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RESEARCH ARTICLE

Strongly I-Pseudo Regular I-Spaces

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Abstract
In this paper Strongly I -pseudo regular I- spaces have been defined and a few important properties of these spaces have been justified. This is the third paper of I- Pseudo regular I- spaces. A number of important theorems regarding these spaces have been established. Strongly I -pseudo normal is not defined for I- spaces.

Keywords: Hausdorff I- spaces, Regular I-spaces, Pseudo regular I-spaces, Strongly Pseudo regular I-spaces.

Mathematics Subject Classification: 54D10, 54D15, 54A05, 54A40, 54G12, 54C08.

1. Introduction

A topological space X is said to be regular if for each non- empty closed set F of X and any point $x \in X$, such that$x \notin F$ (i.e. x is the external point of F), there exist two disjoint open sets V and W such that $x \in V$ and $F \subseteq W$. Inpaper [5] a pseudo regular topological spaces has been defined by replacing a closed subset F by a compact subset K in the definition of a regular space. The concept of Strictly I- Pseudo regularity has been extended to I- spaces in previous paper [2]. These spaces were introduced and studied in [4] and [5]. Here the concept of Strongly I- Pseudo regularity of [1] has been extended to I- spaces. In this paper we have given examples of Strongly I-pseudo regular I- spaces. A number of important theorems regarding these spaces have been established.

Here we have introduced strongly pseudo regular I- spaces and studied their important properties. Many results have been proved about these I-spaces. Strongly I- pseudo normal I –spaces is not established. We have also established characterizations of such I-spaces.

II. Preliminaries

We begin with some basic definitions and examples related to I-spaces.

I-spaces

Definition 2.1 [7]: Let X be a non-empty set. A collection I of subsets of X is called an I-structure on X if
(i) X, ∅∈ I
(ii) $G_1, G_2, G_3, ..., G_n \in$ I implies $G_1 \cap G_2 \cap G_3 \cap ... \cap G_n \in$ I.
Then (X, I) is called an I-space.
The members of I are called I-open sets and the complement of an I-open set is called an I-closed set.

Example 2.1: Let X = {a, b, c, d}, I = {X, ∅, {a}, {a, b}, {c}, {a, c}, {b, c, d}}. Here (X, I) is an I-space but not a topological space and U-space.

I-Pseudo Regular I-Spaces

Definition 2.2 [3]: An I-space X is said to be **I-pseudo regular** if for every I-compact subset K of X and for every point $x \in X$ such that $x \notin K$, there exist two I-open sets $U, V \in I$ with $K \subseteq U$, $x \in V$, and $U \cap V = \emptyset$.

Example 2.2: (An example of an I-pseudo regular space)

Consider the real numbers ℝ with an appropriate I-structure. Let $K=[x_1-a_1,x_1+a_1]\cup[x_2-a_2,x_2+a_2]\cup\cdots\cup[x_n-a_n,x_n+a_n]$ be a union of nn closed intervals, forming an I-compact set. Suppose these intervals are chosen such that for each i,j,l, the following holds

$$\alpha_l < \frac{1}{2} \mid x_i - x_j \mid and \alpha l < \mid x - x_i \mid,$$

Where$x$ is a point not in K.

Let $\alpha = \frac{1}{2}\min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Then, the interval $[x-a,x+a]$ is disjoint from the set K, i.e.,

$[x-a,x+a]\cap([x1-a_1,x_1+a_1]\cup\cdots\cup[x_n-a_n,x_n+a_n])=\emptyset$.

Now, if the I-structure I on R contains such intervals and their unions as I-open sets, then we have:

- $x\in[x-a,x+a]=V$
- $K\subseteq[x_1-a_1,x_1+a_1]\cup\cdots\cup[x_n-a_n,x_n+a_n]=U$
- $U\cap V=\emptyset$

Therefore, the I-space (ℝ,I) is I-pseudo regular.

Strictly I-Pseudo Regular I–Spaces

Definition 2.3 [2]: An I–space X is called Strictly I-pseudo regular if for every I-compact set $K \subset X$ and for every point $x \in X$ with $x \notin K$, there exists a continuous function $f: X \to [0,1]$ such that $f(x)=0$ and $f(K)=1$.

Example 2.3: Let K be an I-compact subset of $\mathbb{R}$ and let $x \in \mathbb{R}$ such that $x \notin K$. Since $\mathbb{R}$ is I–Hausdorff, K is closed, and since $\mathbb{R}$ is I–completely regular, there exists a continuous function $f: X \to [0,1]$ such that $f(x)=0$ and $f(K)=1$. Thus, $\mathbb{R}$ is strictly I–pseudo regular.

III. STRONGLY I-PSEUDO REGULAR I-SPACES

Definition 3.1: An I-space X is called **strongly I-pseudo regular** if for each I-compact set K of X and for every $x \in X$ with $x \notin K$, there exist I-open sets G and H such that $x \in G$, $K \subseteq H$, and $\bar{G} \cap \bar{H} = \emptyset$.

Example 3.1: The space X=R with the usual I-structure is strongly I-pseudo regular.

Proof: Let X=R with the usual I-structure. Let K be a non-empty I-compact subset of X and let $x \in X$ with $x \notin K$. By the Heine-Borel Theorem, K is closed and bounded. Hence, K can be written as a finite union of disjoint closed intervals:

$$K = \bigcup_{i=1}^{n} [a_1, b_1], where\ [a_1, b_1] \cap [a_j, b_j] = \emptyset\ for\ i \neq j.$$

Let $a_0=\min\{a_i\}$ and $b_0=\max\{b_i\}$. One of the following three conditions must hold:
(i) $x<a_0$

(ii) $x>b_s$,
(iii) there exist intervals $[a_j, b_j]$ and $[a_k, b_k]$ in K such that $b_j<x<a_k$, and these intervals are consecutive in K (meaning no point of K lies between b_j and a_k).

In each case, it is straightforward to find disjoint I-open sets G and H separating x and K. For instance, in case (i), one can take $G=(-\infty, a_0)$ and $H=(a_0-\epsilon, b_0+\epsilon)$ for a sufficiently small $\epsilon>0$. The other cases follow similarly. Therefore, $\mathbb{R}$ is strongly I-pseudo regular.

If (i) holds, let $\delta_1=\frac{1}{3}(a_0-x)$, and let $U_1=(x-\delta_1, x+\delta_1)$, $V_1=(a_0-\delta_1, b_0+\delta_1)$. Then U_1, V_1 are open, and $U_1 \cap V_1=\emptyset$. Also $x \in U_1$ and $K \subseteq V_1$.

If (ii) holds, let $\delta_2=\frac{1}{3}(x-b_0)$, and let $U_2=(x-\delta_2, x+\delta_2)$, $V_2=(a_0-\delta_2, b_0+\delta_2)$. Then U_2, V_2 are open, $x \in U_2$and $K \subseteq V_2$ and $U_2 \cap V_2=\emptyset$.

If (iii) holds, let $\delta_3=\frac{1}{3}\min\{x-b^j, a_k-x\}$, and let $U_3=(x-\delta_3, x+\delta_3)$, $V^3=(a^0-\delta^3, b^j+\delta_3) \cup (a_k-\delta_3, b_0+\delta_3)$. Then$U_3$, V_3 are open, $x \in U_3$ and $K \subseteq V_3$ and $\bar{U}_3 \cap \bar{V}_3=\emptyset$. Thus, X is strongly I-pseudo regular.

Theorem 3.1: Every strongly I-pseudo regular I-space is I-pseudo regular but the converse is not true in general.

Proof: The first part is obvious. To prove the converse, Let $X = \{a, b, c, d\}$ and $I = \{X, \emptyset, \{b\}, \{a, d\}, \{b, c\}\}$. Then (X,I) is an I-space. The closed sets are $X, \emptyset, \{a,c,d\}, \{b,c\}, \{a,d\}$. Let $K=\{b\}$. Then K isI-compact and $a \notin K$. Then we have open sets $G = \{b\}$, $H = \{a, d\}$ such that $K \subseteq G$, $a \in H$ and $G \cap H = \{b\} \cap \{a, d\} = \emptyset$. Hence X is I-pseudo regular. G and H are the only disjoint open sets which contain K and a respectively.

Now we have $G=\{a,c,d\}$, $H=\{b,c\}$ and $G\cap H=\{a,c,d\}\cap\{b,c\}=\{c\}\neq\emptyset$. Hence X is not strongly I-pseudo regular.

Theorem 3.2 [1]: Any subspace of a strongly I-pseudo regular I-space is strongly I-pseudo regular.

Proof: Let X be a strongly I-pseudo regular I-space and $Y\subseteq X$. Let $y\in Y$ and K be an I-compact subset of Y such that $y\notin K$. Since K is I-compact in Y, so K is I-compact in X. Since X is strongly I-pseudo regular, there exist open sets G and H of X such that $y\in G$ and $K\subseteq H$ and $G\cap H=\emptyset$.

Let $U=G\cap Y$ and $V=H\cap Y$.

Then U and V are open sets of Y where $y\in U$ and $K\subseteq V$ and $U\cap V=\emptyset$. Hence Y is strongly I-pseudo regular.

Theorem 3.3 [1]: Let X be an I-space and A,B are two strongly I-pseudo regular subspaces of X. Then $A\cap B$ is strongly I-pseudo regular.

Proof: Since $A\cap B$ being a subspace of both A and B, $A\cap B$ is strongly I-pseudo regular by the above Theorem 3.2.

Theorem 3.4 [1]: An I-space X is strongly I-pseudo regular if for each $x\in X$ and any I-compact set K not containing x, there exists an I-open set H of X such that $x\in H\subseteq\overline{H}\subseteq K^C$.

Proof: Let X be a strongly I-pseudo regular space and let K be I-compact in X. Let $x\notin K$ i.e., $x\in K^C$. Since X is strongly I-pseudo regular, there exist open sets U,V such that $x\in U$, $K\subseteq V$ and $U\cap V=\emptyset$ andso $\overline{U}\cap V=\emptyset$. Then $\overline{U}\subseteq V^C\subseteq K^C$So $U\subseteq\overline{U}\subseteq V^C$. Writing $U=H$, we have $x\in H\subseteq\overline{H}\subseteq K^C$.

Theorem 3.5 [1]: An I-space X is strongly I-pseudo regular if X is completely I-Hausdorff.

Proof: Let X be a completely I-Hausdorff space and K be an I-compact subset of X. Let x,y be two distinct points of X with $y\in K$ and $x\notin K$. Since X is completely I-Hausdorff there exist open sets Gy and Hy such that $x\in Gy$ and $y\in Hy$ and $\bar{G}y\cap\overline{H}y=\emptyset$. The collection $\{Hy:y\in K\}$ is an I-open cover of K. Since K is I-compact, there exists a finite subcover $\{Hy_1,Hy_2,Hy_3,\ldots,Hy_n\}$ of K. Let $H=Hy_1\cup Hy_2\cup Hy_3\cup\cdots\cup Hy_n$ and $G=Gy_1\cap Gy_2\cap Gy_3\cap\cdots\cap Gy_n$. Then $K\subseteq H$, $x\in G$ and we claim that $G\cap H=\emptyset$. If $G\cap H\neq\emptyset$, let $z\in G\cap H$. Then $z\in G\Rightarrow x\in Gy_1\cap Gy_2\cap\cdots\cap Gy_n$ and $z\in H\Rightarrow z\in Hyi$ for some yi. This implies $z\in Gy_i\cap Hy_i$, which is a contradiction. Therefore $G\cap H=\emptyset$. Hence X is strongly I-pseudo regular.

Theorem 3.6 [1]: The product I-space X of any non-empty collection $\{Xi\}$ of I-spaces is strongly I-pseudo regular if and only if each Xi is strongly I-pseudo regular.

Proof: Let $\{Xi\}$ be a non-empty collection of strongly I-pseudo regular spaces and $X=\prod Xi$. We show that X is strongly I-pseudo regular I-space. Let K be an I-compact set not containing a point $x\in X$. Let $Ki=\pi_i(K)$, $x_i\notin K_i$. Since the projection maps are I-continuous, $\pi_i(K)=Ki$ is an I-compact subset of Xi. Since $x\notin K$, there exists i_0 such that $xi_0\notin Ki_0$. Since Xi_0 is strongly I-pseudo regular, there exist I-open sets Gi_0, Hi_0 in Xi_0 such that $xi_0\in Hi_0$, $Ki_0\subseteq Gi_0$ and $Gi_0\cap Hi_0=\emptyset$. For each $i\neq i_0$, let Gi, Hi be I-open sets such that $xi\in Hi$, $Ki\subseteq Gi$. Let $G=\prod_i Gi$ and $H=\prod_i Hi$. Then $G\cap H=\emptyset$, since $Gi_0\cap Hi_0=\emptyset$ and $x\in H$, $K\subseteq G$. Hence X is strongly I-pseudo regular.

Theorem 3.7 [1]: Let X be a strongly I-pseudo regular I-space and R be an equivalence relation on X. Then R is an I-closed subset of $X\times X$.

Proof: We shall prove that R is an I-closed set or Rc is an I-open set of $X\times X$. So, let $(x,y)\in R^c$. It is sufficient to show that there exist two I-open sets G and H of X such that $x\in G$ and $y\in H$ and $G\times H\subseteq R^c$.

Let Rp:$X\to\frac{X}{R}$be the projection map. Since $(x,y)\in R^c$, $p(x)\neq p(y)$ i.e., $x\in p^{-1}(p(y))$. Again, since $\{y\}$ is I-compact and p is an I-continuous mapping, $p(y)$ is I-compact. Also, let $\{Gi\}$ be an I-open cover of $p^{-1}(p(y))$ in X, and let $G=p(Gi)$. Then $\{Gi\}$ is an I-open cover of $p(y)$ in $\frac{X}{R}$. Since $p(y)$ is a singleton element in $\frac{X}{R}$, there exists Gi_0 such that $p(y)\in Gi_0$ in $\frac{X}{R}$. Then by the definition of the I-space in $\frac{X}{R}$ and the nature of the map p, (i) Gi is an I-open in X, (ii) $G=p^{-1}(Gi$) and (iii) $p^{-1}(p(y))\subseteq G$ in X. Here $p^{-1}(p(y))$ is I-compact in X. So by the strongly I-pseudo regularity of X there exist I-open sets G and H in X such that $x\in G$ and $p^{-1}(p(y))\subseteq H$ and $G\cap H=\emptyset$. Hence $y\in p^{-1}(p(y))\subseteq H$ i.e., $y\in H$. $G\cap H=\emptyset$, $p(G)\cap p(H)=\emptyset$. Therefore $G\times H\subseteq R^c$ and so $(x,y)\in G\times H\subseteq R^c$. Hence R is an I-closed subset of $X\times X$.

Results and Discussion

The following results have been established:

1. Every strongly I-pseudo regular I-space is I-pseudo regular, but the converse is not true in general.
2. Any subspace of a strongly I-pseudo regular I-space is strongly I-pseudo regular.
3. An I-space X is strongly I-pseudo regular if for each $x\in X$ and any I-compact set K not containing x, there exists an I-open set H of X such that $x\in H\subseteq\overline{H}\subseteq K^C$
4. An I-space X is strongly I-pseudo regular if X is completely I-Hausdorff.
5. The product I-space X of any non-empty collection $\{Xi\}$ of I-spaces is strongly I-pseudo regular if and only if each Xi is strongly I-pseudo regular.
6. If X is a strongly I-pseudo regular I-space and R is an equivalence relation on X, then R is an I-closed subset of $X\times X$.
7. If X is a strongly I-pseudo regular I-space and R is an equivalence relation on X, then the quotient space $\frac{X}{R}$ is I-completely Hausdorff.

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