$$

1.1 Measures of Distance (Divergence) Metric

If $(i) D(Q,R)\geq 0$,

$(ii) D(Q,R)=0$iff $q_i=r_i$, for each i

$(iii) D(Q,R)$ is a convex function of both $q_1, \ldots, q_n$ and $r_1, \ldots, r_n$, then $D(Q,R)$ will be taken into consideration as a measure of directed divergence of the probability distribution Q from the probability distribution R.

1.2 Intuitionistic Fuzzy Set

If F be a fixed set then an intuitionistic fuzzy set [10] S in F is an object having the form $S = \{< x, \mu_s(x), \nu_s(x) > / x \in F\}$.Where the function $\mu_s(x)$ and $\nu_s(x)$ define the degree of membership and degree of non membership of the element $x \in S$ to $S \subset F$ respectively. The function $\mu_s(x)$ and $\nu_s(x)$ satisfy the condition $(\forall x \in F)(0 \leq \mu_s(x) + \nu_s(x) \leq 1)$.

2. Our Results

In light of the fact that Havrda and Charvat [3] measure

$D(B,C) = \frac{1}{\alpha-1}\sum_{i=1}^{n}(\mu_B(x_i))^\alpha\left((\mu_C(x_i))^{1-\alpha}-1\right) + \frac{1}{\alpha-1}\sum_{i=1}^{n}(\nu_B(x_i))^\alpha\left((\nu_C(x_i))^{1-\alpha}-1\right), \alpha \neq 1$

satisfies each of these requirements.

2.1 Arrangement of the Problem

The aforementioned issue must be reduced, subject to

$$\frac{1}{\alpha-1}\sum_{i=1}^{n}(\mu_A(x_i))^\alpha\left((\mu_B(x_i))^{1-\alpha}-1\right) + \frac{1}{\alpha-1}\sum_{i=1}^{n}(\nu_A(x_i))^\alpha\left((\nu_B(x_i))^{1-\alpha}-1\right) =$$

$$\frac{1}{\alpha-1}\sum_{i=1}^{n}(\mu_A(x_i))^\alpha\left((\mu_C(x_i))^{1-\alpha}-1\right) + \frac{1}{\alpha-1}\sum_{i=1}^{n}(\nu_A(x_i))^\alpha\left((\nu_C(x_i))^{1-\alpha}-1\right)$$

$$i.e. \frac{1}{\alpha-1}\sum_{i=1}^{n}(\mu_A(x_i))^\alpha\left((\mu_B(x_i))^{1-\alpha}-(\mu_C(x_i))^{1-\alpha}\right) +$$

$$\frac{1}{\alpha-1}\sum_{i=1}^{n}(\nu_A(x_i))^\alpha\left((\nu_B(x_i))^{1-\alpha}-(\nu_C(x_i))^{1-\alpha}\right) = 0$$

and $\sum_{i=1}^{n}(\mu_A(x_i) + \nu_A(x_i)) = 1.$

Using Lagrange's method

$$L \equiv \frac{1}{\alpha-1}\sum_{i=1}^{n}(\mu_A(x_i))^\alpha\left((\mu_B(x_i))^{1-\alpha}-1\right) + \frac{1}{\alpha-1}\sum_{i=1}^{n}(\nu_A(x_i))^\alpha\left((\nu_B(x_i))^{1-\alpha}-1\right) +$$

$$\lambda_1(\alpha-1)^{-1}\sum_{i=1}^{n}(\mu_A(x_i))^\alpha\left((\mu_B(x_i))^{1-\alpha}-(\mu_C(x_i))^{1-\alpha}\right) +$$

$$\lambda_1(\alpha-1)^{-1}\sum_{i=1}^{n}(\nu_A(x_i))^\alpha\left((\nu_B(x_i))^{1-\alpha}-(\nu_C(x_i))^{1-\alpha}\right)$$

$$\lambda_2\left(\sum_{i=1}^{n}\mu_A(x_i)-1\right) + \lambda_2\left(\sum_{i=1}^{n}\nu_A(x_i)-1\right)$$

Now,

$$\frac{\partial L}{\partial \mu_A(x_1)} = \frac{1}{\alpha-1}\sum_{i=1}^{n}\left(\left(\mu_A(x_i)\right)^{\alpha-1}\left(\mu_B(x_i)\right)^{1-\alpha}\right) + \frac{1}{\alpha-1}\sum_{i=1}^{n}\left(\left(\nu_A(x_i)\right)^{\alpha-1}\left(\nu_B(x_i)\right)^{1-\alpha}\right) +$$

$$\lambda_1\alpha(\alpha-1)^{-1}\sum_{i=1}^{n}\left(\mu_A(x_i)\right)^{\alpha-1}\left(\left(\mu_B(x_i)\right)^{1-\alpha} - \left(\mu_C(x_i)\right)^{1-\alpha}\right) +$$

$\lambda_1\alpha(\alpha-1)^{-1}\sum_{i=1}^{n}\left(\nu_A(x_i)\right)^{\alpha-1}\left(\left(\nu_B(x_i)\right)^{1-\alpha} - \left(\nu_C(x_i)\right)^{1-\alpha}\right) + \lambda_2 = 0.$

Then, we achieve

$$\mu_A(x_i) = \frac{\left[(1+\lambda_1)\left(\mu_B(x_i)\right)^{\alpha-1} - \lambda_1\left(\mu_C(x_i)\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}}{\sum_{i=1}^{n}\left[(1+\lambda_1)\left(\mu_B(x_i)\right)^{\alpha-1} - \lambda_1\left(\mu_C(x_i)\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}}$$

and $$\nu_A(x_i) = \frac{\left[(1+\lambda_1)\left(\nu_B(x_i)\right)^{\alpha-1} - \lambda_1\left(\nu_C(x_i)\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}}{\sum_{i=1}^{n}\left[(1+\lambda_1)\left(\nu_B(x_i)\right)^{\alpha-1} - \lambda_1\left(\nu_C(x_i)\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}}.$$

Now, setting $1+\lambda_1 = \beta \Rightarrow -\lambda_1 = 1-\beta$ and this implies that

$$\mu_A(x_i) = \frac{\left[\beta\left(\mu_B(x_i)\right)^{\alpha-1} + (1-\beta)\left(\mu_C(x_i)\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}}{\sum_{i=1}^{n}\left[\beta\left(\mu_B(x_i)\right)^{\alpha-1} + (1-\beta)\left(\mu_C(x_i)\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}}$$

and $$\nu_A(x_i) = \frac{\left[\beta\left(\nu_B(x_i)\right)^{\alpha-1} + (1-\beta)\left(\nu_C(x_i)\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}}{\sum_{i=1}^{n}\left[\beta\left(\nu_B(x_i)\right)^{\alpha-1} + (1-\beta)\left(\nu_C(x_i)\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}}.$$

Thus out of all the distribution

$$\sum_{i=1}^{n}\left[\frac{\left[\beta\left(\mu_B(x_i)\right)^{\alpha-1} + (1-\beta)\left(\mu_C(x_i)\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}}{\sum_{i=1}^{n}\left[\beta\left(\mu_B(x_i)\right)^{\alpha-1} + (1-\beta)\left(\mu_C(x_i)\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}}\right]^{\alpha}\left(\left(\mu_B(x_i)\right)^{1-\alpha} - \left(\mu_C(x_i)\right)^{1-\alpha}\right) = 0$$

and $$\sum_{i=1}^{n}\left[\frac{\left[\beta\left(\nu_B(x_i)\right)^{\alpha-1} + (1-\beta)\left(\nu_C(x_i)\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}}{\sum_{i=1}^{n}\left[\beta\left(\nu_B(x_i)\right)^{\alpha-1} + (1-\beta)\left(\nu_C(x_i)\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}}\right]^{\alpha}\left(\left(\nu_B(x_i)\right)^{1-\alpha} - \left(\nu_C(x_i)\right)^{1-\alpha}\right) = 0.$$

Letting,

$$G(\beta) \equiv \sum_{i=1}^{n}\left[\frac{\left[\beta\left(\mu_B(x_i)\right)^{\alpha-1} + (1-\beta)\left(\mu_C(x_i)\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}}{\sum_{i=1}^{n}\left[\beta\left(\mu_B(x_i)\right)^{\alpha-1} + (1-\beta)\left(\mu_C(x_i)\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}}\right]^{\alpha}\left(\left(\mu_B(x_i)\right)^{1-\alpha} - \left(\mu_C(x_i)\right)^{1-\alpha}\right)$$

$$+\sum_{i=1}^{n}\left[\frac{\left[\beta\left(\nu_B(x_i)\right)^{\alpha-1} + (1-\beta)\left(\nu_C(x_i)\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}}{\sum_{i=1}^{n}\left[\beta\left(\nu_B(x_i)\right)^{\alpha-1} + (1-\beta)\left(\nu_C(x_i)\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}}\right]^{\alpha}\left(\left(\nu_B(x_i)\right)^{1-\alpha} - \left(\nu_C(x_i)\right)^{1-\alpha}\right) = 0$$

Then, $G(0) = (\alpha - 1)D(C, B)$, and $G(1) = -(\alpha - 1)D(B, C)$. If $0 < \alpha < 1$ then $G(0) < 0$and $G(1) > 0$. Obviously, $G(\beta)$ has a root lying between 0and 1, assuming that it is β_0, then we have a solution of our problem

$$\left(D(A,B)\right)_{min} = \frac{1}{\alpha-1}\left[\frac{\sum_{i=1}^{n}\left[\beta_0\left(\mu_B(x_i)\right)^{\alpha-1} + (1-\beta_0)\left(\mu_C(x_i)\right)^{1-\alpha}\right]^{\frac{\alpha}{\alpha-1}}}{\left[\sum_{i=1}^{n}\left[\beta_0\left(\mu_B(x_i)\right)^{\alpha-1} + (1-\beta_0)\left(\mu_C(x_i)\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}\right]^{\alpha}} - 1\right]$$

$$+\frac{1}{\alpha-1}\left[\frac{\sum_{i=1}^{n}\left[\beta_0\left(\nu_B(x_i)\right)^{\alpha-1} + (1-\beta_0)\left(\nu_C(x_i)\right)^{1-\alpha}\right]^{\frac{\alpha}{\alpha-1}}}{\left[\sum_{i=1}^{n}\left[\beta_0\left(\nu_B(x_i)\right)^{\alpha-1} + (1-\beta_0)\left(\nu_C(x_i)\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}\right]^{\alpha}} - 1\right]$$

On taking $\alpha \to 1$, then we achieve

$$\left(D(A,B)\right)_{min} = \frac{\sum_{i=1}^{n}\left(\mu_B(x_i)\right)^{\beta_0}\left(\mu_C(x_i)\right)^{1-\beta_0}\log_D\left(\mu_B(x_i)\right)^{\beta_0}\left(\mu_C(x_i)\right)^{1-\beta_0}}{\sum_{i=1}^{n}\left(\mu_B(x_i)\right)^{\beta_0}\left(\mu_C(x_i)\right)^{1-\beta_0}}$$

$$+\frac{\sum_{i=1}^{n}\left(\nu_B(x_i)\right)^{\beta_0}\left(\nu_C(x_i)\right)^{1-\beta_0}\log_D\left(\nu_B(x_i)\right)^{\beta_0}\left(\nu_C(x_i)\right)^{1-\beta_0}}{\sum_{i=1}^{n}\left(\nu_B(x_i)\right)^{\beta_0}\left(\nu_C(x_i)\right)^{1-\beta_0}}$$

Now, applying the limiting condition $\beta_0 \to 0$, then we have $\left(D(A,B)\right)_{min} = -S(R)$ where $S(R)$ is Shannon's measure of entropy for the probability distribution $R = r_1, \dots, r_n$.

3. Conclusion

In this communication, we looked at Kapur's [8] approaches to the optimization issues. We have resolved the first of seven optimization problems by employing the Lagrangian policy in IF-criterion. The convexity characteristics of the measure of distance (divergence) resulting from Havrda and Charvat's measure of entropy determine problems and their solutions.

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