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RESEARCH ARTICLE

A COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE MAPS IN COMPLEX VALUED METRIC SPACES

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ABSTRACT

The existence of coincidence points and common fixed points for four mappings satisfying generalized contractive conditions without exploiting the notion of continuity of any map involved therein, in a complex valued metric space is proved.

Keywords: Common fixed point, coincidence point, weakly compatible maps, complex valued metric space.

Mathematics subject classification: 47H10, 54H25.

INTRODUCTION

The study of metric spaces expressed the most important role to many fields both in pure and applied science. Many authors generalized and extended the notion of a metric space such as pseudo metric spaces, fuzzy metric spaces, vector valued metric spaces, G-metric spaces, cone metric spaces, etc.

Recently A. Azam, B. Fisher and M. khan [1], introduced the concept of complex valued metric spaces and obtained sufficient conditions for the existence of common fixed points of a pairs of mappings satisfying contractive type condition. The idea of complex valued metric spaces can be exploited to define complex valued normed spaces and complex valued Hilbert spaces. Several authors studied many common fixed point results on complex valued metric spaces (see [2-7]). In this paper we prove a common fixed point theorem for four self-maps f,g, S and T in complex valued metric spaces, where both {f, S} and {g,T} are weakly compatible maps of a nonempty set X.

Let C be the set of complex numbers and let z_1 , $z_2 \in$ C. Define a partial order $\precsim$ on C as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$. It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$,

(iii) $Re(z_1)<Re(z_2)$, $Im(z_1)<Im(z_2)$,
(iv) $Re(z_1)=Re(z_2)$, $Im(z_1)=Im(z_2)$.

In particular, we will write $z_1 \leq z_2$ if one of (i),(ii) and (iii) is satisfied and we will write $z_1 < z_2$ if only (iii) is satisfied.

Definition1.1. Let X be a non-empty set. Suppose that the mapping $d: X \times X \to C$ satisfies:

(a) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

A point $x \in X$ is called an interior point of a set $A \subseteq X$ whenever there exists $0 < r \in C$ such that $B(x, r) = \{y \in X: d(x, y) < r\} \subseteq A$. A subset A in X is called open whenever each point of A is an interior point of A. The family $F = \{B(x, r): x \in X, 0 < r\}$ is a sub-basis for a Hausdorff topology τ on X. A point $x \in X$ is called a limit point of A whenever for every $0 < r \in C$, $B(x, r) \cap (A \backslash X) \neq \phi$.

A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in C$, with $0 < c$ there is $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x) < c$, then x is called the limit point of $\{x_n\}$ and we write $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

If for every $c \in C$, with $0 < c$ there is $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x_{n+m}) < c$, then $\{x_n\}$ is called a Cauchy sequence in (X, d) is called a complete complex valued metric space.

Lemma1.2. Let (X, d) be a complex valued metric space and $\{x_n\}$ is a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma1.3. Let (X, d) be a complex valued metric space and $\{x_n\}$ is a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.

Definition1.4. Let f and g be self-maps on a set X, if $w = fx = gx$ for some x in X, then x is called coincidence point of f and g, w is called a point of coincidence of f and g.

Definition1.5. Let f and g be two self-maps defined on a set X, then f and g are said to be weakly compatible if they commute at coincidence points.

2. RESULT

We need the following lemma to prove theorem 2.2.

Lemma 2.1. Let f, g, S and T be self-maps on a complex valued metric space, satisfying $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$. Define $\{x_n\}$ and $\{y_n\}$ by $y_{2n+1} = fx_{2n} = Tx_{2n+1}$, $y_{2n+2} = gx_{2n+1} = Sx_{2n+2}$, $n \geq 0$. Suppose that there exist a $\lambda \in [0,1)$ such that

$d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n)$ for each $n \geq 1$. (2.1)

Then either

(a) $\{f, S\}$ and $\{g, T\}$ have coincidence points, and $\{y_n\}$ converges, or
(b) $\{y_n\}$ is Cauchy.

Proof. To prove part (a), suppose that there exists an n such that $y_{2n} = y_{2n+1}$. Then, from the definition of $\{y_n\}$, $gx_{2n-1} = Sx_{2n} = fx_{2n} = Tx_{2n+1}$, f and S have a coincidence point. Moreover, from (2.1),

$d(y_{2n+1}, y_{2n+2}) \leq \lambda d(y_{2n}, y_{2n+1}) = 0$,

So that $y_{2n+1} = y_{2n+2}$; i.e., $fx_{2n} = Tx_{2n+1} = gx_{2n+1} = Sx_{2n+2}$, g and T have a coincidence point. In addition, repeated use of (2.1) yields $y_{2n} = y_m$ for each $m > 2n$, hence $\{y_n\}$ converges.

The same conclusion holds if $y_{2n+1} = y_{2n+2}$ for some n.

For part (b), assume that $y_{2n} \neq y_{2n+1}$ for all n. Then (2.1) implies that

$d(y_n, y_{n+1}) \leq \lambda^n d(y_0, y_1)$.

For any $m, n \in N$ with $m > n$ it follows that

$$d(y_n, y_m) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ \dots\dots\dots + d(y_{m-1}, y_m)$$
$$\leq [\lambda^n + \lambda^{n+1} + \ \dots\dots\dots . + \lambda^{m-1}]\, d(y_0, y_1)$$
$$\leq \frac{\lambda^n}{1-\lambda} d(y_0, y_1)$$

And so

$$|d(y_n, y_m)| \leq \frac{\lambda^n}{1-\lambda} |d(y_0, y_1)|,$$

which implies that $|d(y_n, y_m)| \to 0$ as $n, m \to \infty$. Hence $\{y_n\}$ is a Cauchy sequence.

Theorem 2.2. Let (X, d) be a Complete Complex valued metric space and let f, g, S and T are four self-maps of X such that $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$, satisfying

$$d(fx, gy) \leq h u_{x,y}(f, g, S, T), \tag{2.2}$$

where $h \in (0, \frac{1}{2})$ and

$$u_{x,y}(f, g, S, T) \in \left\{ d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{d(fx, Ty) + d(gy, Sx)}{2}, \frac{d(fx, Sx) + d(gy, Ty)}{2} \right\}$$

for all $x, y \in X$. Suppose that the pairs $\{f, S\}$ and $\{g, T\}$ are weakly compatible and $g(X)$ is closed. Then f, g, S and T have a unique common fixed point.

Proof. For any arbitrary point x_0 in X. Construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n-1} = Tx_{2n-1} = fx_{2n-2}, \text{ and } y_{2n} = Sx_{2n} = gx_{2n-1}$$

we have from (2.2), $d(y_{2n+1}, y_{2n+2}) = d(fx_{2n}, gx_{2n+1}) \leq h u_{x_{2n}, x_{2n+1}}(f, g, S, T)$

for $n = 1, 2, \dots\dots..$ where

$$u_{x_{2n}, x_{2n+1}}(f, g, S, T)$$
$$\in \left\{ d(Sx_{2n}, Tx_{2n+1}), d(fx_{2n}, Sx_{2n}), d(gx_{2n+1}, Tx_{2n+1}), \frac{d(fx_{2n}, Tx_{2n+1}) + d(gx_{2n+1}, Sx_{2n})}{2}, \frac{d(fx_{2n}, Sx_{2n}) + d(gx_{2n+1}, Tx_{2n+1})}{2} \right\}$$
$$= \left\{ d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), \frac{d(y_{2n+1}, y_{2n+1}) + d(y_{2n+2}, y_{2n})}{2}, \frac{d(y_{2n+1}, y_{2n}) + d(y_{2n+2}, y_{2n+1})}{2} \right\}$$
$$= \left\{ d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), \frac{d(y_{2n}, y_{2n+2})}{2}, \frac{d(y_{2n+1}, y_{2n}) + d(y_{2n+2}, y_{2n+1})}{2} \right\}$$

Now there are four possibilities

(i) if $u_{x_{2n}, x_{2n+1}}(f, g, S, T) = d(y_{2n}, y_{2n+1})$ then $d(y_{2n+1}, y_{2n+2}) \leq h\, d(y_{2n}, y_{2n+1})$

(ii) if $u_{x_{2n}, x_{2n+1}}(f, g, S, T) = d(y_{2n+1}, y_{2n+2})$ then $d(y_{2n+1}, y_{2n+2}) \leq h\, d(y_{2n+1}, y_{2n+2})$

Which implies that $d(y_{2n+1}, y_{2n+2}) = 0$ and $y_{2n+1} = y_{2n+2}$.

(iii) if $u_{x_{2n}, x_{2n+1}}(f, g, S, T) = \frac{d(y_{2n}, y_{2n+2})}{2} \leq \frac{d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})}{2}$

Then $d(y_{2n+1}, y_{2n+2}) \leq \frac{h}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})]$

Which implies that $d(y_{2n+1}, y_{2n+2}) \leq k\, d(y_{2n}, y_{2n+1})$, where $k = \frac{h}{2-h} < 1$.

(iv) if $u_{x_{2n}, x_{2n+1}}(f, g, S, T) = \frac{d(y_{2n+1}, y_{2n}) + d(y_{2n+2}, y_{2n+1})}{2}$

Then $d(y_{2n+1}, y_{2n+2}) \leq \frac{h}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})]$

Which implies that $d(y_{2n+1}, y_{2n+2}) \leq k\, d(y_{2n}, y_{2n+1})$, where $k = \frac{h}{2-h} < 1$.

Hence condition (2.1) of lemma 2.1 is satisfied. Now we show that $\{f,S\}$ and $\{g,T\}$ have coincidence points in X. Without loss of generality we may assume that $y_n \neq y_{n+1}$ for any n, For if we have equality for some n, then (a) of lemma 2.1 applies. Now from lemma 2.1 $\{y_n\}$ is a Cauchy sequence in X.

Since X is complete, there exists a point u in X such that $y_n \to u$ ($i.e., \lim_{n\to\infty} S x_{2n} = \lim_{n\to\infty} g x_{2n-1} = \lim_{n\to\infty} T x_{2n-1} = \lim_{n\to\infty} f x_{2n-2} = u$). Since $g(X)$ is closed, so $u \in g(X)$.

Since $g(X) \subseteq S(X)$ there exist a point v in X such that $Sv = u$. We now claim that $fv = u$. For this, consider

$$d(fv,u) \le d(fv, gx_{2n-1}) + d(gx_{2n-1},u) \le hu_{v,x_{2n-1}}(f,g,S,T) + d(gx_{2n-1},u)$$

Where

$$u_{v,x_{2n-1}}(f,g,S,T) \in \left\{d(Sv,Tx_{2n-1}), d(fv,Sv), d(gx_{2n-1},Tx_{2n-1}), \frac{d(fv,Tx_{2n-1})+d(gx_{2n-1},Sv)}{2}, \frac{d(fv,Sv)+d(gx_{2n-1},Tx_{2n-1})}{2}\right\}$$

For all $n \in N$. There are five possibilities:

(i)If $u_{v,x_{2n-1}}(f,g,S,T) = d(Sv,Tx_{2n-1})$

then $d(fv,u) \le h\, d(Sv,Tx_{2n-1}) + d(gx_{2n-1},u)$,this implies $fv = u$.

(ii) If $u_{v,x_{2n-1}}(f,g,S,T) = d(fv,Sv)$

then $d(fv,u) \le h\, d(fv,Sv) + d(gx_{2n-1},u)$, a contradiction .

This implies $fv = u$.

(iii) If $u_{v,x_{2n-1}}(f,g,S,T) = d(gx_{2n-1},Tx_{2n-1})$

then $d(fv,u) \le h\, d(gx_{2n-1},Tx_{2n-1}) + d(gx_{2n-1},u)$, this implies $fv = u$.

(iv) If $u_{v,x_{2n-1}}(f,g,S,T) = \frac{d(fv,Tx_{2n-1})+d(gx_{2n-1},Sv)}{2}$

then $d(fv,u) \le \frac{h}{2}\ [d(fv,Tx_{2n-1}) + d(gx_{2n-1},Sv)] + d(gx_{2n-1},u) = \frac{h}{2} d(fv,u)$,a contradiction.

This implies $fv = u$.

(v) If $u_{v,x_{2n-1}}(f,g,S,T) = \frac{d(fv,Sv)+d(gx_{2n-1},Tx_{2n-1})}{2}$

then $d(fv,u) \le \frac{h}{2}\ [d(fv,Sv) + d(gx_{2n-1},Tx_{2n-1})] + d(gx_{2n-1},u) = \frac{h}{2} d(fv,u)$, a contradiction.

This implies $fv = u$.

So in all cases we have $fv = u$. Hence $fv = Sv = u$.

Since $u \in f(X) \subseteq T(X)$, there exists a $w \in X$ such that $Tw = u$. Now we shall show that $gw = u$. Consider

$$d(gw,u) \le d(gw,fx_{2n}) + d(fx_{2n},u) = d(fx_{2n},gw) + d(fx_{2n},u) \le hu_{x_{2n,w}}(f,g,S,T) + d(fx_{2n},u)$$

Where

$$u_{x_{2n,w}}(f,g,S,T) \in \left\{d(Sx_{2n},Tw), d(fx_{2n},Sx_{2n}), d(gw,Tw), \frac{d(fx_{2n},Tw)+d(gw,Sx_{2n})}{2}, \frac{d(fx_{2n},Sx_{2n})+d(gw,Tw)}{2}\right\}$$

For all $n \in N$. There are five possibilities:

(i)If $u_{x_{2n,w}}(f,g,S,T) = d(Sx_{2n},Tw)$

then $d(gw,u) \le h\, d(Sx_{2n},Tw) + d(fx_{2n},u)$,this implies $gw = u$

(ii) If $u_{x_{2n,w}}(f,g,S,T) = d(fx_{2n},Sx_{2n})$

then $d(gw,u) \le h\, d(fx_{2n},Sx_{2n}) + d(fx_{2n},u)$,this implies $gw = u$

(iii) If $u_{x_{2n,w}}(f,g,S,T) = d(gw,Tw)$

then $d(gw,u) \le h\, d(gw,Tw) + d(fx_{2n},u)$, a contradiction.

This implies $gw = u$.

(iv) If $u_{x_{2n},w}(f,g,S,T) = \frac{d(fx_{2n},Tw)+d(gw,Sx_{2n})}{2}$

then $d(gw,u) \leq \frac{h}{2}\ [d(fx_{2n},Tw) + d(gw,Sx_{2n})] + d(fx_{2n},u) = \frac{h}{2} d(gw,u)$, a contradiction.

This implies $gw = u$.

(v) If $u_{x_{2n},w}(f,g,S,T) = \frac{d(fx_{2n},Sx_{2n})+d(gw,Tw)}{2}$

then $d(gw,u) \leq \frac{h}{2}\ [d(fx_{2n},Sx_{2n}) + d(gw,Tw)] + d(fx_{2n},u) = \frac{h}{2} d(gw,u)$, a contradiction.

This implies $gw = u$.

So in all cases we have $gw = u$. Hence $Tw = gw = u$. Thus $\{f,S\}$ and $\{g,T\}$ have a common point of coincidence in X. Now, if $\{f,S\}$ and $\{g,T\}$ are weakly compatible, $fu = fSv = Sfv = Su = w_1$(say) and $gu = gTw = Tgw = Tu = w_2$(say). Now

$d(w_1,w_2) = d(fu,gu) \leq hu_{u,u}(f,g,S,T)$

Where $u_{u,u}(f,g,S,T) \in \left\{d(Su,Tu), d(fu,Su), d(gu,Tu), \frac{d(fu,Tu)+d(gu,Su)}{2}, \frac{d(fu,Su)+d(gu,Tu)}{2}\right\} = \{d(w_1,w_2)\}$.

Therefore $d(w_1,w_2) \leq h\, d(w_1,w_2)$ which implies $w_1 = w_2$and hence $fu = gu = Su = Tu$.

Now we shall Show that $u = gu$.

$d(u,gu) = d(fv,gu) \leq hu_{v,u}(f,g,S,T)$

Where $u_{v,u}(f,g,S,T) \in \left\{d(Sv,Tu), d(fv,Sv), d(gu,Tu), \frac{d(fv,Tu)+d(gu,Sv)}{2}, \frac{d(fv,Sv)+d(gu,Tu)}{2}\right\} = \{d(u,gu)\}$.

Thus $d(u,gu) \leq hd(u,gu)$, which implies that $gu = u$, and u is a common fixed point of f, g, S and T.

For uniqueness, suppose that u^* is also a common fixed point of f, g, S and T.From (2.2),

$d(u,u^*) = d(fu,gu^*) \leq hu_{u,u^*}(f,g,S,T)$,

Where

$u_{u,u^*}(f,g,S,T) \in \left\{d(Su,Tu^*), d(fu,Su), d(gu^*,Tu^*), \frac{d(fu,Tu^*)+d(gu^*,Su)}{2}, \frac{d(fu,Su)+d(gu^*,Tu^*)}{2}\right\} = \{d(u,u^*)\}$.

Which is possible only if $u = u^*$.

Corollary 2.3. Let (X,d) be a Complete Complex valued metric space and let f, g and S are three self-maps of X such that $f(X) \subseteq S(X)$ and $g(X) \subseteq S(X)$ and satisfying

$$d(fx,gy) \leq hu_{x,y}(f,g,S),$$

where $h \in (0,\frac{1}{2})$ and

$$u_{x,y}(f,g,S) \in \left\{d(Sx,Sy), d(fx,Sx), d(gy,Sy), \frac{d(fx,Sy)+d(gy,Sx)}{2}, \frac{d(fx,Sx)+d(gy,Sy)}{2}\right\}$$

for all $x, y \in X$. Suppose that the pairs $\{f,S\}$ and $\{g,S\}$ are weakly compatible and $g(X)$ is closed. Then f, g and S have a unique common fixed point.

Proof. The result follows on putting $T = S$ in theorem 2.2.

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