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RESEARCH ARTICLE

INTEGRAL POINTS ON THE TERNARY QUADRATIC DIOPHANTINE EQUATION $2x^2+9y^2=11z^2$

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ABSTRACT

The ternary quadratic homogeneous equation representing homogeneous cone given by $2x^2+9y^2=11z^2$ is analyzed for its non-zero distinct integer points on it . Three different patterns of integer points satisfying the cone under consideration are obtained. A few interesting relations between the solutions and special number patterns namely Polygonal number , Pyramidal number , Octahedral number, Pronic number ,Stella octangular number , Pentatope number and Nasty number are presented. Also knowing an integer solution satisfying the given cone , three triples of integers generated from the given solution are exhibited.

Keywords: Ternary homogeneous quadratic, integral solutions

2010 Mathematics Subject Classification: 11D09

INTRODUCTION

The ternary quadratic Diophantine equations offer an unlimited field for research due to their variety [1, 20]. For an extensive review of various problems, one may refer [2-19]. This communication concerns with yet another interesting ternary quadratic equation $2x^2+9y^2=11z^2$ representing a cone for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

Notations:

P_n^m - Pyramidal number of rank n with size m.

$T_{m,n}$ - Polygonal number of rank n with size m.

$\Pr_n$ - Pronic number of rank n

OH_n - Octahedral number of rank n

SO_n - Stella octangular number of rank n

Pt_n - Pentatope number of rank n

METHOD OF ANALYSIS:

The ternary quadratic equation representing a cone under consideration is

$$2x^2 + 9y^2 = 11z^2 \quad (1)$$

we present below different patterns of non-zero distinct integer solutions of (1)

Pattern :1

Let $z = z(a,b) = 2a^2 + 9b^2$,, (a, b $\neq$ 0) (2)

write 11 as ,

$$11 = (3 + i\sqrt{2})(3 - i\sqrt{2}) \quad (3)$$

using (2) and (3) in (1) and applying the method of factorization, define

$$(3y + i\sqrt{2}x) = (3 + i\sqrt{2})(3b + i\sqrt{2}a)^2$$

Equating the rational and irrational parts we get,

$$\left.\begin{aligned} x = x(a,b) &= 9b^2 - 2a^2 + 18ab \\ y = y(a,b) &= 9b^2 - 2a^2 - 4ab \end{aligned}\right\} \quad (4)$$

Thus (2) and (4) represents non-zero distinct integral solutions of (1) in two parameters.

Properties:

- $x(a+1,a) + z(a+1,a) - T_{38,a} - 18P_{r,a} \equiv 0 (\mathrm{mod} 17)$
- $x(a(a+1),(2a+1)) - y(a(a+1),(2a+1)) = 132P_a^4$
- $x(2a^2+1,a) + z(2a^2+1,a) - T_{38,a} - 17a = 54OH_a$
- $z(a,2a^2+1) - x(a,2a^2+1) - T_{10,a} - 3a + 54OH_a = 0$
- $x(2a^2-1,a) + z(2a^2-1,a) - T_{38,a} - 17a = 18OS_a$

Each of the following represents a nasty number

- $22\{x(a,a) - y(a,a) + z(a,a)\}$
- $3\{z(a,a) - y(a,a)\}$
- $6\{x(a,a) + z(a,a)\}$
- $3\{y(2a^2+1,a) + z(2a^2+1,a) - 12OH_a\}$

Note:

Instead of (3) , 11 is written as

$$11 = \frac{(1 + i7\sqrt{2})(1 - i7\sqrt{2})}{9} \quad (5)$$

Following the procedure as above, the non-zero distinct integer solutions are given by ,

$$\left.\begin{aligned} x = x(a,b) &= 189b^2 - 42a^2 + 18ab \\ y = y(a,b) &= 9b^2 - 2a^2 - 84ab \\ z = z(a,b) &= 81b^2 + 18a^2, \end{aligned}\right\} \quad (6)$$

Properties:

- $3x(a,b)+7z(a,b)\equiv 0(\text{mod}6)$
- $9y(2a^2-1,a)+z(2a^2-1,a)-T_{38,a}+84SO_a\equiv 0(\text{mod}17)$
- $3x(2a^2-1,a)+7z(2a^2-1,a)-T_{254,a}-125a=6SO_a$
- $z(a,(a+1)(a+2))-9y(a,(a+1)(a+2))-T_{10,a}-3a=504P_a^3$

Each of the following represents a nasty number

- $2\{3x(a,a)-9y(a,a)\}$
- $3\{9y(a+1,a)+z(a+1,a)+168\Pr_a\}$

Pattern : 2

The ternary quadratic equation (1) can be written as

$$2x^2=11z^2-9y^2 \tag{7}$$

Assume x= $x(a,b)=11a^2-9b^2$,, (a, b ≠ 0) (8)

write 2 as , $2=(\sqrt{11}+3)(\sqrt{11}-3)$ (9)

using (8) and (9) in (1) and applying the method of factorization, define

$$(\sqrt{11}+3)(\sqrt{11}a+3b)^2=(\sqrt{11}z+3y)$$

Equating the real and imaginary parts we get ,

$$\left.\begin{aligned} y=y(a,b)&=11a^2+9b^2+22ab \\ z=z(a,b)&=11a^2+9b^2+18ab \end{aligned}\right\} \tag{10}$$

Thus (9) and (10) represents non-zero distinct integral solutions of (1) in two parameters.

Properties:

- $x(a,b)+y(a,b)\equiv 0(\text{mod}22)$
- $x(a,a)+y(a,a)+z(a,a)\equiv 0(\text{mod}82)$
- $y(a(a+1),(a+2)(a+3))-z(a(a+1),(a+2)(a+3))=96Pt_a$
- $x(a,a+1)+z(a,a+1)-T_{46,a}-21a=18\Pr_a$

Each of the following represents a nasty number

- $7y(a,a)$
- $3\{y(a(a+1),2a+1)-x(a(a+1),2a+1)-132P_a^4\}$
- $3\{y(2a^2-1,a)-x(2a^2-1,a)-22SO_a\}$
- $y(a,6a)-z(a,6a)$

Note:

Instead of (7) , 2 is written as

$$2=\frac{(3\sqrt{11}+7)(3\sqrt{11}-7)}{25} \tag{11}$$

Following the procedure as above , the non-zero distinct integer solutions are given by ,

$$\left.\begin{aligned} x=x(a,b)&=2475a^2-2025b^2 \\ y=y(a,b)&=1155a^2+945b^2+2970ab \\ z=z(a,b)&=1485a^2+1215b^2+1890ab, \end{aligned}\right\} \tag{12}$$

Properties:

- $7x(a,a)+15y(a,a)\equiv 0(\text{mod}11)$

- $15y(a,b)-7x(a,b)\equiv 0(\mathrm{mod}9)$
- $5z(a+1,a)-3x(a+1,a)-T_{110,a}-126T_{3,a}\equiv 0(\mathrm{mod}53)$
- $15y(a+1,a)-7x(a+1,a)-T_{254,a}-125a=198\mathrm{Pr}_a$
- $15y(2a^2+1,a)-7x(2a^2+1,a)-T_{254,a}-125a=594OH_a$

Each of the following represents a nasty number

- $5z(a,a)-3x(a,a)$
- $5z(a+1,a)-3x(a+1,a)-42\mathrm{Pr}_a$
- $5z(2a^2+1,a)-3x(2a^2+1,a)-126OH_a$

Pattern:3

Introducing the linear transformations ,

$$\left.\begin{aligned} z&=U+2T\\ x&=U+11T\end{aligned}\right\} \quad (13)$$

In equation (1) it is written as ,

$$U^2=y^2+22T^2$$

which is satisfied by ,

$$U=22r^2+s^2 \quad , T=2rs \quad (14)$$

$$y=22r^2-s^2 \quad (15)$$

Substituting the above values of T ,U in (13) the corresponding non-zero distinct integer values of x and z are given by ,

$$\left.\begin{aligned} x=x(r,s)&=22r^2+s^2+22rs\\ z=z(r,s)&=22r^2+s^2+4rs\end{aligned}\right\} \quad (16)$$

Thus (15) and (16) represent the non-zero distinct integer solution of (1) in two parameters.

Properties:

- $x(r,s)+y(r,s)\equiv 0(\mathrm{mod}22)$
- $x(r,2r^2-1)-z(r,2r^2-1)=18SO_r$
- $x(r,(r+1)(r+2))-z(r,(r+1)(r+2))=108P_r^3$
- $x(2r^2-1,r)-y(2r^2-1,r)-T_{6,r}-r=22SO_r$
- $y(r,r+1)+z(r,r+1)-T_{90,r}-43r=4\mathrm{Pr}_r$

Each of the following represents a nasty number

- $3\{x(r,r)-z(r,r)\}$
- $z(r,r)-y(r,r)$
- $3\{x(r,r+1)-y(r,r+1)-22\mathrm{Pr}_r\}$

REMARKABLE OBSERVATIONS:

A: Let (x_0,y_0,z_0) be the positive initial solution of (1) .Then each of the following three triples of integers based on x_0,y_0,z_0 also satisfy (1) .

Triple:1 (x_0,y_n,z_n)

where ,

$$y_n = \frac{1}{2}\{[11(1)^n - 9(-1)^n]y_0 + z_0[-11(1)^n + 11(-1)^n]\}$$

$$z_n = \frac{1}{2}\{[9(1)^n - 9(-1)^n]y_0 + z_0[-9(1)^n + 11(-1)^n]\}$$

Triple:2 (x_n, y_n, z_n)

where ,

$$x_n = \frac{1}{18}\{[22(9)^n - 4(-9)^n]x_0 + z_0[-22(9)^n + 22(-9)^n]\}$$

$$y_n = 9^n y_0$$

$$z_n = \frac{1}{18}\{[4(9)^n - 4(-9)^n]x_0 + z_0[-4(9)^n + 22(-9)^n]\}$$

Triple:3 (x_n, y_n, z_n)

where ,

$$x_n = \frac{1}{22}\{[18(11)^n + 4(-11)^n]x_0 + y_0[-18(11)^n + 18(-11)^n]\}$$

$$y_n = \frac{1}{22}\{[-4(11)^n + 4(-11)^n]x_0 + y_0[4(11)^n + 18(-11)^n]\}$$

$$z_n = 11^n z_0$$

B: Employing the solutions (x ,y ,z) of (1) each of the following relations among the special polygonal and pyramidal numbers are observed.

- ❖ $9\left[\frac{P_y^5}{t_{3,y}}\right]^2 - 11\left[\frac{3P_{z-2}^3}{t_{3,z}}\right]^2 \equiv 0(\text{mod}2)$
- ❖ $2\left[\frac{3P_{x-2}^3}{t_{3,x}}\right]^2 + 9\left[\frac{P_y^5}{t_{3,y}}\right] \equiv 0(\text{mod}11)$
- ❖ $22\left[\frac{P_x^5}{t_{3,x}}\right]^2 + 99\left[\frac{3P_{y-2}^3}{t_{3,y}}\right]^2$ is a perfect square
- ❖ $2\left[\frac{3P_{x-2}^3}{t_{3,x}}\right]^2 - 11\left[\frac{P_z^5}{t_{3,y}}\right]^2$ is a perfect square
- ❖ $6\left\{-2\left[\frac{P_x^5}{t_{3,x}}\right]^2 + 11\left[\frac{3P_{z-2}^3}{t_{3,z}}\right]^2\right\}$ a nasty number

CONCLUSION

In this paper, we have presented three different patterns of non-zero distinct integer solutions of the homogeneous cone given by $2x^2 + 9y^2 = 11^2$.To conclude, one may search for other patterns of solution and their corresponding properties.

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