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RESEARCH ARTICLE

INTEGER SOLUTIONS ON THE HOMOGENEOUS CONE $4x^2 + 3y^2 = 28z^2$

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ABSTRACT

The ternary quadratic homogeneous equation representing homogeneous cone given by $4x^2 + 3y^2 = 28z^2$ is analyzed for its non-zero distinct integer points on it. Six different patterns of integer points satisfying the cone under consideration are obtained. A few interesting relations between the solutions and special number patterns namely Polygonal number, Pyramidal number and Nasty number are presented. Also, knowing an integer solution satisfying the given cone, two triples of integers generated from the given solution are exhibited.

Keywords: Ternary homogeneous quadratic, integral solutions

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INTRODUCTION

The ternary quadratic Diophantine equations offer an unlimited field for research due to their variety [1, 20]. For an extensive review of various problems, one may refer [2-19]. This communication concerns with yet another interesting ternary quadratic equation $4x^2 + 3y^2 = 28z^2$ representing a cone for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

NOTATIONS:

P_n^m - Pyramidal number of rank n with size m.

$T_{m,n}$ -Polygonal number of rank n with size m.

2. METHOD OF ANALYSIS:

The ternary quadratic equation to be solved for its non-zero distinct integer solutions is

$$4x^2+3y^2=28z^2 \tag{1}$$

Assume $z=z(a,b)=4a^2+3b^2\,;a,b>0$ (2)

We illustrate below six different patterns of non-zero distinct integer solutions to (1)

2.1 PATTERN: 1

The ternary quadratic equation (1) can be written as

$$4x^2-25z^2=3z^2-3y^2 \tag{3}$$

Factorizing (3) we have

$$(2x+5z)(2x-5z)=3(z+y)(z-y) \tag{4}$$

which is equivalent to the system of double equations

$$\left.\begin{aligned}2bx-ay+(5b-a)z=0\\2ax+3by-(3b+5a)z=0\end{aligned}\right\} \tag{5}$$

Applying the method of cross multiplication, we get

$$\left.\begin{aligned}x&=x(a,b)=5a^2-15b^2+6ab\\y&=y(a,b)=-2a^2+6b^2+20ab\\z&=z(a,b)=2a^2+6b^2\end{aligned}\right\} \tag{6}$$

Thus (6) represents non-zero distinct integral solutions of (1) in two parameters.

PROPERTIES:

1. $y(1,b)+z(1,b)-T_{16,b}-T_{12,b}\equiv 0(\mathrm{mod}30)$
2. $2x(a,a+1)+5y(a,a+1)=224T_{3,a}$
3. $2x(a,a(a+1))+5y(a,a(a+1))=224P_a^5$
4. Each of the following represents a Nasty Number
   - **a)** $y(1,5\alpha^2)+z(1,5\alpha^2)-T_{16,5\alpha^2}-T_{12,5\alpha^2}$
   - **b)** $3\{z(a,1)-6\}$
   - **c)** $z(1,b)-2$

2.2 PATTERN: 2

Also (4) is equivalent to the following two equations

$$\left.\begin{aligned}2bx-3ay-(5b+3a)z=0\\2ax+by+(5a-b)z=0\end{aligned}\right\} \tag{7}$$

Repeating the process as in pattern: 1, the corresponding non-zero distinct integer solutions of (1) are given by

$$\left.\begin{aligned}x&=x(a,b)=-15a^2+5b^2+6ab\\y&=y(a,b)=-6a^2+2b^2-20ab\\z&=z(a,b)=6a^2+2b^2\end{aligned}\right\} \tag{8}$$

Thus (8) represents non-zero distinct integer solutions of (1)

PROPERTIES:

1. $y(a,1)+z(a,1)\equiv 0(\mathrm{mod}4)$

2. $y(1,b)+z(1,b)-T_{10,b}\equiv 0(\mathrm{mod}17)$
3. $2x(a,1)-5y(a,1)=0(\mathrm{mod}112)$
4. $2x(a,a+1)-5y(a,a+1)=224T_{3,a}$
5. $2x(a^2,a+1)-5y(a^2,a+1)=224P_a^5$
6. $x(a,1)+30T_{3,a}\equiv 5(\mathrm{mod}21)$
7. Each of the following represents a Nasty Number
   a) $6\{-y(a,a)-z(a,a)\}$
   b) $z(a,1)-2$
   c) $12\{z(1,b)-6\}$
   d) $12\{z(b,b)-y(b,b)\}$
   e) $42\{2x(a,a)-5y(a,a)\}$
   f) $14\{10x(b,3b)+3y(b,3b)\}$

2.3 PATTERN: 3

Write 28 as

$$28=\left(4+i2\sqrt{3}\right)\left(4-i2\sqrt{3}\right) \tag{9}$$

Substituting (2) and (9) in (1) and employing the method of factorization, define

$$\left(2x+i\sqrt{3}y\right)\left(2x-i\sqrt{3}y\right)=\left(4+i2\sqrt{3}\right)\left(4-i2\sqrt{3}\right)\left(2a+i\sqrt{3}b\right)^2\left(2a-i\sqrt{3}b\right)^2 \tag{10}$$

Equating real and imaginary parts, we have

$$\left.\begin{aligned} x=x(a,b)&=8a^2-6b^2-12ab\\ y=y(a,b)&=8a^2-6b^2+16ab\end{aligned}\right\} \tag{11}$$

Thus (11) and (2) represent non-zero distinct integral solutions of (1) in two parameters.

PROPERTIES:

1. $x(a,1)-T_{18,a}+6\equiv 0(\mathrm{mod}5)$
2. $y(a,1)-x(a,1)\equiv 0(\mathrm{mod}28)$
3. $2z(b^2,b)-y(b^2,b)+32P_b^5-T_{58,b}\equiv 0(\mathrm{mod}27)$
4. $y(1,b)-2z(1,b)+6T_{6,b}\equiv 0(\mathrm{mod}10)$
5. $x(1,b)+2z(1,b)\equiv 0(\mathrm{mod}4)$
6. Each of the following represents a Nasty Number
   a) $3\{x(a,a)+y(a,a)\}$
   b) $6\{y(a,a)+z(a,a)\}$
   c) $14\{z(a,-a)-y(a,-a)\}$
   d) $42\{x(a,-a)-y(a,-a)\}$

2.4 PATTERN: 4

Instead of (9), write 28 as

$$28=\left(5+i\sqrt{3}\right)\left(5-i\sqrt{3}\right) \tag{12}$$

Following the procedure presented in pattern: 3, the corresponding values of x and y obtained from (1) are

$$\left.\begin{aligned} x &= x(a,b) = \frac{1}{2}\left[20a^2 - 15b^2 - 12ab\right] \\ y &= y(a,b) = 4a^2 - 3b^2 + 20ab \end{aligned}\right\} \quad (13)$$

The choice b=2B in (2) and (13) leads to

$$\left.\begin{aligned} x &= x(a,B) = 10a^2 - 30B^2 - 12aB \\ y &= y(a,B) = 4a^2 - 12B^2 + 40aB \\ z &= z(a,B) = 4a^2 + 12B^2 \end{aligned}\right\} \quad (14)$$

which represent non-zero distinct integer solutions of (1) in two parameters.

PROPERTIES:

1. $2x(-a,1) - 5y(-a,1) \equiv 0(\mathrm{mod}\,224)$
2. $2x(1,B) - 5z(1,B) + 60T_{6,B} \equiv 0(\mathrm{mod}\,84)$
3. $y(a,1) + z(a,1) - T_{18,a} \equiv 0(\mathrm{mod}\,47)$
4. $5y(a,a+1) - 2x(a,a+1) = 448T_{3,a}$
5. $2x(-a,a(a+1)) - 5y(-a,a(a+1)) = 448P_a^5$
6. $5y(a,(a+1)(a+2)) - 2x(a,(a+1)(a+2)) = 1344P_a^3$
7. $2x(a,1) - 5z(a,1) \equiv 0(\mathrm{mod}\,24)$
8. Each of the following represents a Nasty Number
   a) $6\{z(a,1) - 12\}$
   b) $2\{z(1,B) - 4\}$

2.5 PATTERN: 5

Instead of (9), write 28 as

$$28 = \left(1 + i3\sqrt{3}\right)\left(1 - i3\sqrt{3}\right) \quad (15)$$

Following the procedure presented in pattern:3, the corresponding values of x and y obtained from (1) are

$$\left.\begin{aligned} x &= x(a,b) = \frac{1}{2}\left[4a^2 - 3b^2 - 36ab\right] \\ y &= y(a,b) = 12a^2 - 9b^2 + 4ab \end{aligned}\right\} \quad (16)$$

The choice b=2B in (2) and (16) leads to

$$\left.\begin{aligned} x &= x(a,B) = 2a^2 - 6B^2 - 36aB \\ y &= y(a,B) = 12a^2 - 36B^2 + 8aB \\ z &= z(a,B) = 4a^2 + 12B^2 \end{aligned}\right\} \quad (17)$$

which represents non-zero distinct integer solutions of (1) in two parameters.

PROPERTIES:

1. $6x(a,1) - y(a,1) \equiv 0(\mathrm{mod}\,224)$
2. $2x(a,1) + z(a,1) - T_{18,a} \equiv 0(\mathrm{mod}\,65)$
3. $z(1,B) - 2x(1,B) - 8T_{8,B} \equiv 0(\mathrm{mod}\,88)$
4. Each of the following represents a Nasty Number
   a) $6\{-2x(a,a) - z(a,a)\}$

b) $6\{x(a,-a)-z(a,-a)\}$

c) $6\{-y(a,-a)-z(a,-a)\}$

d) $3\{y(a,a)+3z(a,a)\}$

2.6 PATTERN: 6

Introducing the linear transformation

$$\left.\begin{array}{l} x=5\overline{X} \\ y=X\pm 28T \\ z=X\pm 3T \end{array}\right\} \qquad (18)$$

in (1), it gives

$$X^2=84T^2+\left(2\overline{X}\right)^2$$

which is satisfied by

$$T=4rs$$
$$\overline{X}=42r^2-2s^2$$
$$X=84r^2+4s^2$$

Thus, in view of (18), the corresponding two sets of solution of (1) are given by

Set 1:

$$x=210r^2-10s^2$$
$$y=84r^2+4s^2+112rs$$
$$z=84r^2+4s^2+12rs$$

Set 2:

$$x=210r^2-10s^2$$
$$y=84r^2+4s^2-112rs$$
$$z=84r^2+4s^2-12rs$$

3. REMARKABLE OBSERVATION:

If the non-zero integer triple (x_0, y_0, z_0) is any solution of (1) then each of the following two triples of integers also satisfies (1)

Triple: 1 (x_0, y_n, z_n)

Where,

$$y_n=\frac{1}{2}\left\{\left[56-54(-1)^n\right]y_o+\left[-168+168(-1)^n\right]z_o\right\}$$

$$z_n=\frac{1}{2}\left\{\left[18-18(-1)^n\right]y_o+\left[-54+56(-1)^n\right]z_o\right\}$$

Triple: 2 (x_n, y_0, z_n)

Where,

$$x_n=\frac{1}{2}\left\{\left[-7+9(-1)^n\right]x_o+\left[21-21(-1)^n\right]z_o\right\}$$

$$z_n=\frac{1}{2}\left\{\left[-3+3(-1)^n\right]x_o+\left[9-7(-1)^n\right]z_o\right\}$$

4. GENERATION OF SOLUTION:

Let x_0, y_0, z_0 be any given solution of (1).Let

$$\left.\begin{aligned} x_1 &= 4x_0 + h \\ y_1 &= 4y_0 + 2h \\ z_1 &= 4z_0 \end{aligned}\right\} \quad (19)$$

be the second solution of (1).

Substituting (19) in (1) and simplifying, we get

$$h = -2x_0 - 3y_0$$

Thus the second solution of (1) is given by

$$x_1 = 2x_0 - 3y_0$$

$$y_1 = -2y_0 - 4x_0$$

$$z_1 = 4z_0$$

Repeating the above process, the formula for generating infinitely many integer solutions based on the given solution is given by

$$x_n = 2^{2n-2}(2x_0 - 3y_0)$$

$$y_n = -2^{2n-1}(2x_0 + y_0)$$

$$z_n = 2^{2n} z_0$$

5. CONCLUSION

In this paper, we have presented six different patterns of non-zero distinct integer solutions of the homogeneous cone given by $4x^2 + 3y^2 = 28z^2$ .To conclude, one may search for other patterns of non-zero integer distinct solution and their corresponding properties

REFERENCES:

[1]. Dickson, L.E., History of Theory of Numbers, Vol.2, Chelsea Publishing company,NewYork, 1952

[2]. Gopalan, M.A., Pandichelvi, V., Integral solution of ternary quadratic equation $z(x+y) = 4xy$, Actociencia Indica, 2008, Vol. XXXIVM, No.3, 1353-1358.

[3]. Gopalan, M.A., Kalinga Rani, J., Observation on the Diophantine equation, $y^2 = Dx^2 + z^2$ Impact J.sci tech; 2008, Vol (2), 91-95.

[4]. Gopalan, M.A., Pandichelvi,V., on ternary quadratic equation $x^2 + y^2 = z^2 + 1$, Impact J.sci tech; 2008, Vol 2(2), 55-58.

[5]. Gopalan, M.A., Manju somanath, Vanitha, N., Integral solutions of ternary quadratic Diophantine equation $x^2 + y^2 = (k^2 + 1)^n z^2$, Impact J.sci tech; 2008, Vol 2(4), 175-178.

[6]. Gopalan, M.A., Manju somanath, Integral solution of ternary quadratic Diophantine equation $xy + yz = zx$ AntarticaJ,Math, 2008,1-5,5(1).

[7]. Gopalan, M.A.,and Gnanam, A., Pythagorean triangles and special polygonal numbers, International Journal of Mathematical Science, Jan-Jun 2010,Vol.(9),No.1-2,211-215.

[8]. Gopalan, M.A., and Vijayasankar, A., Observations on a Pythagorean problem, Acta Ciencia Indica, 2010,Vol.XXXVIM, No.4,517-520.

[9]. Gopalan.M.A., and pandichelvi, V., Integral solutions of ternary quadratic equation $z(x-y) = 4xy$, Impact J.sci TSech; 2011, Vol (5),No.1,01-06.

[10]. Gopalan, M.A., Kalinga Rani, J. On ternary quadratic equation $x^2+y^2=z^2+8$,Impact J.sci tech ; 2011, Vol (5), no.1,39-43.

[11]. Gopalan, M.A., Geetha, D., Lattice points on the hyperbolid of two sheets $x^2-6xy+y^2+6x-2y+5=z^2+4$, Impact J.sci tech; 2010, Vol(4),No.1,23-32.

[12]. Gopalan, M.A., Vidhyalakshmi, S., and Kavitha, A., Integral points on the homogeneous Cone $z^2=2x^2-7y^2$, DiophantusJ.Math., 2012,1(2),127-136.

[13]. Gopalan, M.A., Vidhyalakshmi, S., Sumathi,G., Lattice points on the hyperboloid one Sheet $4z^2=2x^2+3y^2-4$, DiophantusJ.math., 2012,1(2),109-115.

[14]. Gopalan, M.A., Vidhyalakshmi, S., and Lakshmi,K., Integral points on the hyperboloid of two sheets $3y^2=7x^2-z^2+21$, DiophantusJ.math., 2012,1(2),99-107.

[15]. Gopalan, M.A., and Srividhya,G., Observations on $y^2=2x^2+z^2$ Archimedes J.Math, 2012, 2(1), 7-15.

[16]. Gopalan, M.A., Sangeetha,G.,Observation on $y^2=3x^2-2z^2$ Antarctica J.Math, 2012,9(4), 359-362.

[17]. Gopalan, M.A., and Vijayalakshmi, R., On the ternary quadratic equation $x^2=(\alpha^2-1)(y^2-z^2)$, α>1, Bessel J.Math, 2012,2(2),147-151.

[18]. Manju somanath, Sangeetha, G., Gopalan, M.A., On the homogeneous ternary quadratic Diophantine equation $x^2+(2k+1)y^2=(k+1)^2z^2$, Bessel J.Math, 2012,2(2),107-110.

[19]. Manjusomanath, Sangeetha, G., Gopalan, M.A., Observations on the ternary quadratic equation $y^2=3x^2+z^2$, Bessel J.Math, 2012,2(2),101-105.

[20]. Mordell, L.J., Diophantine equations, Academic press, New York, 1969