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PREDICTION OF GENERALIZED ORDER STATISTICS BASED ON ORDINARY ORDER STATISTICS

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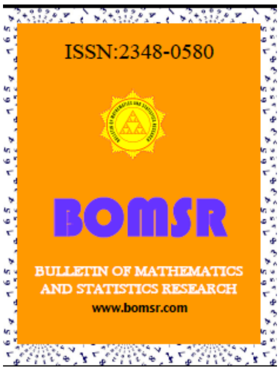

ABSTRACT

This article, discusses the problem of predicting future generalized order statistics based on observed ordinary order statistics. In addition, outer and inner prediction intervals for future generalized order statistics intervals are de-rived based on observed ordinary order statistics. The coverage probabilities of these intervals are exact and are free of the parent distribution F().

Keywords: prediction intervals; generalized order statistics; ordinary order statistics; Coverage probability.

1.INTRODUCTION

The concept of generalized order statistics (gOSs) is introduced by Kamps (1995 [1]) as a general model contains many types of ordered schemes such as, ordinary order statistics (*oOSs*), order statistics with non-integral sample size (*nonI*), sequential or-der statistics (*seque*), Pfeifer's record model (*P f eif*), progressively Type-II censoring (*P COs*), ordering via truncation (*trunc*) and k_n-record values (*record*). table 1, rep-resents these models as special cases of gOSs.

Let X1, $n, \tilde{m}$,k be n gOSs based on the cumulative continuous distribution function F with density function f . The distribution function of gOSs under submission $\gamma_i \neq \gamma_j$, I, j = 1,2, …. n-1 and i ≠ j, are developed by Kamps and Cramer (2001[2]), such that, the marginal pdf and the survival function of $X_{r,n,\tilde{m},k}$ and the joint pdf of any two gOSs, say $X_{r,n,\tilde{m},k}$ and $X_{s,n,\tilde{m},k}$ such that 1 < r < s < n, are expressed in terms of F( ) and f( ) as

$$f\ x_{r,n,\tilde{m},k}\ (x) = c_{r-1} \sum_{i=1}^{r} a_i(r)\ (1 - F(x))^{\gamma_v - 1}\ f(x) \qquad (1)$$

$$F\ x_{r,n,\tilde{m},k}\ (x) = c_{r-1} \sum_{i=1}^{r} \frac{a_r\,(r)}{\gamma_v} (1 - F(x))^{\gamma_v} \qquad (2)$$

Where n ϵ N, = $(m_1,.... m_{n-1}) \epsilon R_{n-1}$ if n≥ 2 (m∈ R , arbitrary if n = 1) k≥ 1, be given constant such that for all $1 \le i \le n-1$, γ_i = k + n – I +M > 0, when $M_i = \sum_{i=1}^{r-1} a_i(r) \prod_{j=1}^{r}$,
$j \neq i, \frac{1}{\gamma_i - \gamma_j}, 1 \le r \le n.$

$$f x_{r,n,\tilde{m},k}\ ,\ X_{r,n,\tilde{m},k}\ (\ x_r\ ,\ x_s)$$
$$= c_{S-1} \left[\sum_{i=r+1}^{s} \left(a^r(\mathrm{s}) \frac{\bar{\mathrm{F}}(\mathrm{x}))^{\gamma_i}}{\bar{\mathrm{F}}(\mathrm{x_r})}\right)\right] \left[\sum_{i=r+1}^{s} (a^r(\mathrm{s})\, \bar{\mathrm{F}}(\mathrm{x_r}))\right] \frac{f(x_r) f(x_s)}{\bar{\mathrm{F}}(x_r)\bar{\mathrm{F}}(\mathrm{x_s}}, \qquad (3)$$

Where $x_r < x$,$1 \le r < s \le n$, $a_i{}^{(r)}(s) = \prod_{j=r+1, j\neq i}^{s}, , \frac{1}{\gamma_i - \gamma_j}$, and $a_i(\mathrm{s}) = a_i{}^{(r)}(s)$

Table 1 : Some special cases of gOSs, where λ_i,α_i ϵ R+

Model	x_r	k	γ_r
oOSs	0	1	n – r + 1
nonI	0	$\alpha - n + 1$	$\alpha - r + 1$
Seque	(n – r +1) α_r - (n – r)$\alpha_{r+1} - 1$	α_n	(n - r + 1) α_r
Pfeif	λ_r - λ_{r+1} -1	λ_n	λ_r
PCOs	0, if r ≠ r_1 α_1, r = r_1	$\nu - n_1$- n + 1	$\nu - r + 1, if\ r\ \le\ r_1$
trunc	$\alpha_r k_r$ - $\alpha_{r+1} k_{r+1}$	$\alpha_n k_n$	$\alpha_r k_r$
records	$\lambda_r k_r$ - $\lambda_{r+1} k_{r+1}$- 1	$\lambda_r k_r$	$\lambda_r k_r$

Distribution-free prediction is useful method in prediction of the future observations without the need to set the parent distribution, there exist some papers of distribution-free prediction intervals (PIs) procedure contained some ordered schemes, such as, distribution-free PIs for an order statistic observation based on either progressive Type-II right censoring, or order data [3], distribution-free PIs for a progressive Type-II right censoring based on ordered observations [4], outer and inner PIs for order statistics based on current records [5], distribution-free PIs for single order statistics based on records, as well as PIs for a future records based on observed order statistics are also obtained [6], PIs for order statistics based on current record coverage [7], and many other contexts have taken place in the distribution-free PIs direction by using several assumptions [see, for example, [8]- [13]]. For the purpose of circular, this paper based on oOSs, discuss the predicting of future gOSs, as well as outer and inner PIs are derived.

Speci cally, the work in our paper organized as follows: In Section 2, we derive distribution-free PIs for single gOSs from a future Y-sequence of iid random variables, based on oOSs from the X-sequence. In Section 3, distribution-free outer and inner PIs for future gOSs intervals from Y-sequence based on also oOSs observations from the X-sequence are derived. In section 4, a real life time data are considered and numerical computations are given to illustrate some of the results which are derived. Finally, a conclusion of this study is given in Section 5.

2 PIs for individual gOSs Based on oOSs

Let $Y_{1,n^+}, \tilde{m}, k$, $Y_{n^+,n^+}, \tilde{m}, k$ be n^* gOSs based on the cumulative continuous distribution function F with density function f from a future Y – sequence of i.i.d random variables, under submission $\gamma_i \neq \gamma_j$, I,j = 1, 2, ……. n^*- 1 and i ≠ j, and Let$X_{J:n}$ be *n* oOSs from another observed random sample, and further let the under-ling distribution of the two samples be the same. Under this set-up, in this sec-tion, we are interested in obtaining two-sided distribution-free PIs for a future r^{th} gOSs $Y_{1,n^+}, \tilde{m}, k$, 1 ≤ r ≤ n^* based on the endpoints of observed oOSs, assuming that P($X_{i:n} \le Y_{n^+,n^*}, \tilde{m}, k \le X_{j:n}$) = π_1(i ,j; r, n^*, m̃, k) then ($X_{i:n}$ $X_{j:n}$) is said to be a 100 π_1(I ,j; r, n^*, m̃, k) % prediction interval for $Y_{1,n^+}, \tilde{m}, k$ from a future sample of size n^*, we show here that the coverage probability of this PIs. i.e., π_1(i ,j, n; r, n^*, m̃, k), does not depend on the sampling distribution.

Theorem 1 : Let $Y_{1,n^*}, \tilde{m}, k$, 1 ≤ r ≤ n^*be r^{th} gOSs from a future random sample size n^* with cdf F(y). If $X_{j:n}$, is the j^{th} oOSs from another observed random sample of size n with the same cdf F(x) then ($X_{i:n}$ $X_{j:n}$), 1 ≤ I < j ≤ n, is a two – sided PI for $Y_{1,n^*}, \tilde{m}, k$; 1 ≤ r ≤ n^*, whose coverage probability is free of F and is given by

$$\pi_1(\text{i ,j, n; r, } n^*, \tilde{m}, \text{k}) = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \frac{\ell!}{(\text{n} + \gamma_v)(\text{n} + \gamma_v - 1) \,\ldots\ldots (\text{n} + \gamma_v - \ell)}$$

Proof: Using the cdf of $X_{j:n}$ (see Arnold et al. [14] and David and Nagaraja [15]),

And for agiven y, we get P($X_{i:n} \le$ y $\le X_{j:n}$) = $\sum_{i=1}^{j-1}\binom{n}{t}$ $(\text{F(y)})^t(1 - (\text{F(y)})^{n-t}$ (4)

Using the conditioning argument, we then have

P($X_{i:n} \le Y_{1,n^*}, \tilde{m}, k \le X_{j:n}$) = $\int_{-\infty}^{\infty}$ P($X_{i:n}$ $\le$ y $\le$ $X_{j:n}$) d F $Y_{1,n^*}, \tilde{m}, k$(y) (5)

Upon substituting (4) and the pdf of $Y_{1,n^*}, \tilde{m}, k$ (see (1) in (5), we obtain

π_1(i ,j, n; r, n^*, m̃, k) = P($X_{i:n} \le Y_{1,n^*}, \tilde{m}, k \le X_{j:n}$)

= $\sum_{\ell=1}^{j-1} a_r$ (r) $\int_{-\infty}^{\infty}(\text{F(y)})^t(1 - (F(y))^{n+\gamma_v-\ell-1}$ f(y) dy

Making the transformation F(y) = u and solving the integration, we obtain

π_1(i ,j, n; r, n^*, m̃, k) = $\sum_{\ell=1}^{j-1}\binom{n}{\ell}$ $\sum_{v=1}^{r} a_v$ (r) B ($\ell + 1, n + \gamma_\vartheta$ - ℓ) (6)

By expanding previous beta constant B(:; :), easily we can obtain the required result. Under the assumptions of theorem 1, we note that $X_{i:n}$ is a lower prediction bound and $X_{j:n}$ is an upper bound for $Y_{1n^*,\tilde{m},k}$, 1 ≤ r ≤ n^* . Note that, the choice of *i* and *j* is not unique. So, for a given confidence level π_v and specied *r* and n^* , we would like to construct a PI as short as possible among all PIs with the same level. $_1$ can be reduced to many special cases, such that, the prediction coefficient of future order statistics based on also order statistics which discussed in theorem 2 by M. M. Mohie El-Din et al. [3] appear here as a special case from $_1$ (*i; j; n; r; n ; m; k*) by setting m_r = 0 , k = 1and α_r= n^*- r +1, 1 ≤ r ≤ n^* and take the form

π_1(i ,j, n; r, n^*, 0, 1) = $\sum_{i=1}^{j-1}\binom{n}{r}$ $\sum_{v=1}^{r}(-1)^{r-v}\, \nu \binom{r}{\nu}$B ($\ell + 1, n - \ell + n^* - \nu + 1$)

By using the different choices of γ_r in table 1 to gaining the special schemes of the future gOSs, tables 2 and 3 presents values of π_1(i ,j, n; r, n^*, 0, 1) for some choices of r, i, j which π_1(i ,j, n; r, n^*, 0, 1) represent consecutively based on oOSs, the coverage probability of future, oOSs, nonI (by setting γ_r = r), PCOs (using r_1 = 5, n_i = 5), trunk (using $\alpha_r = r^2$, k_r = n^*- r) and record (using $\beta_r = r^2$, k_r = r + 2). With π_1(i ,j, n; r, n^*, m̃, k) given by (7) which does not depend on the parent distribution F. Moreover, table 4 presents values of π_1(i ,j, n; r, n^*,m̃, k)

Under previous assumption with for (n = 20, n^* = 15) and confidence level π_0= 0.90 for future oOSs , PCOs and record.

3 PIs for gOSs interval

Under assumption of section 2, in this section, we are interested in obtaining two-sided distribution-free PIs for a future gOSs interval ($y_{r,n,\tilde{m},k}$; $y_{r,n,\tilde{m},k}$) based on the endpoints of observed oOSs. Using (3) the joint pdf of $y_{r,n,\tilde{m},k}$ *and* $y_{r,n,\tilde{m},k}$) can be expressed as

$f y_{r,n,\tilde{m},k}$, $y_{8,n,\tilde{m},k}$ (x_r , x_s) = $\sum_{v=r+1}^{S}\sum_{\eta=1}^{r} ak_r \left(\frac{\overline{\text{F}}(\text{x}))^{\gamma_i}}{\overline{\text{F}}(\text{x}_\text{r})}\right)\left(\text{F}(\text{x}_\text{j})\right)^{i+k}$ (7)

Where $b_{v,\eta}$ = $e_{s-1}a_n{}^{(r)}$, $a_\eta{}^{(r)}$,($y_r < y_8$, 1 ≤ r < s ≤ n^*.

And by using the binomial expansion we can form the joint pdf of any two order statistics $x_{i,n} < x_{j,n}$ < i < j < n (see . Arnold et al [14] and David and Nagaraja [15], as

$$f x_{i,n}\,,\, x_{j,n} = = \sum_{v=r+1}^{S}\sum_{\eta=1}^{r} ak_r \left(\frac{\overline{\mathrm{F}}(\mathrm{x}))^{\gamma_i}}{\overline{\mathrm{F}}(\mathrm{x_r})}\right)\left(\mathrm{F}(\mathrm{x_j})\right)^{i+k} \frac{f(x_i)f(x_j)}{\overline{F}(x_i)\overline{F}(x_j} \qquad (8)$$

where

$$ak_r = \frac{n!(-1)^{k+z}}{(\mathrm{i}-!)!(\mathrm{j}-\mathrm{i}-\mathrm{k}-1)!\,\mathrm{k}\,!(\,\mathrm{n}+\gamma_v-\ell)!\mathrm{z}!}$$

Table 2: Values of $_1$ for *n* = 20*; n* = 15*;* 25 and some choices of *i; j* and *r:*

n	*r*	*i*	*j*	$_1$ *oOSs*	*nonI*	*Seque*	*P f eif*	*P COs*	*trunc*	*record*
15	5	1	8	0:6667	0:7061	0:7627	0:0149	0:6667	0:7788	0:6163
			12	0:9386	0:9495	0:7720	0:0822	0:9386	0:7910	0:8412
			18	0:9906	0:9886	0:7723	0:4956	0:9906	0:7913	0:9624
		3	8	0:5752	0:5994	0:2848	0:0145	0:5752	0:3098	0:4647
			12	0:8472	0:8428	0:2942	0:0818	0:8472	0:3219	0:6896
			18	0:8991	0:8819	0:2944	0:4953	0:8991	0:3223	0:8107
	8	1	8	0:2310	0:2793	0:7907	0:0021	0:0663	0:8070	0:6126
			12	0:6700	0:7304	0:8011	0:0243	0:3146	0:8206	0:8423
			18	0:9925	0:9955	0:8014	0:3480	0:8963	0:8210	0:9662
		3	8	0:2226	0:2670	0:3106	0:0021	0:0650	0:3386	0:4683
			12	0:6631	0:7206	0:3210	0:0243	0:3134	0:3522	0:6980
			18	0:9856	0:9857	0:3213	0:3480	0:8941	0:3526	0:8218
	10	1	16	0:8207	0:8745	0:8144	0:1088	0:1660	0:8347	0:9469
			20	0:9969	0:9985	0:8144	0:6667	0:7253	0:8347	0:9743
		6	16	0:8042	0:8492	0:1087	0:0499	0:1656	0:0612	0:5078
			20	0:9804	0:9732	0:0499	0:6666	0:7249	0:0612	0:5351
	15	1	16	0:0478	0:1355	0:8540	0:0478	-	0:0000	0:9478
			20	0:5714	0:7938	0:8540	0:5714	-	0:0000	0:8253
		6	16	0:0477	0:1355	0:0600	0:0478	-	0:0000	0:5107
			20	0:5714	0:7937	0:0600	0:5714	-	0:0000	0:5382
25	10	6	13	0:7194	0:7024	0:0093	0:0214	0:7439	0:0108	0:4395
			16	0:7406	0:7406	0:0093	0:1087	0:8512	0:0108	0:5078
	19	7	16	0:6015	0:6724	0:0037	0:0269	0:1042	0:0044	0:4204
		11	20	0:9222	0:8999	0:0000	0:5125	0:7700	0:0000	0:1774
		15		0:5366	0:4645	0:0000	0:5009	0:7182	0:0000	0:0446
	21	6	20	0:9564	0:9738	0:0105	0:4878	-	0:0122	0:5395
		15		0:7337	0:6756	0:0000	0:4792	-	0:0000	0:0446
	23	6	20	0:8379	0:9031	0:0108	0:4651	-	0:0127	0:5398
		10		0:8372	0:9013	0:0001	0:4651	-	0:0000	0:2309
	25	6	20	0:4444	0:6678	0:0000	0:4444	-	0:0000	0:5375
		10		0:4444	0:6677	0:0000	0:4444	-	0:0000	0:2310

Table 3: Values of $_1$ for n = 50; n = 25; 40 and for some choices of i; j and r:

n	r	i	j	$_1$ oOSs	nonI	Seque	P f eif	P COs	trunc	record
25	9	1	12	0:1549	0:1781	0:8752	0:0000	0:0877	0:8848	0:3633
			20	0:5308	0:5756	0:8876	0:0003	0:3422	0:8955	0:6668
			30	0:9572	0:9678	0:8877	0:0043	0:8708	0:8957	0:8901
		8	20	0:6298	0:6620	0:0610	0:0005	0:4852	0:0684	0:4802
		12	30	0:8249	0:8071	0:0090	0:0130	0:8505	0:0107	0:5257
	10	6	18	0:3930	0:4349	0:1518	0:0001	0:2328	0:1653	0:4959
			20	0:5272	0:5711	0:1521	0:0002	0:3411	0:1656	0:5613
			30	0:9537	0:9633	0:1522	0:0084	0:8696	0:1658	0:7847
			50	0:9965	0:9955	0:1522	0:8333	0:9988	0:1658	0:8923
		12	35	0:9073	0:8924	0:0091	0:0329	0:9333	0:0109	0:5849
	13	6	30	0:7705	0:8137	0:1578	0:0024	0:4702	0:1721	0:7856
			40	0:9943	0:9964	0:1578	0:0607	0:9435	0:1721	0:8772
		12	35	0:9352	0:9475	0:0094	0:0134	0:6708	0:0113	0:5873
	17	14	35	0:6066	0:6807	0:0035	0:0044	0:1516	0:0044	0:5029
			45	0:9902	0:9940	0:0035	0:1592	0:8033	0:0044	0:5494
	19	3	35	0:3310	0:4177	0:5438	0:0026	0:0178	0:5667	0:9284
			40	0:6915	0:7710	0:5438	0:0205	0:0965	0:5667	0:9609
			45	0:9498	0:9708	0:5437	0:1326	0:3776	0:5667	0:9750
	21	3	45	0:8019	0:8756	0:5517	0:1110		0:5761	0:9751
		12	45	0:8019	0:8758	0:0102	0:1110		0:0124	0:6362
		14	50	0:9970	0:9988	0:0102	0:7042		0:0124	0:6370
	25	8	50	0:6667	0:8732	0:0000	0:0076		0:0000	0:8002
		14	50	0:6667	0:8732	0:0000	0:6667		0:0000	0:5540
40	30	25	40	0:7134	0:7643	0:0000	0:0036	0:2866	0:0000	0:1836
		30	45	0:8990	0:8852	0:0000	0:0529	0:7459	0:0000	0:1054
	35	25	50	0:9938	0:9956	0:0000	0:5882	0:5932	0:0000	0:2012
		30	50	0:9920	0:9934	0:0000	0:5882	0:5932	0:0000	0:1088
	40	25	50	0:5556	0:7833	0:0000	0:5556		0:0000	0:2012
		30	50	0:5556	0:7833	0:0000	0:5556		0:0000	0:1088

3.1 Outer PIs based on order statistics

Suppose ($y_{r,n,\tilde{m},k}$; $y_{s,n,\tilde{m},k}$) is a future gOSs interval from the Y-sequence and $X_{J}:n$, $1\leq j \leq n$ be the j[th] order statistics from X-sequence, we are interested here in obtaining 100(1)% PIs for it of the form ($X_{i:n}$; $X_{j:n}$).

Theorem 2. Suppose the conditions of Theorem 1 hold, Then, ($X_{i:n}$; $X_{j:n}$) is an outer PIs for the future generalized order interval ($y_{r,n,\tilde{m},k}$; $y_{s,n,\tilde{m},k}$) for the Y–sequence with the corresponding prediction coefficient, being free of F, given by

π_1(i ,j, n; r, n^*, m̃, k) = $\sum_{k=0}^{j-i-1}\sum_{z=0}^{n-1}\frac{a_k}{(j-i-(i+k)}$ $b_{y-\eta}$ $\binom{\gamma_{v-1}}{\lambda}$

X $\frac{B(i+k+1)-B(\lambda+i+k+2)/}{\lambda+1}$ - $\frac{B(i+k+1)-B(j+z+\lambda+2)}{j-i-k+z+\lambda+1)}$, (9)

where B(.) = B(., γη − γv) is complete beta constant.

Proof: Under the assumption that X's are i.i.d continous r.v's and for a given $y_1 \le y_2$

We can write

$$P(X_{i:n} \le y_r \le y_s \le X_{j:n}) = \int_{y_r}^{\infty}\int_{-\infty}^{y_r} f x_{i,n}, x_{j,n}, (x_i, x_j)\, dx_i . dx_j \quad (10)$$

Upon the earlier reformed Equation (8), by making the transformation F($x_{i,}$) =$u_{i,}$.F(x_j) = v, we can easily reduce Equation (10) to

$$P(X_{i:n} \le y_r \le y_s \le X_{j:n}) = \sum_{k=0}^{j-i-1}\sum_{z=0}^{n-j}\int_{f_{y_8}}^{1} v^{j-i-k+z-1}\int_0^{f y_r} u^{t-k-1}.du$$

$$\sum_{k=0}^{j-i}\sum_{z=0}^{n-j} a_{k,2}\frac{1-(F(y_8))^{j-t-k+z}(F(y_r))^{i+k}}{(j-i-k+z)(i+k)} \quad (11)$$

By using (11) and (7), we obtain

P (X

$P(X_{i:n} \le y_{r,n^*\tilde{m},k} \le y_{r,} \le y_{8,n^*\tilde{m},k} \le X_{j:n})$

$= \int_{-\infty}^{\infty}\int_{y^r}^{\infty} f\, y_{r,n^*\tilde{m},k}, y_{r,n^*\tilde{m},k}(y_r, y_8)\, dy_8, d\, y_r$

$=\sum_{k=0}^{j-i-1}\sum_{z=0}^{n-1}\frac{a_k}{(j-i-k+z)(i+k)}$ $\sum_{v=r+1}^{8}\sum_{\eta-1}^{r} b_{v,n}\sum_{x=0}^{\gamma_v}(-1)^{\lambda}\binom{\gamma_{v-1}}{\lambda}$

X $\int_0^1[\int_u^1(v^{\lambda} - v^{j-i-k+z+\lambda})dv]u^{i+k}$ $(1-\mu)^{\gamma_n-\gamma_{v-1}}dv$

$=\sum_{k=0}^{j-i-1}\sum_{z=0}^{n-1}\frac{a_{k,2}}{(j-i-k+z)(i+k)}$ $\sum_{v=r+1}^{8}\sum_{\eta-1}^{r} b_{v,n}\sum_{x=0}^{\gamma_v}(-1)^{\lambda}\binom{\gamma_{v-1}}{\lambda}$

$X\ \frac{1-u_y}{\lambda+11}$ - $\frac{1-u^{j-i-k+z+\lambda+1}}{j-i-k+z+\lambda+1}u^{i+k}$ $(1-u)^{\gamma_n-\gamma_{v-1}}du$. (12)

We can get π_1(i ,j, n; r, n^*, m̃, k) that does not depend on F and given by (9) , by completing the previous integration. Where ($x_{i,n}$, $x_{j,n}$) be a (π_2)100% outer prediction interval for ($y_{r,n^*\tilde{m},k}, y_{8,n^*\tilde{m},k}$). Some values of π_1(i ,j, n; r, n^*, m̃, k) are present in table 5, for (*n* = 20*;* n^*= 15) , (*n* = 20*;* n^* = 10) and (*n* = 18*;* n^* = 10) consecutively, for some choices of *r; s*; *i; j*, such

π_1(i ,j, n; r, n^*, m̃, k) represent the coverage probability of future interval of oOSs, *nonl* (by setting γ_v = (n^* 0*:*9) *r*+1), *seque* (using $\alpha_1 = 3$) *P f eif* (by settingr_1 = 5 , $\eta_1 - r$), *P COs* (using r_1 = 5*;* n_1 = r), *trunc* (using $_r$ = 5*; k*

Table 4: Some values of $_1$ 0*:*90 for *n* = 20*; n* = 15 and some choices of *r* and for some *i; j*.

r	$_1$ (*i; j*)		
	oOSs	*P COs*	*record*
3	0*:*9041 (1*;* 10)	0*:*9041 (1*;* 10)	0*:*9182 (1*;* 15)
4	0*:*9037 (1*;* 10)	0*:*9037 (1*;* 10)	0*:*9022 (1*;* 14)
	0*:*9023 (2*;* 14)	0*:*9023 (2*;* 14)	
5	0*:*9386 (1*;* 12)	0*:*9386 (1*;* 12)	0*:*9045 (1*;* 14)
	0*:*9078	0*:*9078	0*:*9031

	(2; 12)	(2; 12)	(2; 19)
6	0:9323	0:9139	0:9058
	(1; 13)	(1; 14)	(1; 14)
	0:9195	0:9059	0:9024
	(2; 13)	(2; 14)	(2; 18)
	0:9194	0:9218	
	(3; 14)	(3; 15)	
7	0:9228	0:9422	0:9065
	(1; 14)	(1; 17)	(1; 14)
8	0:9132	0:9139	0:9058
	(1; 15)	(1; 14)	(1; 14)

Table 5: Values of π_2 for some choices of $i; j; n; r; s$ and n :

$n; n$	$(r; s)$	i	j	2 oOSs	Seque	P f eif	P COs	trunc	record
20; 15	(3; 6)	1	15	0:8604	0:6629	0:8588	0:8312	0:5371	0:3993
		2		0:6984	0:3527	0:7082	0:6723	0:2183	0:1142
		3		0:5140	0:1613	0:5271	0:4955	0:0744	0:0269
		1	16	0:8729	0:6647	0:8759	0:8560	0:5383	0:3999
			17	0:8800	0:6663	0:8856	0:8716	0:5392	0:4004
			18	0:8844	0:6676	0:8914	0:8809	0:5401	0:4009
	(3; 8)	1	15	0:7965	0:6628	0:7656	0:5363	0:5371	0:3993
		2		0:6407	0:3526	0:6241	0:4233	0:2183	0:1142
		3		0:4732	0:1613	0:4677	0:3061	0:0744	0:0269
		1	16	0:8379	0:6647	0:8208	0:6309	0:5383	0:3999
			17	0:8638	0:6662	0:8579	0:7174	0:5393	0:4004
			18	0:8784	0:6676	0:8800	0:7898	0:5401	0:4008
	(5; 8)	1	15	0:8444	0:8303	0:8069	0:5734	0:7336	0:5888
		2		0:7787	0:5921	0:7528	0:5187	0:4262	0:2506
		3		0:5702	0:3664	0:5252	0:4166	0:2056	0:0850
		1	16	0:8890	0:8333	0:8659	0:8560	0:7630	0:7357
			17	0:5140	0:1613	0:5271	0:4955	0:0744	0:0269
			18	0:9147	0:8359	0:9026	0:7629	0:7376	0:5912
20; 10	(3; 6)	1	15	0:7522	0:7722	0:6670	0:0000	0:6777	0:5160
		2		0:6593	0:5104	0:5932	0:0000	0:2183	0:1998
		3		0:4880	0:2990	0:4292	0:0000	0:1760	0:0647

		1	16	0:8170	0:7751	0:7519	0:0000	0:6800	0:5170
			17	0:8636	0:7774	0:8201	0:0000	0:6813	0:5180
			18	0:8957	0:7795	0:8730	0:0000	0:6827	0:8188
18; 10	(3; 6)	1	15	0:7522	0:7722	0:6670	0:0000	0:6777	0:5160
		2		0:6593	0:5103	0:5932	0:0000	0:3717	0:1998
		3		0:4880	0:2990	0:5932	0:0000	0:3717	0:1998
		1	16	0:8170	0:7751	0:7519	0:0000	0:6800	0:5170
			17	0:8636	0:7775	0:8202	0:0000	0:6813	0:5180
			18	0:8987	0:7795	0:8730	0:0000	0:6827	0:5188

3.2 Inner PIs based on order statistics

Suppose we are interested in obtaining 100(1-α) % inner two-sided distribution-free. PIs for a future gOSs interval $(y_{r,n^*\tilde{m},k}, y_{8,n^*\tilde{m},k})$ based on oOSs of the form $(X_{i,n}, X_{j,n})$ such that P ($X_{i:n} \le y_{r,n^*\tilde{m},k} \le X_{i,n,} \le X_{j:n} \le y_{8,n^*\tilde{m},k}$) = π_3(i ,j, n; r, n^*, m̃, k).Here, we describe how such distribution free inner pls can be constructed.

Theorem 3 : Suppose the conditions of Theorem 1 hold, then $(X_{i,n}, X_{j,n})$ inner PIs fir gOSs interval $(y_{r,n^*\tilde{m},k}, y_{8,n^*\tilde{m},k})$ from the Y – sequence with the corresponding prediction coefficient that does not depend on the sampling distribution, given by

π_3(i ,j, n; r, n^*, m̃, k)

$$= \sum_{k=0}^{j-i-1} \sum_{z=0}^{n-1} \frac{a_{k,2}}{(j-i-k+z)(i+k)} \quad \sum_{v=r+1}^{8} \sum_{\eta-1}^{r} b_{v,n} \sum_{x=0}^{\gamma_v} (-1)^{\lambda} \binom{\gamma_{v-1}}{\lambda}$$

$$X \frac{B(i+k+1) - B(\lambda+i+k+2)}{\lambda+1} - \frac{B(i+k+1) - B(j+z+\lambda+2)}{j-i-k+z+\lambda+1},$$

$$- \quad \frac{(I+k)B\,(j+z+\lambda+2}{(\lambda+1)(1-i+j-k+Z+\lambda} + \frac{1}{(j+2)(1+j+z+\lambda)(\gamma_n-\gamma_v} \qquad (13)$$

Proof: For a given $y_r \le y_8$, under the assumption that the j th observed oOSs $x_{j,n}$, $1 \le j \le n$ are continous r.v.s we can write

P ($y_r \le X_{i,n,} \le X_{j:n} \le y_8$) = $\int_{y_v}^{y_x} \int_{y_r}^{x_r} f\, x_{i,n} x_{j,n}(x_i, x_j)\, dx_i dx_j$

Proceeding similarly, from (8) in (14), we obtain

P ($y_r \le X_{i,n,} \le X_{j:n} \le y_8$) = $\int_{y_v}^{y_x} \int_{y_r}^{x_r} u^{i+k-1} v^{j-i-k-z-1}\, du\, dv$

$$= \sum_{k=0}^{j-i-1} \sum_{z=0}^{n-1} \frac{a_{k,2}}{(i+k)} \left(\frac{\overline{F}(x))^{j+2} - \overline{F}(y_r))^{j+z}}{j+z} - \frac{\overline{F}(x))^{j+2} - \overline{F}(y_8))^{j-i-k+z+z} - \overline{F}(y_r))^{j-i-k+z+z}}{j-i-k+z} \right) \quad (15)$$

By using (15) and (8), it it now obvious that

P ($y_r \le X_{i,n,} \le X_{j:n} \le y_8$) = $f x_{i,n}, x_{j,n}$

$$= \int_{-\infty}^{\infty} \int_{y^r}^{\infty} f P\,(y_r \le X_{i,n,} \le Xj{:}\,n \le y_8) f_{r,s,n^*,\tilde{m},k}(y_r, y_8) dy_r\, dy_8$$

$$= \sum_{k=0}^{j-i-1} \sum_{z=0}^{n-1} \frac{a_{k,2}}{(i+k)} \quad \sum_{v=r+1}^{8} \sum_{\eta-1}^{r} b_{v,n} \sum_{x=0}^{\gamma_{v-1}} (-1)^{\lambda} \binom{\gamma_{v-1}}{\lambda}$$

$$X \int_0^1 \int_0^1 \left(\frac{u^{j+2} - u^{j+2}}{j+z} - \frac{u^{j+2} - u^{j-i-k+z+z} - u^{j-i-k+z+z}}{j-i-k+z} \right) u^{\lambda} (1-u)^{\gamma_n - \gamma_{n-1}}\, dv\, du \qquad (16)$$

The prediction coefficient π_3(i ,j, n; r, n^*, m̃, k) that is given by (13) and does not depend on *F* can be obtained by completing the previous integration in (16), where ($X_{i:n}$; $X_{j:n}$) be a (π_3)100% inner prediction interval for ($Y_{r:m}$; $Y_{s:m}$). The outer and inner coverage probability of future oOSs intervals based on also oOSs which discussed in theorem 3 and 4 by M. M. Mohie El-Din et al. [3] appear as a special case from π_2 and π_3 respectively, by setting m_r = 0,*k* = 1 and

$\gamma_r = n^*$- *r* + 1. Some values of $_3$ are presents in table 6, for *n* = 20 and for some choices of *i; j*; *r; s* and *n* which based on oOSs, π_3(i,j,n,r,s,n.m.k) represent consecutively, the inner probability of future intervalof oOSs, seque (using α_r = 2) Pfef (by setting $\gamma_r = n^*$- *r* + 1), PCOs (using r_1 = 3 , η_1= 4), using (α_r = 2, $\gamma_r = n^*$- *r* + 1 and record (using α_r = 2, $\gamma_r = n^*$- *r* + 1).

Table 6: Values ofπ_3 for *n* = 20 and for some choices of *i; j; r; s* and n^* *:*

n	(*r; s*)	*i*	*j*	π_3					
				oOSs	*Seque*	*P f eif*	*P COs*	*trunc*	*record*
15	(2*;* 9)	5	7	0*:*7573	0*:*5048	0*:*7591	0*:*8213	0*:*7520	0*:*7475
		6	7	0*:*8258	0*:*5234	0*:*8336	0*:*8904	0*:*8326	0*:*8104
		5	8	0*:*7054	0*:*3713	0*:*7209	0*:*8104	0*:*7262	0*:*6817
		4	8	0*:*6068	0*:*3401	0*:*6158	0*:*7067	0*:*6152	0*:*5898
	(2*;* 12)	5	7	0*:*8286	0*:*8519	0*:*8081		0*:*7825	0*:*8440
		6	7	0*:*8978	0*:*8719	0*:*8830		0*:*8634	0*:*9076
		5	8	0*:*8232	0*:*7753	0*:*8057		0*:*7818	0*:*8337
		4	9	0*:*7091	0*:*6366	0*:*6920		0*:*6665	0*:*7173
	(5*;* 12)	5	7	0*:*3021	0*:*5993	0*:*2622		0*:*2211	0*:*3394
		6	7	0*:*4287	0*:*7136	0*:*3805		0*:*3288	0*:*4722
		5	8	0*:*2975	0*:*5303	0*:*2602		0*:*2204	0*:*3308
		4	9	0*:*1788	0*:*3265	0*:*1554		0*:*1299	0*:*1986
20	(2*;* 9)	5	7	0*:*6538	0*:*2347	0*:*6831	0*:*7800	0*:*7092	0*:*6224
		6	7	0*:*6973	0*:*2421	0*:*7308	0*:*8245	0*:*7616	0*:*6619
		5	8	0*:*5413	0*:*1358	0*:*5710	0*:*7043	0*:*6172	0*:*5024
		4	8	0*:*4760	0*:*1247	0*:*5083	0*:*6311	0*:*5390	0*:*4430
	(2*;* 12)	5	7	0*:*8538	0*:*5446	0*:*8589	0*:*8980	0*:*8591	0*:*8441
		6	7	0*:*8988	0*:*5534	0*:*9081	0*:*9431	0*:*9127	0*:*8853
		5	8	0*:*8074	0*:*4008	0*:*4236	0*:*8855	0*:*83372	0*:*7860
		4	9	0*:*6657	0*:*2570	0*:*6895	0*:*7849	0*:*7067	0*:*6371
	(5*;* 12)	5	7	0*:*8538	0*:*5446	0*:*8589	0*:*8980	0*:*8591	0*:*8441
		6	7	0*:*8988	0*:*5534	0*:*9081	0*:*9431	0*:*9127	0*:*8853
		5	8	0*:*8074	0*:*4008	0*:*8236	0*:*8855	0*:*8337	0*:*7860
		4	9	0*:*6658	0*:*2570	0*:*6895	0*:*7849	0*:*7067	0*:*6371

4 **Illustrative example**

In this section, to illustrate the inferential procedures developed in the preceding sec-tions and to compare the results developed, we use the data representing the all causes deaths in children 1-59 months (total deaths), in middle east and north Africa that ob-served in year 2008 (source link, http://www.gapminder.org, http://www.lancet.org), and the extracted order statistics are presented in table 7.

Table 7: oOSs of total children 1-59 months deaths.

r	1	2	3	4	5	6	7
Country	Qatar	Bahrain	Emirate	kuwait	Oman	Lebanon	Libya
$X_{r:18}$	65	89	166	262	268	361	1071
r	8	9	10	11	12	13	14
Country	Jordan	Djibouti	Tunisia	Syria	Saudi	Morocco	Algeria
$X_{r:18}$	1077	1380	1527	4932	5527	8505	12925
r	15	16	17	18			
Country	Egypt	Iraq	Iran	Yamen			
$X_{r:18}$	17631	18472	19824	29554			

Based on the observed oOSs in table 7, prediction intervals for future sample Y of size 15 of *oOSs* and *P COs* with r_1 = 3 and n_1 = 5 were obtained. These intervals are presented in table 8.

Table 8: Prediction intervals for *oOSs* and PCOs

r	*i*	*j*	$(X_{i:18}; X_{j:18})$	1	
				$Y_{r:15}$	$Y_{r:10:15}$
1	1	9	(65; 1380)	0:5442	0:5442
2	1	9	(65; 1380)	0:7928	0:7928
3	1	9	(65; 1380)	0:8867	0:8867
4	1	10	(65; 1527)	0:9256	0:8941
5	1	10	(65; 1527)	0:8942	0:7713
6	2	12	(89; 5527)	0:9183	0:7695
7	3	13	(166; 8507)	0:9063	0:6829
8	3	14	(166; 12925)	0:9200	0:5838
9	8	17	(1077; 19824)	0:7956	0:7400
10	8	17	(1077; 19824)	0:8702	0:4399
11	8	16	(1077; 18472)	0:8189	0:0000
12	8	17	(1077; 19824)	0:8636	0:0000
13	7	17	(1071; 19824)	0:8189	0:0000
14	8	18	(1077; 29554)	0:8000	0:0000
15	8	18	(1077; 29554)	0:54529	0:0000

5 Concluding remark

In this article, we have derived distribution-free PIs for future gOSs based on oOSs. Also, outer and inner PIs are derived based on oOSs observations. In general, the numerical results consistent with the fact that says, the coverage probability increase as the length of the PIs increase, else the prediction coefficients values of the inner prediction case increase as the PIs length decrease. The generality of our work enabled us to compare the values of different future sampling schemes at the same time, and choose the best one corresponding with the practical work, for example, tables 2 and 3 show that, under the same assumption and based on oOSs, the change on *i* is more impact than *j* for predicting future trunc and record, on the contrary, the change on *j* is more impact than *i* on predicting of future oOSs and PCOs .

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