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RESEARCH ARTICLE

INTEGRAL POINTS ON THE TERNARY QUADRATIC DIOPHANTINE EQUATION $3x^2+5y^2=128z^2$

M.A.GOPALAN[1],S.VIDHYALAKSHMI[2], C.NITHYA[3]*

[1,2] Professor, Department of Mathematics , Shrimati Indira Gandhi College,Trichy, Tamilnadu , India.

[3] M.Phil student, Department of Mathematics,Shrimati Indira Gandhi College,Trichy,Tamilnadu ,India

*** C.NITHYA**

Author for Correspondence

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ABSTRACT

The ternary quadratic homogeneous equation representing homogeneous cone given by $3x^2+5y^2=128z^2$ is analyzed for its non-zero distinct integer points on it. Six different patterns of integer points satisfying the cone under consideration are obtained. A few interesting relation between the solutions and special number patterns namely Polygonal number , Pyramidal number , and Nasty number are presented. Also knowing an integer solution satisfying the given cone , three triples of integers generated from the given solution are exhibited.

Keywords: Ternary homogeneous quadratic, integral solutions

2010 Mathematics Subject Classification: 11D09

INTRODUCTION

The ternary quadratic Diophantine equations offer an unlimited field for research due to their variety [1, 20]. For an extensive review of various problems, one may refer [2-19]. This communication concerns with yet another interesting ternary quadratic equation $3x^2+5y^2=128z^2$ representing a cone for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

NOTATIONS:

P_n^m - Pyramidal umber of rank n with size m.

$T_{m,n}$ -Polygonal number of rank n with size m.

2. METHOD OF ANALYSIS:

The ternary quadratic equation to be solved for its non-integer solutions is

$$3x^2 + 5y^2 = 128z^2 \quad (1)$$

The substitution of linear transformations

$$x = X + 5T, y = X - 3T \quad (2)$$

in (1) leads to

$$X^2 + 15T^2 = 16z^2 \quad (3)$$

We illustrate below six different patterns of non-zero distinct integer solutions to (1)

2.1 PATTERN: 1

Assume $$z = z(a,b) = a^2 + 15b^2, a,b \neq 0 \quad (4)$$

Write $$16 = (1 + i\sqrt{15})(1 - i\sqrt{15}) \quad (5)$$

Substituting (4) and (5) in (3) and employing the method of factorization and equating real and imaginary parts, we have

$$\left.\begin{aligned} X &= X(a,b) = a^2 - 15b^2 - 30ab \\ T &= T(a,b) = a^2 - 15b^2 + 2ab \end{aligned}\right\} \quad (6)$$

Using (6) in (2), we have

$$\left.\begin{aligned} x &= x(a,b) = 6a^2 - 90b^2 - 20ab \\ y &= y(a,b) = -2a^2 + 30b^2 - 36ab \end{aligned}\right\} \quad (7)$$

Thus (7) and (4) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:

- $x(a,a+1) + 3y(a,a+1) + 256T_{3,a} = 0$
- $x(a,a(a+1)) + 3y(a,a(a+1)) + 256P_a^5 = 0$
- $x(a,1) + 6z(a,1) - T_{16,a} - T_{12,a} \equiv 0 \pmod{10}$
- $2z(a,1) - y(a,1) - T_{10,a} \equiv 0 \pmod{39}$
- $x(a,a) + 3y(a,a) + 128T_{4,a} = 0$
- $6\{z(a,a)\}$ a nasty number
- $-\{x(a,a) + y(a,a) + z(a,a)\}$ a nasty number
- $-3\{y(b,b)\}$ a nasty number

2.2 PATTERN: 2

Instead of (5), write 16 as

$$16 = \frac{(-7 + i\sqrt{15})(-7 - i\sqrt{15})}{4} \quad (8)$$

Following the procedure presented in pattern:1 ,the corresponding values of x and y are given by

$$\left.\begin{aligned} x &= x(a,b) = -a^2 + 15b^2 - 50ab \\ y &= y(a,b) = -5a^2 + 75b^2 + 6ab \end{aligned}\right\} \quad (9)$$

Thus (9) and (4) represents non-zero distinct integer solutions of (1) in two parameters.

Properties:

- $y(a+2,a-2) - 4T_{40,a} \equiv 8 \pmod{248}$
- $x(1,b) + z(1,b) - T_{62,b} \equiv 0 \pmod{21}$
- $z(4a,a) - T_{64,a} \equiv 0 \pmod{30}$

- $y(3a,1)+T_{92,a}\equiv 23(\text{mod}26)$
- $x(a,-1)+y(a,-1)+z(a,-1)+T_{12,a}\equiv 25(\text{mod}40)$
- $6\{x(a,-a)+y(a,-a)+z(a,-a)\}$ a nasty number
- $-6\{x(b,b)\}$ a nasty number
- $10\{y(b,b)-z(b,b)\}$ a nasty number

2.3 PATTERN: 3

The ternary quadratic equation (3) can be written as

$$X^2-16Z^2=-15T^2 \tag{10}$$

Write (10) in the form of ratio as

$$\frac{(X+4z)}{-3T}=\frac{5T}{(X-4z)}=\frac{a}{b}\text{ , where } b\neq 0$$

which is equivalent to the system of double equations

$$bX+3aT+4bz=0$$
$$aX-5bT-4az=0$$

Solving the aboue system by the method of cross multiplication, we get

$$X=X(a,b)=-12a^2+20b^2$$
$$T=T(a,b)=8ab$$
$$z=z(a,b)=-3a^2-5b^2$$

Inview of (2), the solutions of (1) are given by

$$\left.\begin{aligned} x&=x(a,b)=-12a^2+20b^2+40ab\\ y&=y(a,b)=-12a^2+20b^2-24ab\\ z&=z(a,b)=-3a^2-5b^2\end{aligned}\right\} \tag{11}$$

Properties:

- $y(1,b)-4z(1,b)-T_{52,b}-T_{32,b}\equiv 0(\text{mod}14)$
- $y(a+2,a+2)+T_{34,a}\equiv -64(\text{mod}79)$
- $x(2a,-2)-y(2a,-2)-z(2a,-2)-T_{26,a}\equiv 20(\text{mod}245)$
- $x(1,-b)-T_{42,b}\equiv -12(\text{mod}21)$
- $x(a,a(a+1))-y(a,a(a+1))-128P_a^5=0$
- $x(a,a+1)-y(a,a+1)-128T_{3,a}=0$
- $-3\{z(a,a)\}$ a nasty number
- $6\{y(a,a)-4z(a,a)\}$ a nasty number

NOTE :

(10) may also be written in the form of ratio in two ways as follows

$$\frac{(X+4z)}{-15T}=\frac{T}{(X-4z)}=\frac{a}{b}\text{ , where } b\neq 0$$

$$\frac{(X+4z)}{-T}=\frac{15T}{(X-4z)}=\frac{a}{b}\text{ , where } b\neq 0$$

Applying the procedure presented in pattern: 3, the corresponding two sets of integer solutions are presented below

SET :1

$$\left.\begin{aligned} x &= x(a,b) = -60a^2 + 4b^2 + 40ab \\ y &= y(a,b) = -60a^2 + 4b^2 - 24ab \\ z &= z(a,b) = -15a^2 - b^2 \end{aligned}\right\} \quad (12)$$

Properties:

- $y(1,b) - 4z(1,b) - T_{18,b} \equiv 0(\text{mod}17)$
- $y(a,-5) + T_{122,a} \equiv 39(\text{mod}61)$
- $x(a,-1) - y(a,-1) - z(a,-1) - T_{16,a} - T_{18,a} \equiv 1(\text{mod}51)$
- $x(a,a) - y(a,a) - 64T_{4,a} = 0$

SET: 2

$$\left.\begin{aligned} x &= x(a,b) = -4a^2 + 60b^2 + 40ab \\ y &= y(a,b) = -4a^2 + 60b^2 - 24ab \\ z &= z(a,b) = -a^2 - 15b^2 \end{aligned}\right\} \quad (13)$$

Properties:

- $x(a,a(a+1)) - 240T_{3,a}^2 - 80P_a^5 + T_{10,a} \equiv 0(\text{mod}3)$
- $y(5a,-1) + T_{202,a} \equiv 18(\text{mod}21)$
- $x(-2a,-1) + 8T_{6,a} \equiv 60(\text{mod}72)$
- $x(a,a+1) - y(a,a+1) - 128P_a^2 = 0$

2.4 PATTERN: 4

The ternary quadratic equation (3) can be written as

$$X^2 = 16z^2 - 15T^2 \quad (14)$$

Assume $X = X(a,b) = 16a^2 - 15b^2$; a, b>0 (15)

Substituting (15) in (14) and employing the method of factorization,equating rational and irrational factors,we get

$$z = z(a,b) = \frac{1}{4}(16a^2 + 15b^2)$$

$$T = T(a,b) = 8ab$$

Thus, inview of (15) and (2), the corresponding solutions of (1) are found to be

$$x = x(a,b) = 16a^2 - 15b^2 + 40ab$$

$$y = y(a,b) = 16a^2 - 15b^2 - 24ab$$

$$z = z(a,b) = \frac{1}{4}(16a^2 + 15b^2)$$

As our interest is on finding integer solutions,replacing b by 2b,we have

$$\left.\begin{aligned} x &= x(a,b) = 16a^2 - 60b^2 + 80ab \\ y &= y(a,b) = 16a^2 - 60b^2 - 48ab \\ z &= z(a,b) = 4a^2 + 15b^2 \end{aligned}\right\} \quad (16)$$

Properties:

- $y(-3,a) + T_{58,a} + T_{66,a} \equiv 58(\text{mod}86)$
- $z(a+1,a-1) - T_{10,a} - T_{32,a} \equiv 4(\text{mod}5)$

- $z(3a-2,a)-T_{104,a}\equiv 0 \pmod 2$
- $x(a,-3)+z(a,-3)-T_{42,a}\equiv -184 \pmod{221}$
- $x(a,a-1)-y(a,a-1)-32T_{10,a}\equiv 0 \pmod{32}$
- $z(a+3,a+1)-T_{12,a}-T_{30,a}\equiv 51 \pmod{71}$
- $y(a,a)+4z(a,a)+T_{34,a}\equiv 0 \pmod{15}$
- $38\{y(-3a,a)\}$ a nasty number
- $6\{z(a,2a)\}$ a nasty number

2.5 PATTERN: 5

The ternary quadratic equation (14) can be written as

$$16z^2-15T^2=X^2*1 \tag{17}$$

Write 1 as

$$1=(4+\sqrt{15})(4-\sqrt{15}) \tag{18}$$

Substituting (15) and (18) in (17) and employing the method of factorization, following the procedure presented in pattern: 4, the corresponding integer solutions of (1) are represented by

$$\left.\begin{aligned} x &= x(a,b)=96a^2+60b^2+160ab \\ y &= y(a,b)=-32a^2-60b^2-96ab \\ z &= z(a,b)=16a^2+15b^2+30ab \end{aligned}\right\} \tag{19}$$

Properties:

- $x(a,1)+y(a,1)-8T_{16,a}-8T_{4,a}\equiv 0 \pmod{112}$
- $z(1,b-1)-30P_b^2\equiv 1 \pmod{15}$
- $x(-b,b)+T_{10,a}\equiv 0 \pmod 3$
- $y(a,-1)+T_{66,a}\equiv -60 \pmod{65}$
- $z(-2,b)-T_{26,b}-T_{8,b}\equiv 17 \pmod{47}$
- $x(-1,b)-6z(-1,b)+5T_{14,b}\equiv 0 \pmod 5$
- $3\{x(-4a,a)-6z(-4a,a)\}$ a nasty number
- $6\{y(b,-b)\}$ a nasty number

2.6 PATTERN: 6

Instead of (18), write 1 as

$$1=\frac{(8+\sqrt{15})(8-\sqrt{15})}{49} \tag{20}$$

The corresponding integer solutions of (1) are

$$\left.\begin{aligned} x &= x(a,b)=1344a^2-210b^2+2240ab \\ y &= y(a,b)=448a^2-1050b^2-1344ab \\ z &= z(a,b)=224a^2+210b^2+210ab \end{aligned}\right\} \tag{21}$$

Properties:

- $y(a,1)-2z(a,1)\equiv -1470(\mathrm{mod}1764)$
- $y(-1,b)-2z(-1,b)+42T_{72,b}\equiv 0(\mathrm{mod}336)$
- $y(a,a(a+1))-2z(a,a(a+1))+1470*4T_{3,a}^{2}+1764*2P_{a}^{5}=0$
- $y(2a,-1)-448T_{4,a}-448*6P_{a}^{2}\equiv -1050(\mathrm{mod}1344)$
- $x(1,b)-6z(1,b)+5T_{290,b}+5T_{302,b}\equiv 0(\mathrm{mod}480)$
- $z(a,-a)-T_{138,a}-T_{314,a}\equiv 0(\mathrm{mod}222)$
- $x(a,a(a+1))-3y(a,a(a+1))-2940*4T_{3,a}^{2}-6272*2P_{a}^{5}=0$
- $\{y(-b,b)-2z(-b,b)\}$ a nasty number

3. REMARKABLE OBSERVATION:

If the non-zero integer triple (x_0, y_0, z_0) is any solution of (1) then each of the following three triples also satisfy (1)

Triple: 1 (x_n, y_n, z_n)

$$x_n = 3^n x_0$$

$$y_n = \frac{3^{n-1}}{2}\{(256-(-1)^n 250)y_0 - 1280(1-(-1)^n)z_0\}$$

$$z_n = \frac{3^{n-1}}{2}\{50(1-(-1)^n)y_0 + (-250+(-1)^n 256)z_0\}$$

Triple: 2 (x_n, y_n, z_n)

$$x_n = \frac{5^{n-1}}{2}\{(64-(-1)^n 54)x_0 + 384(1-(-1)^n)z_0\}$$

$$y_n = 5^n y_0$$

$$z_n = \frac{5^{n-1}}{2}\{-9(1-(-1)^n)x_0 + (-54+(-1)^n 64)z_0\}$$

Triple: 3 (x_n, y_n, z_n)

$$x_n = \frac{8^{n-1}}{2}\{(10+(-1)^n 6)x_0 + 10(1-(-1)^n)y_0\}$$

$$y_n = \frac{8^{n-1}}{2}\{6(1-(-1)^n)x_0 + (6+(-1)^n 10)y_0\}$$

$$z_n = 8^n z_0$$

4. CONCLUSION: To conclude, one may search for other patterns of solutions and their corresponding properties.

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