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RESEARCH ARTICLE

TRUNCATED TWO PIECE BIVARIATE LOGNORMAL DISTRIBUTION

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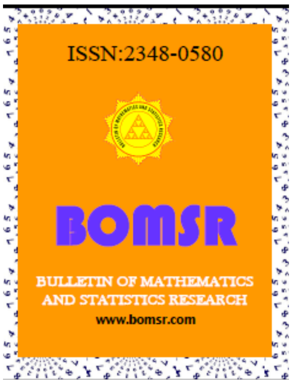

ABSTRACT

In this paper we define doubly truncated two piece bivariate lognormal distribution. We have estimated the mean, variance and correlation coefficient of this distribution by method of moments. Results regarding singly truncated two piece lognormal distribution have been shown as a particular case of doubly truncated two piece bivariate lognormal distribution.

Keywords: Doubly truncated two piece bivariate lognormal distribution, method of moments, lower truncation, upper truncation, Recurrence relation.

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1 INTRODUCTION

Wu (1966) has shown that lognormal distribution can arise as limiting distributions of order statistics, of sample size and order increase in certain relationships. Carrobi, C.F.M. et al (2003) have described the procedure through which the parameters of the lognormal distribution by fitting a given set of experiment outcomes and this procedure is applied to the measured field distribution in the screened room. Sheng Yue (2000) has proposed the bivariate lognormal distribution as a model for the joint distribution of storm peak (maximum rainfall density) and storm amount. The model is found appropriate for representing multiple episodic storm events at the Motoyama meteorological observation station in Japan. A procedure for using the bivariate lognormal distribution to describe the joint distribution of correlated flood peaks and volumes and correlated flood volumes and durations is also considered by him (2000a). Shimizu, K. (1993) have proposed a bivariate mixed lognormal distribution as a probability model for representing rainfalls, containing zeros, measured at two monitoring sites. Nabar and Deshmukh (2002) have proposed bilognormal distribution to model income distribution and life time distribution. Expression of moments of truncated bivariate lognormal distribution was derived by Da Hsiang Donald Lien (1985). He has shown that these results can be applied to test the Houthakker effect in futures markets.

In this chapter we introduce truncated two piece bivariate lognormal (TTPBLN) distribution. In section 2 we have given the p.d.f. of TTPBLN distribution. The general recurrence relation for the moments of this distribution is obtained in section 3. Mean, variance, covariance and correlation coefficient has been obtained in this section. Singly truncated distribution and its mean, variance, covariance and correlation coefficient are derived in section 4.

2. The density function:

The density function of doubly truncated two piece bivariate lognormal distribution is given as

$$f(x,y)=\begin{cases} K(xy)^{-1}\exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{\log x-\mu_1}{\sigma_{11}}\right)^2-2\rho\left(\frac{\log x-\mu_1}{\sigma_{11}}\right)\left(\frac{\log y-\mu_2}{\sigma_{21}}\right)+\left(\frac{\log y-\mu_2}{\sigma_{21}}\right)^2\right\}\right] \\ \qquad h_1<\log x<\mu_1, k_1<\log y<\mu_2 \\ K(xy)^{-1}\exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{\log x-\mu_1}{\sigma_{11}}\right)^2-2\rho\left(\frac{\log x-\mu_1}{\sigma_{11}}\right)\left(\frac{\log y-\mu_2}{k_2\sigma_{22}}\right)+\left(\frac{\log y-\mu_2}{k_2\sigma_{22}}\right)^2\right\}\right] \\ \qquad h_1<\log x<\mu_1, \mu_2<\log y<m_1 \\ K(xy)^{-1}\exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{\log x-\mu_1}{k_1\sigma_{12}}\right)^2-2\rho\left(\frac{\log x-\mu_1}{k_1\sigma_{12}}\right)\left(\frac{\log y-\mu_2}{k_1\sigma_{21}}\right)+\left(\frac{\log y-\mu_2}{k_1\sigma_{21}}\right)^2\right\}\right] \\ \qquad \mu_1<\log x<l_1, k_1<\log y<\mu_2 \\ K(xy)^{-1}\exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{\log x-\mu_1}{k_1\sigma_{12}}\right)^2-2\rho\left(\frac{\log x-\mu_1}{k_1\sigma_{12}}\right)\left(\frac{\log y-\mu_2}{k_2\sigma_{22}}\right)+\left(\frac{\log y-\mu_2}{k_2\sigma_{22}}\right)^2\right\}\right] \\ \qquad \mu_1<\log x<l_1, \mu_2<\log y<m_1 \end{cases} \quad \textbf{(2.1)}$$

where $K=\frac{2}{\pi}\left[(1+k_1)(1+k_2)\,\sigma_{11}\sigma_{22}\,\sqrt{1-\rho^2}\right]^{-1}$

Considering $z_{11}=\frac{\log x-\mu_1}{\sigma_{11}},\ z_{21}=\frac{\log y-\mu_2}{\sigma_{21}}$

and $h=\frac{h_1-\mu_1}{\sigma_{11}},\ k=\frac{k_1-\mu_1}{\sigma_{11}},\ m=\frac{m_1-\mu_2}{\sigma_{21}},\ l=\frac{l_1-\mu_2}{\sigma_{21}}$

we get the doubly truncated standard two piece bivariate lognormal distribution as

$$f(z_{11},z_{21})=\begin{cases} C_1K\ \exp\left\{\frac{-1}{2(1-\rho^2)}\left[z_{11}^2-2\rho z_{11}z_{21}+z_{21}^2\right]\right\} & h<z_{11}<0,\ k<z_{21}<0 \\ C_2K\ \exp\left\{\frac{-1}{2(1-\rho^2)}\left[z_{11}^2-2\rho z_{11}\frac{z_{21}}{k_2}+\left(\frac{z_{21}}{k_2}\right)^2\right]\right\} & h<z_{11}<0,\ 0<z_{21}<m \\ C_3K\ \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{z_{11}}{k_1}\right)^2-2\rho\left(\frac{z_{11}}{k_1}\right)z_{21}+z_{21}^2\right]\right\} & 0<z_{11}<l,\ k<z_{21}<0 \\ C_4K\ \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{z_{11}}{k_1}\right)^2-2\rho\left(\frac{z_{11}}{k_1}\right)\left(\frac{z_{21}}{k_2}\right)+\left(\frac{z_{21}}{k_2}\right)^2\right]\right\} & 0<z_{11}<l,\ 0<z_{21}<m \end{cases}$$

(2.2)

where $K = \frac{2}{\pi}\left[(1+k_1)(1+k_2)\sigma_{11}\sigma_{21}\sqrt{1-\rho^2}\right]^{-1}$

$$C_1^{-1} = 8K\sigma_{11}\sigma_{21}(2\pi)\left[\frac{1}{2}-\Phi\left(\frac{h-\rho z_{21}}{\sqrt{1-\rho^2}}\right)\right]\left[\frac{1}{2}-\Phi(k)\right]$$

$$C_2^{-1} = 8K\sigma_{11}\sigma_{21}(2\pi)\left[\frac{1}{2}-\Phi\left(\frac{h-\rho \frac{z_{21}}{k_2}}{\sqrt{1-\rho^2}}\right)\right]\left[\Phi\left(\frac{m}{k_2}\right)-\frac{1}{2}\right]$$

$$C_3^{-1} = 8K\sigma_{11}\sigma_{21}(2\pi)\left[\Phi\left(\frac{\frac{l}{k_1}-\rho z_{21}}{\sqrt{1-\rho^2}}\right)-\frac{1}{2}\right]\left[\frac{1}{2}-\Phi(k)\right]$$

$$C_4^{-1} = 8K\sigma_{11}\sigma_{21}(2\pi)\left[\Phi\left(\frac{\frac{l}{k_1}-\rho z_{21}}{\sqrt{1-\rho^2}}\right)-\frac{1}{2}\right]\left[\Phi\left(\frac{m}{k_2}\right)-\frac{1}{2}\right]$$

Let

$$G_S(t,\mu,\sigma) = \int_t^\infty \frac{x^S}{\sigma} Z\left(\frac{x-\mu}{\sigma}\right)\, dx$$

$$= t^{S-1}\sigma Z\left(\frac{t-\mu}{\sigma}\right) + (s-1)\sigma G_{S-2}(t,\mu,\sigma) + \mu G_{S-1}(t,\mu,\sigma) \tag{2.3}$$

and $G_O(t,\mu,\sigma) = \int_t^\infty \frac{1}{\sigma} Z\left(\frac{x-\mu}{\sigma}\right)\, dx = \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\, dx = \Phi\left(\frac{t-\mu}{\sigma}\right)$ (2.4)

Also

$$G_S(t,p,\mu,\sigma) = \int_t^p \frac{1}{\sqrt{2\pi}\sigma} x^S e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\, dx$$

$$= \sigma\left[p^{S-1}Z\left(\frac{p-\mu}{\sigma}\right) - t^{S-1}Z\left(\frac{t-\mu}{\sigma}\right)\right] + (s-1)\sigma G_{S-2}(t,p,\mu,\sigma) + \mu G_{S-1}(t,p,\mu,\sigma) \tag{2.5}$$

and

$$G_O(t,p,\mu,\sigma) = \int_t^p \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\, dx = \Phi\left(\frac{t-\mu}{\sigma}\right) - \Phi\left(\frac{p-\mu}{\sigma}\right) \tag{2.6}$$

3 Estimation of Parameters

$$V_{r,s} = E\left[z_{11}^{r}\ z_{21}^{s} \right]$$

$$= K(2\pi)(1-\rho^2)^{3/2}\Big[h^{r-1}Z(h)\Big\{C_1G_s\left(k,o,h\rho,\sqrt{1-\rho^2}\right)+C_2k_2^5G_s\left(o,m,h\rho,\sqrt{1-\rho^2}\right)$$

$$-l^{r-1}Z\left(\frac{l}{k_1}\right)k_1^2\left\{C_3G_s\left(k,o,\frac{\rho l}{k_1},\sqrt{1-\rho^2}\right)+C_4k_2^2G_s\left(o,m,\frac{\rho l}{k_1},\sqrt{1-\rho^2}\right)\right\}\Big]$$

$$+(r-1)(1-\rho^2)\ \left[V_{r-2,s}^1 + V_{r-2,s}^2 + k_1^2\left(V_{r-2,s}^3 + V_{r-2,s}^4\right)\right]$$

$$+\ \rho\left[V_{r-1,s+1}^1 + \frac{1}{k_2}V_{r-1,s+1}^2 + k_1\left(V_{r-1,s+1}^3 + \frac{1}{k_2}\ V_{r-1,s+1}^4\right)\right] \qquad (3.1)$$

(3.1) can be rewritten as follows.

$$V_{r,s} = C_1K(2\pi)\sqrt{1-\rho^2}\Big[h^{r-1}Z(h)\ G_s\left(k,o,h\rho,\sqrt{1-\rho^2}\right)$$

$$+k^{s-1}\ Z(k)(r-1)(1-\rho^2)G_{r-2}\left(h,o,k\rho,\sqrt{1-\rho^2}\right)$$

$$+k^s\ Z(k)\rho\ G_{r-1}\left(h,o,k\rho,\sqrt{1-\rho^2}\right)\Big]$$

$$+C_2K\ (2\pi)\sqrt{1-\rho^2}\Big[h^{r-1}\ Z(h)k_2^5G_s\left(o,m,h\rho,\sqrt{1-\rho^2}\right)$$

$$-k_2^2\ m^{s-1}\ Z\left(\frac{m}{k_2}\right)(r-1)(1-\rho^2)G_{r-2}\left(h,o,\frac{\rho m}{k_2},\sqrt{1-\rho^2}\right)$$

$$-k_2\ m^s\ Z\left(\frac{m}{k_2}\right)\rho\ G_{r-1}\left(h,o,\frac{\rho m}{k_2},\sqrt{1-\rho^2}\right)\Big]$$

$$-C_3K\ (2\pi)\sqrt{1-\rho^2}\Big[\ l^{r-1}k_1^2\ Z\left(\frac{l}{k_1}\right)\ G_s\left(k,o,\frac{\rho l}{k_1},\sqrt{1-\rho^2}\right)$$

$$-k_1^r k^{s-1}Z(k)(1-\rho^2)(r-1)G_{r-2}\left(o,l,\rho k,\sqrt{1-\rho^2}\right)$$

$$-k_1^r\ k^s Z(k)\ \rho\ G_{r-1}\left(o,l,\rho k,\sqrt{1-\rho^2}\right)\Big]$$

$$-C_4K\ (2\pi)\sqrt{1-\rho^2}\ \Big[k_1^2k_2^5\ l^{r-1}\ Z\left(\frac{l}{k_1}\right)\ G_s\left(o,m,\frac{\rho l}{k_1},\sqrt{1-\rho^2}\right)$$

$$+k_1^r\ k_2^2\ m^{s-2}\ Z\left(\frac{m}{k_2}\right)(1-\rho^2)(r-1)\ G_{r-2}\left(o,l,\frac{\rho m}{k_2},\sqrt{1-\rho^2}\right)$$

$$+k_1^r\frac{m^s}{k_2}\ Z\left(\frac{m}{k_2}\right)\ \rho\ G_{r-1}\left(o,l,\frac{\rho m}{k_2},\sqrt{1-\rho^2}\right)\Big].$$

$$+\rho(r+s-1)\left[V_{r-1,s-1}^1 + k_2V_{r-1,s-1}^2 + k_1V_{r-1,s-1}^3 + k_1k_2V_{r-1,s-1}^4\right]$$

$$+(r-1)(s-1)(1-\rho^2)\left[V_{r-2,s-2}^1 + k_2^2V_{r-2,s-2}^2 + k_1^2V_{r-2,s-2}^3 + k_1^2k_2^2V_{r-2,s-2}^4\right] \qquad (3.2)$$

Considering s = 0 in (3.2), we get $V_{r,0}$ and by symmetry we get V_{0s}

We can obtain the first and second order moments V_{10} , V_{01}, V_{2o}, V_{02} using $V_{r,0}$ and V_{0s}.

Putting r = 1, s = 1, in (3.2), we get $V_{1,1}$

Now, $E(z_{11}) = V_{10}$, $E(z_{21}) = V_{01}$, $V(z_{11}) = V_{20} - (V_{10})^2$, $V(z_{21}) = V_{02} - (V_{01})^2$.

Also, $Cov(z_{11}, z_{21}) = V_{11} - V_{10}V_{01}$ and Correlation coefficient (ρ) is $\frac{V_{11} - V_{10}V_{01}}{\sqrt{V(z_{11})}\sqrt{V(z_{11})}}$.

4. Singly truncated two piece bivariate lognormal distributions

Singly TTPBLN distribution can be shown as a particular case of doubly truncated distribution.
If we consider $h = -\infty$ *and* $k = -\infty$ and then the upper truncated TPBLN distribution will be given as

$$f(z_{11}, z_{21}) = \begin{cases} C_1K \exp\left\{\frac{-1}{2(1-\rho^2)}\left[z_{11}^2 - 2\rho z_{11}z_{21} + z_{21}^2\right]\right\} & -\infty < z_{11} < 0,\ -\infty < z_{21} < 0 \\ C_2K \exp\left\{\frac{-1}{2(1-\rho^2)}\left[z_{11}^2 - 2\rho z_{11}\frac{z_{21}}{k_2} + \left(\frac{z_{21}}{k_2}\right)^2\right]\right\} & -\infty < z_{11} < 0,\ 0 < z_{21} < m \\ C_3K \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{z_{11}}{k_1}\right)^2 - 2\rho\left(\frac{z_{11}}{k_1}\right)z_{21} + z_{21}^2\right]\right\} & 0 < z_{11} < l,\ -\infty < z_{21} < 0 \\ C_4K \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{z_{11}}{k_1}\right)^2 - 2\rho\left(\frac{z_{11}}{k_1}\right)\left(\frac{z_{21}}{k_2}\right) + \left(\frac{z_{21}}{k_2}\right)^2\right]\right\} & 0 < z_{11} < l,\ 0 < z_{21} < m \end{cases}$$

Estimation of Parameters of upper TTPBLN distribution

Let $V_{r,s} = E\left[z_{11}^r\ z_{21}^s\right]$

$$\begin{aligned} \therefore V_{r,s} = & -C_2K(2\pi)\sqrt{1-\rho^2}\left[k_2^2 m^{s-1} Z\left(\frac{m}{k_1}\right)(r-1)(1-\rho^2)G_{r-2}\left(-\infty, o, \frac{\rho m}{k_2}, \sqrt{1-\rho^2}\right)\right. \\ & \left. - k_2\ m^s\ Z\left(\frac{m}{k_2}\right)\rho\ G_{r-1}\left(-\infty, o, \frac{\rho m}{k_2}, \sqrt{1-\rho^2}\right)\right] \\ & - C_3K\ (2\pi)\sqrt{1-\rho^2}\ \left[l^{r-1}k_1^2\ Z\left(\frac{l}{k_1}\right)\ G_s\left(-\infty, o, \frac{\rho l}{k_1}, \sqrt{1-\rho^2}\right)\right] \\ & - C_4K\ (2\pi)\sqrt{1-\rho^2}\ \left[k_1^2 k_2^s\ l^{r-1}\ Z\left(\frac{l}{k_1}\right)\ G_s\left(o, m, \frac{\rho l}{k_1}, \sqrt{1-\rho^2}\right)\right. \\ & + k_1^r\ k_2^2\ m^{s-2}\ Z\left(\frac{m}{k_2}\right)(1-\rho^2)(r-1)\ G_{r-2}\left(o, l, \frac{\rho m}{k_2}, \sqrt{1-\rho^2}\right) \\ & \left. + k_1^r\frac{m^s}{k_2}\ Z\left(\frac{m}{k_2}\right)\ \rho\ G_{r-1}\left(o, l, \frac{\rho m}{k_2}, \sqrt{1-\rho^2}\right)\right]. \end{aligned} \qquad \textbf{(4.1)}$$

By considering s = 0 in (4.1), we get $V_{r,0}$ and by symmetry we obtain V_{os}

We can obtain the first and second order moments V_{10} , V_{01}, V_{2o}, V_{02} using $V_{r,0}$ and V_{0s}.

Putting r = 1, s = 1, in (4.1), we get $V_{1,1}$

Now, $E(z_{11}) = V_{10}$, $E(z_{21}) = V_{01}$, $V(z_{11}) = V_{20} - (V_{10})^2$, $V(z_{21}) = V_{02} - (V_{01})^2$.

Also, $Cov(z_{11}, z_{21}) = V_{11} - V_{10}V_{01}$ and Correlation coefficient (ρ) is $\dfrac{V_{11} - V_{10}V_{01}}{\sqrt{V(z_{11})}\sqrt{V(z_{11})}}$.

If we consider $l = \infty$ *and* $m = \infty$ *Then* the lower truncated TPBLN distribution will be given as

$$f(z_{11}, z_{21}) = \begin{cases} C_1 K \exp\left\{\dfrac{-1}{2(1-\rho^2)}\left[z_{11}^2 - 2\rho z_{11} z_{21} + z_{21}^2\right]\right\} & h < z_{11} < 0,\ k < z_{21} < 0 \\ C_2 K \exp\left\{\dfrac{-1}{2(1-\rho^2)}\left[z_{11}^2 - 2\rho z_{11}\dfrac{z_{21}}{k_2} + \left(\dfrac{z_{21}}{k_2}\right)^2\right]\right\} & h < z_{11} < 0,\ 0 < z_{21} < \infty \\ C_3 K \exp\left\{\dfrac{-1}{2(1-\rho^2)}\left[\left(\dfrac{z_{11}}{k_1}\right)^2 - 2\rho\left(\dfrac{z_{11}}{k_1}\right) z_{21} + z_{21}^2\right]\right\} & 0 < z_{11} < \infty,\ k < z_{21} < 0 \\ C_4 K \exp\left\{\dfrac{-1}{2(1-\rho^2)}\left[\left(\dfrac{z_{11}}{k_1}\right)^2 - 2\rho\left(\dfrac{z_{11}}{k_1}\right)\left(\dfrac{z_{21}}{k_2}\right) + \left(\dfrac{z_{21}}{k_2}\right)^2\right]\right\} & 0 < z_{11} < \infty,\ 0 < z_{21} < \infty \end{cases}$$

Estimation of Parameters of lower TTPBLN distribution

Let $V_{r,s} = E\left[z_{11}^r\ z_{21}^s\right]$

$$\begin{aligned} \therefore V_{r,s} = {} & C_1 K (2\pi)\sqrt{1-\rho^2}\Big[h^{r-1} Z(h)\ G_s\left(k, o, h\rho, \sqrt{1-\rho^2}\right) \\ & + k^{s-1}\ Z(k)(r-1)(1-\rho^2) G_{r-2}\left(h, o, k\rho, \sqrt{1-\rho^2}\right) \\ & + k^s\ Z(k)\rho\ G_{r-1}\left(h, o, k\rho, \sqrt{1-\rho^2}\right)\Big] \\ & + C_2 K\ (2\pi)\sqrt{1-\rho^2} h^{r-1} Z(h) k_2^s G_s\left(o, \infty, h\rho, \sqrt{1-\rho^2}\right) \\ & + C_3 K\ (2\pi)\sqrt{1-\rho^2}\Big[\ k_1^r k^{s-1}\ Z(k)(1-\rho^2)(r-1) G_{r-2}\left(o, \infty, \rho k, \sqrt{1-\rho^2}\right) \\ & + k_1^r\ k^s\ Z(k)\ \rho\ G_{r-1}\left(o, \infty, \rho k, \sqrt{1-\rho^2}\right)\Big] \end{aligned} \qquad \textbf{(4.2)}$$

As done before we get,

$E(z_{11}) = V_{10}$, $E(z_{21}) = V_{01}$, $V(z_{11}) = V_{20} - (V_{10})^2$, $V(z_{21}) = V_{02} - (V_{01})^2$.

Also, $Cov(z_{11}, z_{21}) = V_{11} - V_{10}V_{01}$ and Correlation coefficient (ρ) is $\dfrac{V_{11} - V_{10}V_{01}}{\sqrt{V(z_{11})}\sqrt{V(z_{11})}}$.

REFERENCES

[1]. Carobbi, C.F.M, Coti, M. and Millanta, L.M. (2003) Using the lognormal distribution in the statistical treatment of experimental data affected by large dispersion. Electromagnetic compatibility, 2003 IEEE International Symposium 2, 812-816.

[2]. Da Hsiang Donald Lien (1985) Moments of truncated bivariate log-normal distribution, Economics letters, Vol. 19,30, 243-247.

[3]. Nabar, S.P. and Deshmukh, S.C. (2002). On estimation of parameters of bilognormal distribution, JISA, 40, 1, 59-70.

[4]. Sheng Yue (2000) The bivariate lognormal distribution for describing joint statistical properties of a multivariate storm event, Environmetrics, 13, 8, 811-819.

[5]. Sheng Yue (2000a) The bivariate lognormal distribution to model a multivariate flood episode, Hydrological Processes, 14, 14, 2575-2588.

[6]. Shimizu, K. (1993). A bivariate mixed lognormal distribution with an analysis of rainfall data Journal of Applied Meterlogy 322, 161-171.

[7]. Wu, Chuan-Yi (1966) The types of limit distributions for some terms of Variational Series, Scientia Sinica, 15, 749-762.