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RESEARCH ARTICLE

ON TERNARY QUADRATIC DIOPHANTINE EQUATION $7x^2 + 9y^2 = z^2$

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ABSTRACT

The ternary quadratic diophantine equation $7x^2 + 9y^2 = z^2$ is analyzed for its non-zero distinct integral points on it. A few interesting properties among the solutions are presented.

Key words: Integral points, Ternary quadratic, Polygonal numbers, Pyramidal numbers and Special numbers.

Notation: $t_{m,n}$ = Polygonal number of rank n with sides m

p_m^n = Pyramidal number of rank n with sides m

$ct_{m,n}$ = Centered Polygonal number of rank n with sides m

cp_m^n = Centered Pyramidal number of rank n with sides m

p_n = Pronic number

g_n = Gnomonic number

Tha_n = Thabit-ibn-Kurrah number

$carl_n$ = Carol number

mer_n = Mersenne number

ky_n = Kynea number

H_n = Hilbert number

PEN_n = Pentatope number

INTRODUCTION

The ternary quadratic diophantine equation offers an unlimited field for research because of their variety [1-2]. For an extensive review of various problems one may refer [3-11]. In this context one may also see [12-19]. This communication concerns with yet another interesting ternary quadratic diophantine equation $7x^2+9y^2=z^2$ for determining its infinitely many non-zero integral solutions. Also a few interesting properties among the solutions are presented.

2. Method of analysis:

The ternary quadratic diophantine equation is

$$7x^2+9y^2=z^2 \tag{1}$$

We present below different patterns of non-zero distinct integral solutions to (1).

Pattern 1:

Assume $z=9a^2+7b^2$ (2)

where $a,b\succ 0$

using (2) in (1), we get

$$9y^2+7x^2=9a^2+7b^2$$

On employing the method of factorization and on equating real and imaginary parts, we get

$$\left.\begin{aligned} x &= x(a,b)=6ab \\ y &= y(a,b)=\frac{1}{3}\left(9a^2-7b^2\right) \end{aligned}\right\} \tag{3}$$

Thus (2), and (3) represents non-zero distinct integral solutions of (1).

As our interest centers on finding integer solutions, it is seen that y is an integer for suitable choices of a and b . A few illustrations are given below:

Case 1:

Assume $a=3A$, $b=3B$

The corresponding solutions of (1) are

$$x=x(A,B)=54AB$$

$$y=y(A,B)=27A^2-21B^2$$

$$z=z(A,B)=81A^2+63B^2$$

Properties:

1) $x(A,1)+2y(A,1)+42=108p_n^5$

2) $z(A,1)-x(A,1)=3t_{56,A}+12g_A+75$

3) $x\left(n,5n^2+1\right)=324cp_n^5$

4) $x\left(2^n,1\right)=18\left(Tha_n+1\right)$

5) $y\left(2^n,2^n\right)-3\left(ky_n+carl_n\right)$ is Nasty number

Case 2:

Assume b=3na

The corresponding solutions of (1) are

$$x = x(a,n) = 18na^2$$
$$y = y(a,n) = 3a^2(1-7n^2)$$
$$z = z(a,n) = 9a^2(1+7n^2)$$

Properties:

1) $x(n,1) - y(n,1) - s_n - 16t_{3,n+1} - n^2 + 20$
2) $z(a,1) - x(a,1) = x(a,1) + y(a,1)$
3) $x(1,a+1) - 2ct_{18,n} \equiv 16(\text{mod}18)$
4) $x(1,2^n) - Tha_n$ is Nasty number
5) $3y(a,n) + z(a,n) - 17a = t_{38,a}$

Pattern 2:

(1) is written as

$$7x^2 + 9y^2 = z^2 * 1 \quad (4)$$

Assume $z = 7a^2 + 9b^2$ (5)

where $a, b \succ 0$

write 1 as

$$1 = \frac{(\sqrt{7}+i3)(\sqrt{7}-i3)}{16} \quad (6)$$

Substituting (5) and (6) in (4) and on employing the method of factorization, we get

$$(\sqrt{7}x + i3y)(\sqrt{7}x - i3y) = (\sqrt{7}a + i3b)^2 (\sqrt{7}a - i3b)^2 \frac{(\sqrt{7}+i3)(\sqrt{7}-i3)}{16}$$

On equating positive and negative factors and on comparing real and imaginary parts we get,

$$x = x(a,b) = \frac{1}{4}(7a^2 - 9b^2 - 18ab)$$
$$y = y(a,b) = \frac{1}{4}(7a^2 - 9b^2 + 14ab)$$

Case 1:

Let a=2A, b= 2B

The corresponding solutions are

$$x = x(A,B) = 7A^2 - 9B^2 - 18AB$$
$$y = y(A,B) = 7A^2 - 9B^2 + 14AB$$
$$z = z(A,B) = 28A^2 + 36B^2$$

Properties:

1) $x(A,1) - y(A,1) + z(A,1) = 2t_{30,A} - 3g_A + 33$
2) $x(A^2, A) - y(A^2, A) = 32cp_A^6$
3) $x(A, A^2+1) - y(A, A^2+1) = 64cp_A^3$

4) $y(1,B)+t_{20,B}-3g_B=10$

Case 2:

Let $a=(2n-1)b$

The corresponding solutions are

$$x=x(a,b)=b^2\left(7n^2+4-16n\right)$$

$$y=y(a,b)=b^2\left(7n^2-4\right)$$

$$z=z(a,b)=28n^2b^2+16b^2-28nb^2$$

Properties:

1) $y(1,n+1)-2ct_{7,n}\equiv 1 \pmod 7$

2) $z(n,n)=56t_{3,n^2}-24p_n^5-16cp_n^6$

3) $2x(1,n)-s_n-2t_{3,n}-t_{16,n}\equiv 7 \pmod{21}$

Pattern 3:

Write (1) as $z^2-(3y)^2=7x^2$ (7)

write (7) as

$$\frac{z+3y}{7x}=\frac{x}{z-3y}=\frac{p}{q} \quad (8)$$

This is equivalent to the following equations

$$\left.\begin{aligned} 7zp-3yq-qz&=0 \\ qx+3yp-pz&=0 \end{aligned}\right\} \quad (9)$$

Applying the method of cross multiplication we get the values of x, y, z represents non-zero distinct values of (1) we get

$$x=x(p,q)=6pq$$

$$y=y(p,q)=7p^2-q^2$$

$$z=z(p,q)=21p^2+3q^2$$

Properties:

1) $x(2^n,1)-4=Tha_n+mer_n$

2) $3y(p,q)-z(p,q)+x(p,q)=0$

3) $x(p^2,p)-y(p,1)-2p_p^{20}+t_{18,p}\equiv 1 \pmod 2$

Pattern 4:

Write (1) as $z^2-7x^2=9y^2$ (10)

Let $y=a^2-7b^2$ (11)

where $a,b\succ 0$

write 9 as

$$9=\left(4+\sqrt{7}\right)\left(4-\sqrt{7}\right) \quad (12)$$

Substitute (11) and (12) in (10) we get,

$$\left(z+\sqrt{7}x\right)\left(z-\sqrt{7}x\right)=\left(4+\sqrt{7}\right)\left(4-\sqrt{7}\right)\left(a+\sqrt{7}b\right)^2\left(a-\sqrt{7}b\right)^2$$

On equating the positive and negative factors, we get

$$\left(z+\sqrt{7}x\right)=\left(4+\sqrt{7}\right)\left(a+\sqrt{7}b\right)^2 \tag{13}$$

$$\left(z-\sqrt{7}x\right)=\left(4-\sqrt{7}\right)\left(a-\sqrt{7}b\right)^2 \tag{14}$$

in (13), on equating the rational and irrational parts, we have

$$\left.\begin{aligned} x &= x(a,b) = a^2+7b^2+8ab \\ z &= z(a,b) = 4a^2+28b^2+14ab \end{aligned}\right\} \tag{15}$$

Thus (11) and (15) represents non-zero distinct integral solutions of (1).

Properties

1) $x\left(a^2,a\right)-y\left(a,1\right)-24PEN_a-4p_a^5+t_{40,a}\equiv 7\left(\mathrm{mod}\,24\right)$
2) $y\left(a^2,a\right)+14a^2-p_{a^2}$ is Nasty number
3) $z\left(a,1\right)-ct_{8,a}-5g_a=32$

Pattern 5:

Consider $x=X+9T$
and $y=X-7T$ (16)

Substituting (16) in (1), we get

$$z^2=16X^2+16\left(63T^2\right) \tag{17}$$

Taking $z=4w$ (18)

in (20), we get

$$w^2=X^2+63T^2 \tag{19}$$

Thus, we have the following integer solutions to (19) as represented below:

$$\left.\begin{aligned} w &= 63r^2+s^2 \\ T &= 2rs \\ X &= 63r^2-s^2 \end{aligned}\right\} \tag{20}$$

Substituting (20) in (16) and (18) we get the values of x, y, z represents non-zero distinct integral solutions to (1).

$$x=63r^2-s^2+18rs \tag{26}$$

$$y=63r^2-s^2-14rs \tag{27}$$

$$z=4\left(63r^2+s^2\right) \tag{28}$$

Properties:

1) $z\left(1,s\right)-4x\left(1,s\right)-ct_{16,s}\equiv -1\left(\mathrm{mod}\,26\right)$
2) $x\left(r,1\right)+y\left(r,1\right)-21ct_{12,n}\equiv -3\left(\mathrm{mod}\,122\right)$
3) $z\left(1,s\right)-8t_{3,s}+H_s=253$

3.Generation in solutions:

Let (x_0,y_0,z_0) be the initial solution of (1). Then, each of the following triples of non-zero distinct integers based on x_0,y_0 and z_0 also satisfies (1)

Triple 1: $\left(16^{2n-2}\left(-2x_0+18y_0\right),16^{2n-2}\left(14x_0+2y_0\right),16^{2n-1}z_0\right)$

Triple 2: $\left(x_n, y_0, z_n\right)$

Here $x_n = \frac{1}{2\sqrt{7}}\left[\sqrt{7}\left(\alpha^n+\beta^n\right)x_0+\left(\alpha^n-\beta^n\right)z_0\right]$

$z_n = \frac{1}{2\sqrt{7}}\left[7\left(\alpha^n-\beta^n\right)x_0+\sqrt{7}\left(\alpha^n+\beta^n\right)z_0\right]$

where $\alpha = 8+3\sqrt{7}$ and $\beta = 8-3\sqrt{7}$

Triple 3: $\left(x_n, y_n, z_n\right)$

Here $x_n = 4^{2n-1}x_0$

$y_n = \frac{1}{6}\left[3\left(\alpha^n+\beta^n\right)y_0+\left(\alpha^n-\beta^n\right)z_0\right]$

$z_n = \frac{1}{2}\left[3\left(\alpha^n-\beta^n\right)y_0+\left(\alpha^n+\beta^n\right)z_0\right]$

where $\alpha = 8$ and $\beta = 2$

Triple 4: $\left(x_n, y_n, z_n\right)$

Here $x_n = \frac{1}{2\sqrt{7}}\left[\sqrt{7}\left(\alpha^n+\beta^n\right)x_0+\left(\alpha^n-\beta^n\right)z_0\right]$

$y_n = 4^{2n-1}y_0$

$z_n = \frac{1}{2\sqrt{7}}\left[7\left(\alpha^n-\beta^n\right)x_0+\sqrt{7}\left(\alpha^n+\beta^n\right)z_0\right]$

where $\alpha = 4+\sqrt{7}$ and $\beta = 4-\sqrt{7}$

4. CONCLUSION: One may search for other patterns of solution and their corresponding properties.

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