Vol.3.Issue.3.2015

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

http://www.bomsr.com

A Peer Reviewed International Research Journal

ISSN:2348-0580

RESEARCH ARTICLE

SOME FIXED POINT THEOREMS OF EXPANSION MAPPING SATISFYING IMPLICIT RELATION

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ABSTRACT

In this paper, we prove some fixed point theorem for expansion mapping satisfying implicit relation.

Mathematical Subject Classification: 47H10 and 54H25.

Key words-Semi compatible mappings, Weak compatibles mappings, implicit relation, Common fixed point.

1. INTRODUCTION

Wang, Li, Gao and lseki [14] proved some fixed point theorems on expansion mappings which correspond to some contractive mappings. In a paper Rhoades [9] generalized the results for pairs of mapping. Some theorems on unique fixed point for expansion mapping are proved by Popa [6]. Popa [7] further extended results [6], [9] for compatible mappings

In 1999, Popa [8] proved some fixed point theorems for compatible mappings satisfying an implicit relation.

Let S and T be two self mappings of a metric space (X,d). Sessa [10] defines $S\&T$ to be weakly commuting if $d(STx,TSx)\le d(Tx,Sx)$ for all $x\in X$. Jungck [1] defines S and T to be compatible if $\lim_{n\to\infty} d(STx_n,TSx_n)=0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} Tx_n=x$ for some x in X inclearly, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but implications are not reversible [11,Ex1]and[1,Ex 2.2] .

Many authores have proved common fixed point theorems for compatible mappings for this we refer to Jungck [1],[2] and [3], Sessa, Rhoades and Khan [12], Kang, Cho and Jungck [4], Kang and Ray [5] and Sharma and Patidar [13] .

In this paper, we prove common fixed point theorems for semi and weak compatible mapping in Metric spaces, satisfying an implicit relation.

Lemma1.1-Let S and T be compatible self mappings on a metric space (X,d). If $\lim Sx_n = \lim Tx_n$ then $\lim STx_n = \lim TSx_n$.

Lemma1.2-Let S and T are the self map of metric space (X,d).Then pair (S,T) be semi compatible if $\lim Sx_n = \lim Tx_n = z$ then $\lim STx_n = Tz$.

Lemma1.3- Let S and T are the self map of metric space (X,d).Then pair (S,T) be weak compatible if $Tz = Sz \Rightarrow TSz = STz$.

2. Implicit Relations-

Let Φ be the set of all real continuous functions $\phi(t_1,t_2,t_3..t_6): R_+^6 \to R$ satisfying the following conditions:

$\phi_1 : \phi$ is non increasing in variable t_6.

ϕ_2 :there exist $h>1$ such that for every $u,v \ge 0$ with

$$(\phi_a): \phi\left(u,v,v,u,\frac{u+v}{2},0\right) \ge 0$$

$$(\phi_b): \phi\left(u,v,u,v,\frac{u+v}{2},u+v\right) \ge 0$$

We have $v \ge hu$

ϕ_3: $\phi(u,u,0,0,0,u) < 0$ For all $u>0$.

Example 2.1- $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_2 - h\ \max\left(t_1,t_3,t_4,t_5,{t_6}/{2}\right)$ where $h>1$

ϕ_1 : Obviously

ϕ_2 :Let $v>0$

$$(\phi_a): \phi\left(u,v,v,u,\frac{u+v}{2},0\right) = v - h\ \max\left(u,v,u,\frac{u+v}{2},0\right) \ge 0$$

If $u \le v$ then $v - hv \ge 0$

$v \ge hv > v \quad for\, h>1 \Rightarrow v>v$, which is contradiction, thus $u>v$

Therefore we will have $v - hu \ge 0 \Rightarrow v \ge hu$.

$$(\phi_b): \phi\left(u,v,u,v,\frac{u+v}{2},u+v\right) = v - h\ \max\left(u,u,v,\frac{u+v}{2},u+v\right) \ge 0$$

If $v>0$ and $u>v$ then

$v - h(u+v) \ge 0$

$v - hu - hv \ge 0$

$v - hu \ge 0 \Rightarrow v \ge hu$

ϕ_3: $\phi\left(u,u,0,0,0,{u}/{2}\right) = u - h\ \max\left(u,0,0,0,{u}/{2}\right) < 0$

$for\ u>v \therefore \quad = u - hu < 0$

$= u(1-h) < 0$

Since $h>1 \therefore u>0$

3. Main Results

Theorem3.1- Let (X,d) be a complete metric space and $A\&B$ be self maps of Metric space satisfying

(a) $A(X)\subset B(X)$

$$(b)\quad \phi\begin{bmatrix} d(Ax,Ay),d(Bx,By),d(Ax,Bx),d(Ay,By), \\ \tfrac{1}{2}\{d(Ax,Bx)+d(Ay,By)\},d(Ax,By) \end{bmatrix}\geq 0$$

For all $x,y\in X$ & $\phi\in\Phi$

(c) Either $A\ or\ B$ is continuous.

(d) (A,B) is semi compatible and weak compatible

Then A & B have a unique common fixed point in X .

Proof-Let $x_0\in X$ and $A(X)\subset B(X)$ then there exist a point $x_1\in X$ such that $Bx_1=Ax_0=y_0$. Inductively we can define a sequence $Bx_{n+1}=Ax_n=y_n$.

Step (1) - By using (b) with $x=x_n$, $y=x_{n+1}$

$$\phi\begin{bmatrix} d(Ax_n,Ax_{n+1}),d(Bx_n,Bx_{n+1}),d(Ax_n,Bx_n),d(Ax_{n+1},Bx_{n+1}), \\ \tfrac{1}{2}\{d(Ax_n,Bx_n)+d(Ax_{n+1},Bx_{n+1})\},d(Ax_n,Bx_{n+1}) \end{bmatrix}\geq 0$$

$$\phi\begin{bmatrix} d(y_n,y_{n+1}),d(y_{n-1},y_n),d(y_n,y_{n-1}),d(y_{n+1},y_n), \\ \tfrac{1}{2}\{d(y_n,y_{n-1})+d(y_{n+1},y_n)\},d(y_n,y_n) \end{bmatrix}\geq 0$$

$$\phi\begin{bmatrix} d(y_n,y_{n+1}),d(y_{n-1},y_n),d(y_n,y_{n-1}),d(y_{n+1},y_n), \\ \tfrac{1}{2}\{d(y_n,y_{n-1})+d(y_{n+1},y_n)\},0 \end{bmatrix}\geq 0$$

By (ϕ_a), $v\geq hu$

$$d(y_{n-1},y_n)\geq h\ d(y_n,y_{n+1})$$

$$d(y_n,y_{n+1})\leq \tfrac{1}{h}(y_{n-1},y_n)\ ...(1)$$

Again by using (b) with $x=x_n$, $y=x_{n-1}$

$$\phi\begin{bmatrix} d(Ax_n,Ax_{n-1}),d(Bx_n,Bx_{n-1}),d(Ax_n,Bx_n),d(Ax_{n-1},Bx_{n-1}), \\ \tfrac{1}{2}\{d(Ax_n,Bx_n)+d(Ax_{n-1},Bx_{n-1})\},d(Ax_n,Bx_{n-1}) \end{bmatrix}\geq 0$$

$$\phi\begin{bmatrix} d(y_n,y_{n-1}),d(y_{n-1},y_{n-2}),d(y_n,y_{n-1}),d(y_{n-1},y_{n-2}), \\ \tfrac{1}{2}\{d(y_n,y_{n-1})+d(y_{n-1},y_{n-2})\},d(y_n,y_{n-2}) \end{bmatrix}\geq 0$$

Since by triangular inequality we have

$d(y_n,y_{n-2})\leq d(y_n,y_{n-1})+(y_{n-1},y_{n-2})$ Therefore

$$\phi\begin{bmatrix} d(y_n,y_{n-1}),d(y_{n-1},y_{n-2}),d(y_n,y_{n-1}),d(y_{n-1},y_{n-2}), \\ \tfrac{1}{2}\{d(y_n,y_{n-1})+d(y_{n-1},y_{n-2})\},\{d(y_n,y_{n-1})+d(y_{n-1},y_{n-2})\} \end{bmatrix}\geq 0$$

By (ϕ_b), $v\geq hu$

$d(y_{n-1}, y_{n-2}) \geq h\, d(y_n, y_{n-1})$

$d(y_n, y_{n-1}) \leq 1/h\,(y_{n-1}, y_{n-2})$ ...(2)

By equation (1) and (2)

$d(y_n, y_{n+1}) \leq 1/h^2\, d(y_{n-1}, y_{n-2})$

Similarly it can be found

$d(y_n, y_{n+1}) \leq 1/h^n\, d(y_1, y_0)$

$\lim n \to \infty$, $d(y_n, y_{n+1}) < \varepsilon$ where $\varepsilon > 0$

Therefore $\{y_n\}$ is Cauchy sequence.

Since X is complete therefore $\{y_n\} \to z$, and then all of its subsequence also converges to z. Therefore

$\lim Ax_n = z$, $\lim Bx_{n+1} = z$.

Step (2) - A is continuous

Since

$\lim Ax_n = z$, $\lim Bx_n = z$

$\therefore \lim AAx_n = Az$ & $\lim ABx_n = Az$

Also pair (A, B) is semi compatible then, $\lim BAx_n = Az$.

Step (3) -By using (b) with $x = Ax_n$, $y = x_n$

$$\phi\begin{bmatrix} d(AAx_n, Ax_n), d(BAx_n, Bx_n), d(AAx_n, BAx_n), d(Ax_n, Bx_n), \\ 1/2\{d(AAx_n, BAx_n) + d(Ax_n, Bx_n)\}, d(AAx_n, Bx_n) \end{bmatrix} \geq 0$$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(Az, z), d(Az, z), d(Az, Az), d(z, z), \\ 1/2\{d(Az, Az) + d(z, z)\}, d(Az, z) \end{bmatrix} \geq 0$$

$$\phi[d(Az, z), d(Az, z), 0, 0, 0, d(Az, z)] \geq 0$$

Which is contradiction of ϕ_3: $\phi(u, u, 0, 0, 0, u) < 0$

It is only possible when $u = 0$ or $d(Az, z) = 0 \Rightarrow Az = z$

Step (4) –Since $A(X) \subset B(X)$. Then there exist a point $w \in X$ such that $Az = Bw = z$.

By using (b) with $x = x_n$, $y = w$

$$\phi\begin{bmatrix} d(Ax_n, Aw), d(Bx_n, Bw), d(Ax_n, Bx_n), d(Aw, Bw), \\ 1/2\{d(Ax_n, Bx_n) + d(Aw, Bw)\}, d(Ax_n, Bw) \end{bmatrix} \geq 0$$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(z, Aw), d(z, z), d(z, z), d(Aw, z), \\ 1/2\{d(z, z) + d(Aw, z)\}, d(z, z) \end{bmatrix} \geq 0$$

$$\phi\left[d(z, Aw), 0, 0, d(Aw, z), 1/2\, d(Aw, z), 0\right] \geq 0$$

By (ϕ_a) we have $v \geq hu$

$0 \geq h\ d(z, Aw)$ Since $h > 1$ therefore $d(z, Aw) = 0 \Rightarrow Aw = z$,and so $Aw = Bw$

Since (A, B) is weak compatible, therefore $\therefore ABw = BAw \Rightarrow Az = Bz = z$

Therefore z is a common fixed point of $A \& B$.

Step (5)- B is continuous

Since $\lim Ax_n = z$, $\lim Bx_n = z$, therefore

$\lim BAx_n = Bz$, $\lim BBx_n = Bz$

Again pair (A, B) is semi compatible then $\lim ABx_n = Bz$

By using (b) with $x = Bx_n$, $y = x_n$

$$\phi\begin{bmatrix} d(ABx_n, Ax_n), d(BBx_n, Bx_n), d(ABx_n, BBx_n), d(Ax_n, Bx_n), \\ \tfrac{1}{2}\{d(ABx_n, BBx_n) + d(Ax_n, Bx_n)\}, d(ABx_n, Bx_n) \end{bmatrix} \geq 0$$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(Bz, z), d(Bz, z), d(Bz, Bz), d(z, z), \\ \tfrac{1}{2}\{d(Bz, Bz) + d(z, z)\}, d(Bz, z) \end{bmatrix} \geq 0$$

$$\phi[d(Bz, z), d(Bz, z), 0, 0, 0, d(Bz, z)] \geq 0$$

Which is contradiction of ϕ_3: $\phi(u, u, 0, 0, 0, u) < 0$

It is only possible when $u = 0$ or $d(Bz, z) = 0 \Rightarrow Bz = z$

Step (6)- By using (b) with $x = x_n$, $y = z$

$$\phi\begin{bmatrix} d(Ax_n, Az), d(Bx_n, Bz), d(Ax_n, Bx_n), d(Az, Bz), \\ \tfrac{1}{2}\{d(Ax_n, Bx_n) + d(Az, Bz)\}, d(Ax_n, Bz) \end{bmatrix} \geq 0$$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(z, Az), d(z, z), d(z, z), d(Az, z), \\ \tfrac{1}{2}\{d(z, z) + d(Az, z)\}, d(z, Bz) \end{bmatrix} \geq 0$$

By (ϕ_a) we have, $0 \geq h\ d(z, Az)$

Since $h > 1$ therefore $d(z, Az) = 0 \Rightarrow Az = z$

Or $Az = Bz = z$ .Therefore z is a common fixed point of A & B.

Uniqueness- Let u is another fixed point of A & B, therefore $Au = Bu = u$.

Then by using (b) with $x = u$ & $y = z$,

$$\phi\begin{bmatrix} d(Au, Az), d(Bu, Bz), d(Au, Bu), d(Az, Bz), \\ \tfrac{1}{2}\{d(Au, Bu) + d(Az, Bz)\}, d(Au, Bz) \end{bmatrix} \geq 0$$

$$\phi\begin{bmatrix} d(u, z), d(u, z), d(u, u), d(z, z), \\ \tfrac{1}{2}\{d(u, u) + d(z, z)\}, d(u, z) \end{bmatrix} \geq 0$$

$\phi\left[d(u,z),d(u,z),0,0,0,d(u,z)\right]\geq 0$

Which is contradiction of ϕ_3: $\phi(u,u,0,0,0,u)<0$

It is only possible when $u=0$ or $d(u,z)=0\Rightarrow u=z$.

Therefore z is unique common fixed point of A & B.

Corollary 3.2-Let T, F and S be self mapping of metric space (X,d) with

(a) $T(X)\subset F(X)$, $S(X)\subset F(X)$

(b) $\psi\left[d(Tx,Sy)\right]\geq \psi\left[d(Tx,Fx)+d(Tx,Fy)\right]+\varphi\left[d(Tx,Sy)\right]$

(c) Either T or F is continuous function.

(d) (T,F) is semi compatible and weak compatible.

$(e)\, TS=ST$, $FS=SF$

If ψ and φ are monotonic increasing function such that $\psi,\varphi:[0,\infty)\rightarrow[0,\infty)$ and $\psi(t)=\varphi(t)=0\Leftrightarrow t=0$, then z is unique common fixed point of F and T.

Theorem3.3- Let (X,d) be a complete metric space and A,B,G & F be self maps of Metric space satisfying

(a) $A(X)\subset B(X)$, $F(X)\subset G(X)$

$$(b)\quad \phi\begin{bmatrix} d(Ax,Fy),d(Gx,By),d(Ax,Gx),d(Fy,By), \\ \tfrac{1}{2}\{d(Ax,Gx)+d(Fy,By)\},d(Ax,By) \end{bmatrix}\geq 0$$

For all $x,y\in X$ & $\phi\in\Phi$

(c) Either A *or* G is continuous.

(d) (A,G) is semi compatible and (F,B) weak compatible

$(e)\, FG=GF$ & $BG=GB$

Then A,B,F & G have a unique fixed point in X.

Proof-Let $x_0\in X$ and $A(X)\subset B(X)$ and $F(X)\subset G(X)$ then there exist a point $x_1,x_2\in X$ such that $Bx_1=Ax_0=y_0$ & $Gx_2=Fx_1=y_1$. Inductively we can define a sequence $Bx_{n+1}=Ax_n=y_n$ & $Gx_{n+2}=Fx_{n+1}=y_{n+1}$.

Step (1) - By using (b) with $x=x_n$, $y=x_{n+1}$

$$\phi\begin{bmatrix} d(Ax_n,Fx_{n+1}),d(Gx_n,Bx_{n+1}),d(Ax_n,Gx_n),d(Fx_{n+1},Bx_{n+1}), \\ \tfrac{1}{2}\{d(Ax_n,Gx_n)+d(Fx_{n+1},Bx_{n+1})\},d(Ax_n,Bx_{n+1}) \end{bmatrix}\geq 0$$

$$\phi\begin{bmatrix} d(y_n,y_{n+1}),d(y_{n-1},y_n),d(y_n,y_{n-1}),d(y_{n+1},y_n), \\ \tfrac{1}{2}\{d(y_n,y_{n-1})+d(y_{n+1},y_n)\},d(y_n,y_n) \end{bmatrix}\geq 0$$

$$\phi\begin{bmatrix} d(y_n,y_{n+1}),d(y_{n-1},y_n),d(y_n,y_{n-1}),d(y_{n+1},y_n), \\ \tfrac{1}{2}\{d(y_n,y_{n-1})+d(y_{n+1},y_n)\},0 \end{bmatrix}\geq 0$$

By (ϕ_a), $v \geq hu$

$d(y_{n-1}, y_n) \geq h\ d(y_n, y_{n+1})$

$d(y_n, y_{n+1}) \leq \frac{1}{h}(y_{n-1}, y_n)$ ...(1)

Again by using (b) with $x = x_n$, $y = x_{n-1}$

$$\phi\begin{bmatrix} d(Ax_n, Fx_{n-1}), d(Gx_n, Bx_{n-1}), d(Ax_n, Gx_n), d(Fx_{n-1}, Bx_{n-1}), \\ \frac{1}{2}\{d(Ax_n, Gx_n) + d(Fx_{n-1}, Bx_{n-1})\}, d(Ax_n, Bx_{n-1}) \end{bmatrix} \geq 0$$

$$\phi\begin{bmatrix} d(y_n, y_{n-1}), d(y_{n-1}, y_{n-2}), d(y_n, y_{n-1}), d(y_{n-1}, y_{n-2}), \\ \frac{1}{2}\{d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2})\}, d(y_n, y_{n-2}) \end{bmatrix} \geq 0$$

Since by triangular inequality we have

$d(y_n, y_{n-2}) \leq d(y_n, y_{n-1}) + (y_{n-1}, y_{n-2})$ Therefore

$$\phi\begin{bmatrix} d(y_n, y_{n-1}), d(y_{n-1}, y_{n-2}), d(y_n, y_{n-1}), d(y_{n-1}, y_{n-2}), \\ \frac{1}{2}\{d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2})\}, \{d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2})\} \end{bmatrix} \geq 0$$

By (ϕ_b), $v \geq hu$

$d(y_{n-1}, y_{n-2}) \geq h\ d(y_n, y_{n-1})$

$d(y_n, y_{n-1}) \leq \frac{1}{h}(y_{n-1}, y_{n-2})$ ...(2)

By equation (1) and (2)

$d(y_n, y_{n+1}) \leq \frac{1}{h^2} d(y_{n-1}, y_{n-2})$

Similarly it can be found

$d(y_n, y_{n+1}) \leq \frac{1}{h^n} d(y_1, y_0)$

$\lim n \to \infty$, $d(y_n, y_{n+1}) < \varepsilon$ where $\varepsilon > 0$

Therefore $\{y_n\}$ is Cauchy sequence.

Since X is complete therefore $\{y_n\} \to z$, and then all of its subsequence also converges to z.

Therefore

$\lim Ax_n = z$, $\lim Fx_{n+1} = z$, $\lim Bx_{n+1} = z$ & $\lim Gx_{n+2} = z$.

Step (2) - A is continuous

Since

$\lim Ax_n = z$, $\lim Gx_n = z$

$\therefore \lim AAx_n = Az$ & $\lim AGx_n = Az$

Also pair (A, G) is semi compatible then, $\lim GAx_n = Az$.

Step (3) -By using (b) with $x = Ax_n$, $y = x_n$

$$\phi\begin{bmatrix} d(AAx_n, Fx_n), d(GAx_n, Bx_n), d(AAx_n, GAx_n), d(Fx_n, Bx_n), \\ \frac{1}{2}\{d(AAx_n, GAx_n) + d(Fx_n, Bx_n)\}, d(AAx_n, Bx_n) \end{bmatrix} \geq 0$$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(Az,z),d(Az,z),d(Az,Az),d(z,z), \\ \tfrac{1}{2}\{d(Az,Az)+d(z,z)\},d(Az,z) \end{bmatrix} \geq 0$$

$$\phi\left[d(Az,z),d(Az,z),0,0,0,d(Az,z)\right] \geq 0$$

Which is contradiction of ϕ_3: $\phi(u,u,0,0,0,u) < 0$

It is only possible when $u = 0$ or $d(Az,z) = 0 \Rightarrow Az = z$

Step (4) –Since $A(X) \subset B(X)$. Then there exist a point $w \in X$ such that $Az = Bw = z$.

By using (b) with $x = x_n$, $y = w$

$$\phi\begin{bmatrix} d(Ax_n,Fw),d(Gx_n,Bw),d(Ax_n,Gx_n),d(Fw,Bw), \\ \tfrac{1}{2}\{d(Ax_n,Gx_n)+d(Fw,Bw)\},d(Ax_n,Bw) \end{bmatrix} \geq 0$$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(z,Fw),d(z,z),d(z,z),d(Fw,z), \\ \tfrac{1}{2}\{d(z,z)+d(Fw,z)\},d(z,z) \end{bmatrix} \geq 0$$

$$\phi\left[d(z,Fw),0,0,d(Fw,z),\tfrac{1}{2}d(Fw,z),0\right] \geq 0$$

By (ϕ_a) we have $v \geq hu$

$0 \geq h\ d(z,Fw)$ Since $h > 1$ therefore $d(z,Fw) = 0 \Rightarrow Fw = z$.

Since (F,B) is weak compatible, therefore

$Fw = Bw$

$\therefore FBw = BFw$

$or\ Fz = Bz$

Step5-By using (b) with $x = x_n$ & $y = z$

$$\phi\begin{bmatrix} d(Ax_n,Fz),d(Gx_n,Bz),d(Ax_n,Gx_n),d(Fz,Bz), \\ \tfrac{1}{2}\{d(Ax_n,Gx_n)+d(Fz,Bz)\},d(Ax_n,Bz) \end{bmatrix} \geq 0$$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(z,Fz),d(z,Fz),d(z,z),d(Fz,Fz), \\ \tfrac{1}{2}\{d(z,z)+d(Fz,Fz)\},d(z,Fz) \end{bmatrix} \geq 0$$

$$\phi\left[d(z,Fz),d(z,Fz),0,0,0,d(z,Fz)\right] \geq 0$$

Which is contradiction of ϕ_3: $\phi(u,u,0,0,0,u) < 0$

It is only possible when $u = 0$ or $d(Fz,z) = 0 \Rightarrow Fz = z$

$Bz = Fz = z$

Step6-By using (b) with $x = x_n$ & $y = Gz$

$$\phi\begin{bmatrix} d(Ax_n, FGz), d(Gx_n, BGz), d(Ax_n, Gx_n), d(FGz, BGz), \\ \frac{1}{2}\{d(Ax_n, Gx_n) + d(FGz, BGz)\}, d(Ax_n, BGz) \end{bmatrix} \geq 0$$

Since $FG = GF$ & $BG = GB$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(z, GFz), d(z, GBz), d(z,z), d(GFz, GBz), \\ \frac{1}{2}\{d(z,z) + d(GFz, GBz)\}, d(z, GBz) \end{bmatrix} \geq 0$$

$$\phi\begin{bmatrix} d(z, Gz), d(z, Gz), d(z,z), d(Gz, Gz), \\ \frac{1}{2}\{d(z,z) + d(Gz, Gz)\}, d(z, Gz) \end{bmatrix} \geq 0$$

$$\phi\left[d(z, Gz), d(z, Gz), 0, 0, 0, d(z, Gz)\right] \geq 0$$

Which is contradiction of ϕ_3: $\phi(u, u, 0, 0, 0, u) < 0$

It is only possible when $u = 0$ or $d(Gz, z) = 0 \Rightarrow Gz = z$

z is a common fixed point of A, B, F & G.

Step (7)- G is continuous

Since $\lim Ax_n = z$, $\lim Gx_n = z$, therefore

$\lim GAx_n = Gz$, $\lim GGx_n = Gz$

Again pair (A, G) is semi compatible then $\lim AGx_n = Gz$

Step (8)- By using (b) with $x = Gx_n$, $y = x_n$

$$\phi\begin{bmatrix} d(AGx_n, Fx_n), d(GGx_n, Bx_n), d(AGx_n, GGx_n), d(Fx_n, Bx_n), \\ \frac{1}{2}\{d(AGx_n, GGx_n) + d(Fx_n, Bx_n)\}, d(AGx_n, Bx_n) \end{bmatrix} \geq 0$$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(Gz, z), d(Gz, z), d(Gz, Gz), d(z,z), \\ \frac{1}{2}\{d(Gz, Gz) + d(z,z)\}, d(Gz, z) \end{bmatrix} \geq 0$$

$$\phi\left[d(Gz, z), d(Gz, z), 0, 0, 0, d(Gz, z)\right] \geq 0$$

Which is contradiction of ϕ_3: $\phi(u, u, 0, 0, 0, u) < 0$

It is only possible when $u = 0$ or $d(Gz, z) = 0 \Rightarrow Gz = z$

Step (9)-By using (b)with $x = z$, $y = x_n$

$$\phi\begin{bmatrix} d(Az, Fx_n), d(Gz, Bx_n), d(Az, Gz), d(Fx_n, Bx_n), \\ \frac{1}{2}\{d(Az, Gz) + d(Fx_n, Bx_n)\}, d(Az, Bx_n) \end{bmatrix} \geq 0$$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(Az, z), d(z,z), d(Az, z), d(z,z), \\ \frac{1}{2}\{d(Az, z) + d(z,z)\}, d(Az, z) \end{bmatrix} \geq 0$$

$$\phi\left[d(Az, z), 0, d(Az, z), 0, \frac{1}{2}d(Az, z), d(Az, z)\right] \geq 0$$

By (ϕ_b) we have $v \geq hu$

$0 \geq h\ d(Az,z)$

Since $h>1$ therefore $d(Az,z)=0 \Rightarrow Az=z$

Step (10) –Since $A(X) \subset B(X)$. Then there exist a point $w \in X$ such that $Az = Bw = z$.

By using (b) with $x = x_n$, $y = w$

$$\phi\begin{bmatrix} d(Ax_n,Fw), d(Gx_n,Bw), d(Ax_n,Gx_n), d(Fw,Bw), \\ \frac{1}{2}\{d(Ax_n,Gx_n)+d(Fw,Bw)\}, d(Ax_n,Bw) \end{bmatrix} \geq 0$$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(z,Fw), d(z,z), d(z,z), d(Fw,z), \\ \frac{1}{2}\{d(z,z)+d(Fw,z)\}, d(z,z) \end{bmatrix} \geq 0$$

$$\phi\left[d(z,Fw), 0, 0, d(Fw,z), \frac{1}{2}d(Fw,z), 0\right] \geq 0$$

By (ϕ_a) we have $v \geq hu$

$0 \geq h\ d(z,Fw)$ Since $h>1$ therefore $d(z,Fw)=0 \Rightarrow Fw=z$ and so $Bw=Fw$.

Since (F,B) is weak compatible, therefore $FBw = BFw \Rightarrow Fz = Bz = z$

Step(11)-By using (b) with $x = x_n$ & $y = z$

$$\phi\begin{bmatrix} d(Ax_n,Fz), d(Gx_n,Bz), d(Ax_n,Gx_n), d(Fz,Bz), \\ \frac{1}{2}\{d(Ax_n,Gx_n)+d(Fz,Bz)\}, d(Ax_n,Bz) \end{bmatrix} \geq 0$$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(z,Fz), d(z,Fz), d(z,z), d(Fz,Fz), \\ \frac{1}{2}\{d(z,z)+d(Fz,Fz)\}, d(z,Fz) \end{bmatrix} \geq 0$$

$$\phi\left[d(z,Fz), d(z,Fz), 0, 0, 0, d(z,Fz)\right] \geq 0$$

Which is contradiction of ϕ_3: $\phi(u,u,0,0,0,u) < 0$

It is only possible when $u = 0$ or $d(Fz,z) = 0 \Rightarrow Fz = z$

$Bz = Fz = z$

Therefore $Az = Bz = Fz = Gz = z$.

Or z is a unique common fixed point of A, B, F & G.

Uniqueness can be easily proved.

Theorem3.4- Let (X,d) be a complete metric space and A, B, F, G, H & E be self maps of Metric space satisfying

(a) $A(X) \subset BH(X)$, $F(X) \subset GE(X)$

$$(b) \quad \phi\begin{bmatrix} d(Ax,Fy), d(GEx,BHy), d(Ax,GEx), d(Fy,BHy), \\ \frac{1}{2}\{d(Ax,GEx)+d(Fy,BHy)\}, d(Ax,BHy) \end{bmatrix} \geq 0$$

For all $x, y \in X$ & $\phi \in \Phi$

(c) One of the (A,GE) is continuous.

(d) (A,GE) is semi compatible and weak compatible.

$(e)(BH,F)$ is weak compatible

$(f)FH = HF\ , BH = HB, AE = EA\ \&\ GE = EG$

Then $A,B,F,G,H\ \&\ E$ have a unique common fixed point in X .

Proof-Let $x_0 \in X$ and $A(X) \subset BH(X)$ and $F(X) \subset GE(X)$ then there exist a point $x_1, x_2 \in X$ such that $BHx_1 = Ax_0 = y_0\ \&\ GEx_2 = Fx_1 = y_1$. Inductively we can define a sequence $BHx_{n+1} = Ax_n = y_n\ \&\ GEx_{n+2} = Fx_{n+1} = y_{n+1}$.

Step (1) - By using (b) with $x = x_n\ ,\ y = x_{n+1}$

$$\phi\begin{bmatrix} d(Ax_n, Fx_{n+1}), d(GEx_n, BHx_{n+1}), d(Ax_n, GEx_n), d(Fx_{n+1}, BHx_{n+1}), \\ 1/2\{d(Ax_n, GEx_n) + d(Fx_{n+1}, BHx_{n+1})\}, d(Ax_n, BHx_{n+1}) \end{bmatrix} \geq 0$$

$$\phi\begin{bmatrix} d(y_n, y_{n+1}), d(y_{n-1}, y_n), d(y_n, y_{n-1}), d(y_{n+1}, y_n), \\ 1/2\{d(y_n, y_{n-1}) + d(y_{n+1}, y_n)\}, d(y_n, y_n) \end{bmatrix} \geq 0$$

$$\phi\begin{bmatrix} d(y_n, y_{n+1}), d(y_{n-1}, y_n), d(y_n, y_{n-1}), d(y_{n+1}, y_n), \\ 1/2\{d(y_n, y_{n-1}) + d(y_{n+1}, y_n)\}, 0 \end{bmatrix} \geq 0$$

By (ϕ_a), $v \geq hu$

$$d(y_{n-1}, y_n) \geq h\ d(y_n, y_{n+1})$$

$$d(y_n, y_{n+1}) \leq 1/h(y_{n-1}, y_n)\ ...(1)$$

Again by using (b) with $x = x_n\ ,\ y = x_{n-1}$

$$\phi\begin{bmatrix} d(Ax_n, Fx_{n-1}), d(GEx_n, BHx_{n-1}), d(Ax_n, GEx_n), d(Fx_{n-1}, BHx_{n-1}), \\ 1/2\{d(Ax_n, GEx_n) + d(Fx_{n-1}, BHx_{n-1})\}, d(Ax_n, BHx_{n-1}) \end{bmatrix} \geq 0$$

$$\phi\begin{bmatrix} d(y_n, y_{n-1}), d(y_{n-1}, y_{n-2}), d(y_n, y_{n-1}), d(y_{n-1}, y_{n-2}), \\ 1/2\{d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2})\}, d(y_n, y_{n-2}) \end{bmatrix} \geq 0$$

Since by triangular inequality we have

$d(y_n, y_{n-2}) \leq d(y_n, y_{n-1}) + (y_{n-1}, y_{n-2})$ Therefore

$$\phi\begin{bmatrix} d(y_n, y_{n-1}), d(y_{n-1}, y_{n-2}), d(y_n, y_{n-1}), d(y_{n-1}, y_{n-2}), \\ 1/2\{d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2})\}, \{d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2})\} \end{bmatrix} \geq 0$$

By (ϕ_b), $v \geq hu$

$$d(y_{n-1}, y_{n-2}) \geq h\ d(y_n, y_{n-1})$$

$$d(y_n, y_{n-1}) \leq 1/h(y_{n-1}, y_{n-2})\ ...(2)$$

By equation (1) and (2)

$$d(y_n, y_{n+1}) \leq 1/h^2\, d(y_{n-1}, y_{n-2})$$

Similarly it can be found

$$d\left(y_n,y_{n+1}\right)\le \frac{1}{h^n}\, d\left(y_1,y_0\right)$$

$\lim n\to\infty$, $d\left(y_n,y_{n+1}\right)<\varepsilon$ where $\varepsilon>0$

Therefore $\{y_n\}$ is Cauchy sequence.

Since X is complete therefore $\{y_n\}\to z$, and then all of its subsequence also converges to z.

Therefore

$\lim Ax_n=z$, $\lim Fx_{n+1}=z$, $\lim BHx_{n+1}=z$ & $\lim GEx_{n+2}=z$.

Step (2) - A is continuous

Since

$\lim Ax_n=z$, $\lim GEx_n=z$

$\therefore \lim AAx_n=Az$ & $\lim AGEx_n=Az$

Also pair (A,GE) is semi compatible then, $\lim GEAx_n=Az$.

Step (3) -By using (b) with $x=Ax_n$, $y=x_n$

$$\phi\begin{bmatrix} d\left(AAx_n,Fx_n\right),d\left(GEAx_n,BHx_n\right),d\left(AAx_n,GEAx_n\right),d\left(Fx_n,BHx_n\right),\\ \frac{1}{2}\left\{d\left(AAx_n,GEAx_n\right)+d\left(Fx_n,BHx_n\right)\right\},d\left(AAx_n,BHx_n\right)\end{bmatrix}\ge 0$$

Taking $\lim n\to\infty$

$$\phi\begin{bmatrix} d\left(Az,z\right),d\left(Az,z\right),d\left(Az,Az\right),d\left(z,z\right),\\ \frac{1}{2}\left\{d\left(Az,Az\right)+d\left(z,z\right)\right\},d\left(Az,z\right)\end{bmatrix}\ge 0$$

$$\phi\left[d\left(Az,z\right),d\left(Az,z\right),0,0,0,d\left(Az,z\right)\right]\ge 0$$

Which is contradiction of ϕ_3: $\phi(u,u,0,0,0,u)<0$

It is only possible when $u=0$ or $d\left(Az,z\right)=0\Rightarrow Az=z$

Step (4) –Since $A(X)\subset BH(X)$. Then there exist a point $w\in X$ such that $Az=BHw=z$.

By using (b) with $x=x_n$, $y=w$

$$\phi\begin{bmatrix} d\left(Ax_n,Fw\right),d\left(GEx_n,BHw\right),d\left(Ax_n,GEx_n\right),d\left(Fw,BHw\right),\\ \frac{1}{2}\left\{d\left(Ax_n,GEx_n\right)+d\left(Fw,BHw\right)\right\},d\left(Ax_n,BHw\right)\end{bmatrix}\ge 0$$

Taking $\lim n\to\infty$

$$\phi\begin{bmatrix} d\left(z,Fw\right),d\left(z,z\right),d\left(z,z\right),d\left(Fw,z\right),\\ \frac{1}{2}\left\{d\left(z,z\right)+d\left(Fw,z\right)\right\},d\left(z,z\right)\end{bmatrix}\ge 0$$

$$\phi\left[d\left(z,Fw\right),0,0,d\left(Fw,z\right),\frac{1}{2}d\left(Fw,z\right),0\right]\ge 0$$

By (ϕ_a) we have $v\ge hu$

$0\ge h\ d\left(z,Fw\right)$ Since $h>1$ therefore $d\left(z,Fw\right)=0\Rightarrow Fw=z$ and so $Fw=BHw$

Since (F,BH) is weak compatible, therefore $FBHw=BHFw\Rightarrow Fz=BHz$

Step5-By using (b) with $x=x_n$ & $y=z$

$$\phi\begin{bmatrix} d(Ax_n,Fz), d(GEx_n,BHz), d(Ax_n,GEx_n), d(Fz,BHz), \\ \frac{1}{2}\{d(Ax_n,GEx_n)+d(Fz,BHz)\}, d(Ax_n,BHz) \end{bmatrix} \geq 0$$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(z,Fz), d(z,Fz), d(z,z), d(Fz,Fz), \\ \frac{1}{2}\{d(z,z)+d(Fz,Fz)\}, d(z,Fz) \end{bmatrix} \geq 0$$

$$\phi[d(z,Fz), d(z,Fz), 0,0,0, d(z,Fz)] \geq 0$$

Which is contradiction of ϕ_3: $\phi(u,u,0,0,0,u) < 0$

It is only possible when $u = 0$ or $d(Fz,z) = 0 \Rightarrow Fz = z$

Step6-By using (b) with $x = x_n$ & $y = Hz$

$$\phi\begin{bmatrix} d(Ax_n,FHz), d(GEx_n,BHHz), d(Ax_n,GEx_n), d(FHz,BHHz), \\ \frac{1}{2}\{d(Ax_n,GEx_n)+d(FHz,BHHz)\}, d(Ax_n,BHHz) \end{bmatrix} \geq 0$$

Since $FH = HF$ & $BH = HB$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(z,HFz), d(z,HBHz), d(z,z), d(HFz,HBHz), \\ \frac{1}{2}\{d(z,z)+d(HFz,HBHz)\}, d(z,HBHz) \end{bmatrix} \geq 0$$

$$\phi\begin{bmatrix} d(z,Hz), d(z,Hz), d(z,z), d(Hz,Hz), \\ \frac{1}{2}\{d(z,z)+d(Hz,Hz)\}, d(z,Hz) \end{bmatrix} \geq 0$$

$$\phi[d(z,Hz), d(z,Hz), 0,0,0, d(z,Hz)] \geq 0$$

Which is contradiction of ϕ_3: $\phi(u,u,0,0,0,u) < 0$

It is only possible when $u = 0$ or $d(Hz,z) = 0 \Rightarrow Hz = z$

Since $BHz = z \Rightarrow Bz = z$

Therefore $Fz = Bz = z$

Step (7) –Since $F(X) \subset GE(X)$. Then there exist a point $w \in X$ such that $Fz = GEw = z$.

By using (b) with $x = w, y = x_n$

$$\phi\begin{bmatrix} d(Aw,Fx_n), d(GEw,BHx_n), d(Aw,GEw), d(Fx_n,BHx_n), \\ \frac{1}{2}\{d(Aw,GEw)+d(Fx_n,BHx_n)\}, d(Aw,BHx_n) \end{bmatrix} \geq 0$$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(Aw,z), d(z,z), d(Aw,z), d(z,z), \\ \frac{1}{2}\{d(Aw,z)+d(z,z)\}, d(Aw,z) \end{bmatrix} \geq 0$$

$$\phi\left[d(Aw,z), 0, d(Aw,z), 0, \frac{1}{2}d(Aw,z), d(Aw,z)\right] \geq 0$$

By (ϕ_b) we have $v \geq hu$

$0 \geq h\ d(Aw,z)$ Since $h > 1$ therefore $d(Aw,z) = 0 \Rightarrow Aw = z$ and so $Aw = GEw$

Since (A,GE) is weak compatible therefore $AGEw = GEAw \Rightarrow Az = GEz = z$

Step (8)-By using (b) with $x = Ez \,\&\, y = x_n$

$$\phi\begin{bmatrix} d(AEz, Fx_n), d(GEEz, BHx_n), d(AEz, GEEz), d(Fx_n, BHx_n), \\ 1/2\{d(AEz, GEEz) + d(Fx_n, BHx_n)\}, d(AEz, BHx_n) \end{bmatrix} \geq 0$$

Since $AE = EA \,\&\, GE = EG$

$$\phi\begin{bmatrix} d(EAz, z), d(EGEz, z), d(EAz, EGEz), d(z,z), \\ 1/2\{d(EAz, EGEz) + d(z,z)\}, d(EAz, z) \end{bmatrix} \geq 0$$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(Ez, z), d(Ez, z), d(Ez, Ez), d(z,z), \\ 1/2\{d(Ez, Ez) + d(z,z)\}, d(Ez, z) \end{bmatrix} \geq 0$$

$$\phi[d(Ez,z), d(Ez,z), 0, 0, 0, d(Ez,z)] \geq 0$$

Which is contradiction of ϕ_3: $\phi(u,u,0,0,0,u) < 0$

It is only possible when $u = 0$ or $d(Ez, z) = 0 \Rightarrow Ez = z$

Since $GEz = z \Rightarrow Gz = z$

Therefore $Az = Bz = Fz = Gz = Hz = Ez = z$

Therefore z is a common fixed point of all six maps.

Step (9)- GE is continuous

Since

$\lim Ax_n = z$, $\lim GEx_n = z$

$\therefore \lim GEAx_n = GEz$ & $\lim GEGEx_n = GEz$

Also pair (A,GE) is semi compatible then, $\lim AGEx_n = GEz$.

Step (10) -By using (b) with $x = GEx_n$, $y = x_n$

$$\phi\begin{bmatrix} d(AGEx_n, Fx_n), d(GEGEx_n, BHx_n), d(AGEx_n, GEGEx_n), d(Fx_n, BHx_n), \\ 1/2\{d(AGEx_n, GEGEx_n) + d(Fx_n, BHx_n)\}, d(AGEx_n, BHx_n) \end{bmatrix} \geq 0$$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(GEz, z), d(GEz, z), d(GEz, GEz), d(z,z), \\ 1/2\{d(GEz, GEz) + d(z,z)\}, d(GEz, z) \end{bmatrix} \geq 0$$

$$\phi[d(GEz,z), d(GEz,z), 0, 0, 0, d(GEz,z)] \geq 0$$

Which is contradiction of ϕ_3: $\phi(u,u,0,0,0,u) < 0$

It is only possible when $u = 0$ or $d(GEz, z) = 0 \Rightarrow GEz = z$

Step (11) -By using (b) with $x = z$, $y = x_n$

$$\phi\begin{bmatrix} d(Az, Fx_n), d(GEz, BHx_n), d(Az, GEz), d(Fx_n, BHx_n), \\ 1/2\{d(Az, GEz) + d(Fx_n, BHx_n)\}, d(Az, BHx_n) \end{bmatrix} \geq 0$$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(Az,z), d(z,z), d(Az,z), d(z,z), \\ 1/2\{d(Az,z)+d(z,z)\}, d(Az,z) \end{bmatrix} \geq 0$$

$$\phi\left[d(Az,z), 0, d(Az,z), 0, 1/2\, d(Az,z), d(Az,z)\right] \geq 0$$

By (ϕ_a), $v \geq hu$ we have

$0 \geq h\, d(Az,z)$

Since $h > 1$ therefore $d(Az,z) = 0 \Rightarrow Az = z$

Step (12) -By using (b) with $x = Ez$, $y = x_n$

$$\phi\begin{bmatrix} d(AEz, Fx_n), d(GEEz, BHx_n), d(AEz, GEEz), d(Fx_n, BHx_n), \\ 1/2\{d(AEz, GEEz)+d(Fx_n, BHx_n)\}, d(AEz, BHx_n) \end{bmatrix} \geq 0$$

Since $AE = EA \,\&\, GE = EG$ also taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(EAz,z), d(EGEz,z), d(EAz, EGEz), d(z,z), \\ 1/2\{d(EAz, EGEz)+d(z,z)\}, d(EAz,z) \end{bmatrix} \geq 0$$

$$\phi\left[d(Ez,z), d(Ez,z), d(Ez,Ez), 0, 1/2\, d(Ez,Ez), d(Ez,z)\right] \geq 0$$

$$\phi\left[d(Ez,z), d(Ez,z), 0, 0, 0, d(Ez,z)\right] \geq 0$$

Which is contradiction of ϕ_3: $\phi(u,u,0,0,0,u) < 0$

It is only possible when $u = 0$ or $d(Ez,z) = 0 \Rightarrow Ez = z$

Since $GEz = z \Rightarrow Gz = z$

Step (13) –Since $A(X) \subset BH(X)$. Then there exist a point $w \in X$ such that $Az = BHw = z$.

By using (b) with $x = x_n$, $y = w$

$$\phi\begin{bmatrix} d(Ax_n, Fw), d(GEx_n, BHw), d(Ax_n, GEx_n), d(Fw, BHw), \\ 1/2\{d(Ax_n, GEx_n)+d(Fw, BHw)\}, d(Ax_n, BHw) \end{bmatrix} \geq 0$$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(z, Fw), d(z,z), d(z,z), d(Fw,z), \\ 1/2\{d(z,z)+d(Fw,z)\}, d(z,z) \end{bmatrix} \geq 0$$

$$\phi\left[d(z,Fw), 0, 0, d(Fw,z)\, 1/2\, d(Fw,z), 0\right] \geq 0$$

By (ϕ_b) we have $v \geq hu$

$0 \geq h\; d(Fw,z)$ Since $h > 1$ $\therefore$ $d(Fw,z) = 0 \Rightarrow Fw = z$ and so $Fw = BHw$.

Since (BH, F) is weak compatible, therefore $BHFw = FBHw \Rightarrow BHz = Fz = z$.

Step(14)-By using (b) with $x = x_n$ & $y = z$

$$\phi\begin{bmatrix} d(Ax_n, Fz), d(GEx_n, BHz), d(Ax_n, GEx_n), d(Fz, BHz), \\ \tfrac{1}{2}\{d(Ax_n, GEx_n) + d(Fz, BHz)\}, d(Ax_n, BHz) \end{bmatrix} \geq 0$$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(z, Fz), d(z, Fz), d(z, z), d(Fz, Fz), \\ \tfrac{1}{2}\{d(z, z) + d(Fz, Fz)\}, d(z, Fz) \end{bmatrix} \geq 0$$

$$\phi[d(z, Fz), d(z, Fz), 0, 0, 0, d(z, Fz)] \geq 0$$

Which is contradiction of ϕ_3: $\phi(u, u, 0, 0, 0, u) < 0$

It is only possible when $u = 0$ or $d(Fz, z) = 0 \Rightarrow Fz = z$

Step(16)-By using (b) with $x = x_n$ & $y = Hz$

$$\phi\begin{bmatrix} d(Ax_n, FHz), d(GEx_n, BHHz), d(Ax_n, GEx_n), d(FHz, BHHz), \\ \tfrac{1}{2}\{d(Ax_n, GEx_n) + d(FHz, BHHz)\}, d(Ax_n, BHHz) \end{bmatrix} \geq 0$$

Since $FH = HF$ & $BH = HB$

Taking $\lim n \to \infty$

$$\phi\begin{bmatrix} d(z, HFz), d(z, HBHz), d(z, z), d(HFz, HBHz), \\ \tfrac{1}{2}\{d(z, z) + d(HFz, HBHz)\}, d(z, HBHz) \end{bmatrix} \geq 0$$

$$\phi\begin{bmatrix} d(z, Hz), d(z, Hz), d(z, z), d(Hz, Hz), \\ \tfrac{1}{2}\{d(z, z) + d(Hz, Hz)\}, d(z, Hz) \end{bmatrix} \geq 0$$

$$\phi[d(z, Hz), d(z, Hz), 0, 0, 0, d(z, Hz)] \geq 0$$

Which is contradiction of ϕ_3: $\phi(u, u, 0, 0, 0, u) < 0$

It is only possible when $u = 0$ or $d(Hz, z) = 0 \Rightarrow Hz = z$

Since $BHz = z \Rightarrow Bz = z$

Therefore $Fz = Bz = z$

Or $Az = Bz = Fz = Gz = Hz = Ez = z$

Therefore z is a common fixed point of all six maps.

Uniqueness can be easily proved.

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