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RESEARCH ARTICLE

ON THE INTEGRAL SOLUTIONS OF THE BINARY QUADRATIC EQUATION $x^2 = 4(k^2+1)y^2 + 4^t$, $k, t \geq 0$

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ABSTRACT

The binary quadratic Diophantine equation represented by $x^2 = 4(k^2+1)y^2 + 4^t$, k, $t \geq 0$ is analyzed for its non-zero distinct integer solutions. Employing the lemma of Brahmagupta, infinitely many integral solutions of the above Pell equation are obtained. The recurrence relations on the solutions are also presented. A few interesting relations between the solutions and special number patterns namely, Polygonal numbers are also given. Further employing the integer solutions of the considered Pell equation, a special pattern of Pythagorean triangle is obtained.

Key words: Binary quadratic, Pell equation. Integer solutions
2010 Mathematics subject classification: 11D09

Notations:

$t_{m,n}$ - Polygonal number of rank n with size m

So_n - Stella octangular number of rank n

J_n - Jacobsthal number of rank n

j_n -Jacobsthal-Lucas number of rank n

KY_n -keynea number of rank n

INTRODUCTION

It is well known that the Pell equation $x^2 - Dy^2 = \pm 1$, (D>0 and square free) has always positive integer solutions. When $N \neq 1$, the Pell equation $x^2 - Dy^2 = N$ may not have any positive integer solutions. For example the equations $x^2 = 3y^2 - 1$ and $x^2 = 7y^2 - 4$ have no positive integer solutions. When k is a positive integer and $D \in \{k^2 \pm 4, k^2 \pm 1\}$, positive integer solutions of the equations $x^2 - Dy^2 = \pm 4$ and $x^2 - Dy^2 = \pm 1$ have been investigated by Jones in [4]. The same or similar equations are investigated in [3,6,9,10]. In [1,2,5,7,8,11,12,13] some specific Pell equation and their integer solutions are considered. In [14], the integer solutions of Pell equation $x^2 - (k^2 + k)y^2 = 2^t$ has been considered. In [15], the Pell equation $x^2 - (k^2 - k)y^2 = 2^t$ is analyzed for the integer solutions.

This communication concerns with the Pell equation $x^2 = 4(k^2 + 1)y^2 + 4^t$ and infinitely many positive integer solutions are obtained. The recurrence relations on the solutions are also given. A few interesting relations between the solutions and special numbers are presented.

2. METHOD OF ANALYSIS

The binary quadratic Diophantine equation representing a hyperbola to be solved for its distinct non-zero integral solutions is

$$x^2 = 4(k^2 + 1)y^2 + 4^t, \text{ k, t} \geq 0 \tag{1}$$

Let $(X_1, Y_1) = (2^t(2k^2 + 1), 2^t k)$ be the smallest positive integer solution to (1)

Consider the Pell's equation of (1) is given by $x^2 = 4(k^2 + 1)y^2 + 1$ (2)

Let $(\tilde{x}_{0,} \tilde{y}_0) = (2k^2 + 1, K)$ be the smallest positive integer solution to (2).

Then the general solution $(\tilde{x}_{n,} \tilde{y}_n)$ to (2) is given by

$$\tilde{x}_n = \frac{f_n}{2}$$

$$\tilde{y}_n = \frac{g_n}{2\sqrt{k^2 + 1}}$$

where

$$f_n = [((2k^2 + 1) + 2k\sqrt{k^2 + 1})^{n+1} + ((2k^2 + 1) - 2k\sqrt{k^2 + 1})^{n+1}]$$

$$g_n = [((2k^2 + 1) + 2k\sqrt{k^2 + 1})^{n+1} - ((2k^2 + 1) - 2k\sqrt{k^2 + 1})^{n+1}]$$

Employing the lemma of Brahmagupta between the solutions (X_1, Y_1) and $(\tilde{x}_n, \tilde{y}_n)$, the general solutions to (1) are given by

$$Y_{n+2} = 2^{t-1} k f_n + 2^{t-2} \frac{(2k^2 + 1)}{\sqrt{k^2 + 1}} g_n \tag{3}$$

$$X_{n+2} = 2^{t-1}(2k^2 + 1) f_n + 2^t k g_n \sqrt{k^2 + 1} \tag{4}$$

where $n=-1,0,1,2,3.........$

The recurrence relations satisfied by $(X_{n+2,}Y_{n+2})$ are correspondingly exhibited below:

$$X_{n+4}-(4k^2+2)X_{n+3}+X_{n+2}=0 \quad X_1=2^t(2k^2+1),\ X_2=2t(8k^4+8k^2+1)$$

$$Y_{n+4}-(4k^2+2)Y_{n+3}+Y_{n+2}=0 \quad Y_1=2^t k,\ Y_2=2^{t+1}k(2k^2+1)$$

3. Properties

$$(i)X_{n+3}=(2k^2+1)X_{n+2}+4k(k^2+1)Y_{n+2}$$

$$(ii)Y_{n+3}=kX_{n+2}+(2k^2+1)Y_{n+2}$$

$$(iii)X_{n+4}=(8k^4+8k^2+1)X_{n+2}+8k(k^2+1)(2k^2+1)Y_{n+2}$$

$$(iv)Y_{n+4}=2k(2k^2+1)X_{n+2}+(8k^4+8k^2+1)Y_{n+2}$$

$$(v)X_{n+4}=(2k^2+1)X_{n+3}+4k(k^2+1)Y_{n+3}$$

$$(vi)Y_{n+4}=kX_{n+3}+(2k^2+1)Y_{n+3}$$

$$(vii)Y^2{}_{n+3}-Y_{n+2}Y_{n+4}=k[X_{n+2}Y_{n+3}-X_{n+3}Y_{n+2}]$$

$$(viii)X_{n+3}Y_{n+3}-X_{n+2}Y_{n+4}=(2k^2+1)[X_{n+3}Y_{n+2}-X_{n+2}Y_{n+3}]$$

$$(ix)Y_{n+4}-kX_{n+3}=Y_{n+3}(2t_{4,k}+1)$$

$$(x)Y_{n+4}-(8k^4+8k^2+1)Y_{n+2}=X_{n+2}(2SO_k+4k)$$

$$(xi)(Y_{n+3}-kX_{n+2})^2=Y^2{}_{n+2}(8t_{3,k^2}+1)$$

(x) $$[2(2k^2+1)X_{n+2}-8k(k^2+1)Y_{n+1}]^2-(k^2+1)[4(2k^2+1)Y_{n+2}-4kX_{n+2}]^2 = 4(j_{2t}-1)$$

$$(xi)[2(2k^2+1)X_{n+2}-8k(k^2+1)Y_{n+1}]^2-(k^2+1)[4(2k^2+1)Y_{n+2}-4kX_{n+2}]^2 = 4(3J_{2t}+1)$$

$$(xii)[2(2k^2+1)X_{n+2}-8k(k^2+1)Y_{n+1}]^2-(k^2+1)[4(2k^2+1)Y_{n+2}-4kX_{n+2}]^2 = 4(KY_t-j_{t+1})$$ when t is even

$$(xiii)X_{n+4}-8k(k^2+1)(2k^2+1)Y_{n+2}=X_{n+2}(1+16t_{3,k^2})$$

(xiv) Each of the following is a nasty number:

$$(a)6[\frac{kX_{n+4}-k(2k^2+1)X_{n+3}+8k^3Y_{n+3}}{Y_{n+3}}]$$

$$(b)6[\frac{kX_{n+3}-k(2k^2+1)X_{n+2}+8k^3Y_{n+2}}{Y_{n+2}}]$$

$$(c)6[\frac{2Y_{n+4}-2kX_{n+3}-2Y_{n+3}}{Y_{n+3}}]$$

(xv) For the values of k given by $k=\frac{1}{2\sqrt{2}}[(3+2\sqrt{2})^{n+1}-(3-2\sqrt{2})^{n+1}]$ n=0,1,2…….

each of the following expressions is a perfect square

$(a)\frac{Y_{n+4}-kX_{n+3}}{Y_{n+3}}$

$(b)\frac{X_{n+4}-4k(k^2+1)Y_{n+3}}{X_{n+3}}$

$(c)\frac{Y_{n+3}-kX_{n+2}}{Y_{n+2}}$

$(d)\frac{X_{n+3}-4k(k^2+1)Y_{n+2}}{X_{n+2}}$

4. APPLICATIONS

(I) Define $r=X_{n+2}+\frac{Y_{n+2}}{2}$, $s=\frac{Y_{n+2}}{2}$ where (X_{n+2},Y_{n+2}) is any solution of (1). Note that r and s are integers and $r>s>0.$ Treat r and s as the generators of the Pythagorean triangle $T(\alpha,\beta,\gamma)$, where $\alpha=2rs,\ \beta=r^2-s^2,\ \gamma=r^2+s^2$. Let A and P represent its area and perimeter respectively. Then this Pythagorean triangle T is such that

$(i)\left|8\beta(k^2+1)-\alpha-(8k^2+7)\gamma\right|\equiv 0(\mathrm{mod}4^t)$

$(ii)\gamma-\alpha(8k^2+9)+\frac{32A(k^2+1)}{P}\equiv 0(\mathrm{mod}4^t)$

(II) Let x and y be taken as the sides of a rectangle R whose length of the diagonal, Perimeter and area are denoted by L, P and A respectively. Note that,

$(i)\left|L^2-(j_{2t}-1)\right|-5y^2$ is a perfect square

$(ii)P^2-8A=4L^2$

5. CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

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