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ON THE BINARY QUADRATIC DIOPHANTINE EQUATION $x^2 - 6xy + y^2 + 8x = 0$

K.MEENA[1], S.VIDHYALAKSHMI[2], N.SUJITHA[3], M.A.GOPALAN[4]

[1]Former VC, Bharthidasan university,Trichy,Tamilnadu,India

[2,4]Professor, Department of Mathematics, SIGC,Trichy,Tamilnadu,India

[3]M .Phil student , Department of Mathematics,SIGC,Trichy,Tamilnadu,India

*** N.SUJITHA**

Author for Correspondence

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ABSTRACT

The binary quadratic equation $x^2 - 6xy + y^2 + 8x = 0$ representing hyperbola is considered. Different patterns of solutions are obtained. A few interesting recurrence relations satisfied by x and y are exhibited.

Keywords: binary quadratic, hyperbola, integer solutions.

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INTRODUCTION

The binary quadratic equation offers an unlimited field for research because of their variety [1-5]. In this context one may also refer [6-19]. This communication concerns with yet another interesting binary quadratic equation $x^2 - 6xy + y^2 + 8x = 0$ for determining its infinitely many non-zero integral solutions. Also a few interesting relations are presented.

2. METHOD OF ANALYSIS:

The hyperbola under consideration is

$$x^2 - 6xy + y^2 + 8x = 0 \tag{1}$$

Different patterns of solutions for (1) are illustrated below:

2.1 PATTERN: 1

Introducing the linear transformations $(X \neq T \neq 0)$,

$$x = X + T \text{ and } y = X - T$$

(2) in (1), it becomes

$$Y^2 = 2Z^2 - 1 \quad (3)$$

where $Y = 2T + 1$ and $Z = X - 1$ (4)

The smallest positive integer solution of (3) is

$$Z_0 = 1 \text{ and } Y_0 = 1$$

To find the other solution of (3), consider the pellian equation

$$Y^2 = 2Z^2 + 1$$

whose general solution $(\overline{Y_n}, \overline{Z_n})$ is given by

$$\overline{Y_n} = \frac{1}{2}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right]$$

$$\overline{Z_n} = \frac{1}{2\sqrt{2}}\left[\left(3+2\sqrt{2}\right)^{n+1} - \left(3-2\sqrt{2}\right)^{n+1}\right]$$

Applying Brahmagupta Lemma between (Y_0, Z_0) and $(\overline{Y_n}, \overline{Z_n})$, the general solutions to (3) are given by,

$$Y_{n+1} = Y_0 Y_n + 2Z_0 Z_n$$

$$Z_{n+1} = Z_0 Y_n + Y_0 Z_n$$

In view of (4), we have

$$X_{n+1} = Y_{n+1} + Z_n + 1$$

$$T_{n+1} = \frac{1}{2}\left(Y_n + 2Z_n - 1\right)$$

Employing (2), the values of x and y satisfying (1) are given by

$$x_{n+1} = \frac{1}{4}\left[\left(3+2\sqrt{2}\right)^{n+2} + \left(3-2\sqrt{2}\right)^{n+2}\right] + \frac{1}{2}, \quad n = 0,1,2,3......$$

$$y_{n+1} = \frac{1}{4}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] + \frac{3}{2}, \quad n = 0,1,2,3......$$

PROPERTIES:

1. $x_{n+4} - 35x_{n+2} + 6x_{n+1} = -14$
2. $6x_{n+2} - x_{n+1} - x_{n+3} = 2$
3. $34x_{n+3} - x_{n+5} - x_{n+1} = 16$
4. $6x_{n+4} - x_{n+3} - x_{n+5} = 2$
5. $y_{n+5} - 34y_{n+3} + y_{n+1} = -48$
6. $70y_{n+2} - 2y_{n+4} - 12y_{n+1} = 84$
7. $y_{n+4} + y_{n+2} - 6y_{n+3} = -6$
8. $y_{n+5} - 6y_{n+4} + y_{n+3} = -6$
9. Each of the expressions represents a Nasty Number:
   - $24x_{2n+3}$
   - $24y_{2n+2} - 24$

10. Each of the expressions represents a cubical integer:
    - $4x_{3n+5}+12x_{n+1}-8$
    - $4y_{3n+3}+12y_{n+1}-24$
11. Each of the expressions represents a bi-quadratic integer:
    - $4x_{4n+7}+64x_{n+1}^2-64x_{n+1}+12$
    - $4y_{4n+4}+64y_{n+1}^2-192y_{n+1}+136$

Note:

Instead of (2), if we consider the linear transformations $(X \neq T \neq 0)$,

$x = X - T$ and $y = X + T$

Then, the corresponding integer solutions to (1) are obtained as,

$$x_{n+1}=\frac{1}{4}\left[\left(3+2\sqrt{2}\right)^{n+1}+\left(3-2\sqrt{2}\right)^{n+1}\right]+\frac{1}{2}\ ,\quad n=0,1,2,3,,......$$

$$y_{n+1}=\frac{1}{4}\left[\left(3+2\sqrt{2}\right)^{n+2}+\left(3-2\sqrt{2}\right)^{n+2}\right]+\frac{3}{2}\ ,\quad n=0,1,2,3,,......$$

The recurrence relations satisfied by x and y are given by

$$x_{n+1}+2=6x_{n+2}-x_{n+3}\,;\quad x_1=2, x_3=50$$

$$y_{n+1}+6=6y_{n+2}-y_{n+3}\,;\quad y_1=10, y_3=290$$

Some numerical examples of x and y satisfying (1) are given in the following table:

n	x_{n+1}	y_{n+1}
0	2	10
1	9	51
2	50	290
3	289	1683
4	1682	9802
5	9801	57123
6	57122	332930
7	332929	1940451
8	1940450	11309770

From the above table relations observed are as follows:

1. x_{2n} is a perfect square
2. $x_{2n-1} \equiv 0(\mathrm{mod}\,2)$
3. $y_{2n-1} \equiv 0(\mathrm{mod}\,2)$
4. $y_{2n} \equiv 0(\mathrm{mod}\,3)$
5. $x_{n+1}+1=y_n, n>0$

2.2 PATTERN: 2

Treating (1) as a quadratic in x and solving for x, we get

$$x = 3y - 4 \pm 2\sqrt{2y^2 - 6y + 4} \qquad (5)$$

Let $\alpha^2 = 2y^2 - 6y + 4$ (6)

Substituting $y = \frac{Y+3}{2}$ (7)

in (6), we have

$$Y^2 = 2\alpha^2 + 1$$

Whose general solution is given by,

$$Y_n = \frac{1}{2}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] \qquad (8)$$

$$\alpha_n = \frac{1}{2\sqrt{2}}\left[\left(3+2\sqrt{2}\right)^{n+1} - \left(3-2\sqrt{2}\right)^{n+1}\right] \qquad (9)$$

From (7) and (8), we have

$$y_n = \frac{1}{4}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] + \frac{3}{2} \qquad (10)$$

Substituting (9) and (10) in (5) and taking the positive sign, the corresponding integer solutions to (1) are given by

$$x_n = \frac{1}{4}\left[\left(3+2\sqrt{2}\right)^{n+2} + \left(3-2\sqrt{2}\right)^{n+2}\right] + \frac{1}{2}, \quad n = 0,1,2,3,,......$$

$$y_n = \frac{1}{4}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] + \frac{3}{2}, \quad n = 0,1,2,3,,......$$

PROPERTIES:

- $24x_{2n+2}$ is a Nasty Number
- $4x_{3n+4} + 12x_n - 8$ is a Cubical integer
- $4x_{4n+6} + 64x_n^2 - 64x_n + 12$ is a Bi-quadratic integer
- $x_{2n+2} = (2x_n - 1)^2$
- $x_{2n} = (2y_n - 3)^2$

Also, taking the negative sign in (5), the other set of solutions to (1) is given by

$$x_n = \frac{1}{4}\left[\left(3+2\sqrt{2}\right)^{n} + \left(3-2\sqrt{2}\right)^{n}\right] + \frac{1}{2}, \quad n = 0,1,2,3,,......$$

$$y_n = \frac{1}{4}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] + \frac{3}{2}, \quad n = 0,1,2,3,,......$$

PROPERTIES:

- $6x_{2n}$ is a Nasty Number
- $4x_{3n} + 12x_n - 8$ is a Cubical integer
- $4x_{4n} + 64x_n^2 - 64x_n + 12$ is a Bi-quadratic integer
- $4x_{2n}$ is a perfect square

In addition, the above two sets of solutions satisfy the following properties:

1. $35y_{n+2} - 6y_{n+3} - y_n = 60$
2. $35y_{n+1} - y_{n+3} - 6y_n = 42$

3. $y_{n+2} + y_n - 6y_{n+1} = -6$
4. $x_{n+4} + x_{n+2} - 6x_{n+3} = -2$
5. $70x_{n+1} - 2x_{n+3} - 12x_n = 28$
6. $34x_{n+2} - x_{n+4} - x_n = 16$
7. $x_n + x_{n+2} - 6x_{n+1} = -2$
8. $24y_{2n+1} - 24$ is a Nasty Number
9. $4y_{3n+2} + 12y_n - 24$ is a Cubical integer
10. $4y_{4n+3} + 64y_n^2 - 192y_n + 136$ is a Bi-quadratic integer
11. $y_{2n+1} - 1 = (2y_n - 3)^2$

2.3 PATTERN: 3

Treating (1) as a quadratic in y and solving for y, we get

$$y = 3x \pm 2\sqrt{2x^2 - 2x} \tag{11}$$

Let $\alpha^2 = 2x^2 - 2x$ (12)

Substituting $x = \frac{X+1}{2}$ (13)

In (12), we have

$$X^2 = 2\alpha^2 + 1$$

whose general solution is given by,

$$X_n = \frac{1}{2}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] \tag{14}$$

$$\alpha_n = \frac{1}{2\sqrt{2}}\left[\left(3+2\sqrt{2}\right)^{n+1} - \left(3-2\sqrt{2}\right)^{n+1}\right] \tag{15}$$

From (13) and (14), we have

$$x_n = \frac{1}{4}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] + \frac{1}{2} \tag{16}$$

Substituting (15) and (16) in (11) and taking the positive sign, the corresponding integer solutions to (1) are given by

$$x_n = \frac{1}{4}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] + \frac{1}{2}, \quad n = 0,1,2,3,,......$$

$$y_n = \frac{1}{4}\left[\left(3+2\sqrt{2}\right)^{n+2} + \left(3-2\sqrt{2}\right)^{n+2}\right] + \frac{3}{2}, \quad n = 0,1,2,3,,......$$

PROPERTIES:

- $24y_{2n+2} - 24$ is a Nasty Number
- $4y_{3n+4} + 12y_n - 24$ is a Cubical integer
- $4y_{4n+6} + 64y_n^2 - 192y_n + 136$ is a Bi-quadratic integer
- $y_{2n+2} - 1 = (2y_n - 3)^2$
- $y_{2n} - 1 = (2x_n - 1)^2$

Also, taking the negative sign in (11), the other set of solutions to (1) is given by

$$x_n = \frac{1}{4}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] + \frac{1}{2} \quad , \quad n = 0,1,2,3,,......$$

$$y_n = \frac{1}{4}\left[\left(3+2\sqrt{2}\right)^{n} + \left(3-2\sqrt{2}\right)^{n}\right] + \frac{3}{2} \quad , \quad n = 0,1,2,3,,......$$

PROPERTIES:

- $24y_{2n} - 24$ is a Nasty Number
- $4y_{3n} + 12y_n - 24$ is a Cubical integer
- $4y_{4n} + 64y_n^2 - 192y_n + 136$ is a Bi-quadratic integer
- $y_{2n} - 1 = (2y_n - 3)^2$
- $y_{n+1} - 1 = x_n$

In addition, the above two sets of solutions satisfy the following properties:

- $24x_{2n+1}$ is a Nasty Number
- $4x_{3n+2} + 12x_n - 8$ is a Cubical integer
- $4x_{4n+3} + 64x_n^2 - 64x_n + 12$ is a Bi-quadratic integer

3. CONCLUSION: As the binary quadratic equations are rich in variety, one may consider other choices of hyperbolas and search for their non-trivial distinct integral solutions along with the corresponding properties.

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