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RESEARCH ARTICLE

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ON θ-CLOSEDNESS AND H-CLOSEDNESS

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ABSTRACT

In this paper the set of all θ-open sets in a topological space (X, $\Im$) has been shown to form a topology $\Im_\theta$. It has been proved that $\Im_\theta$ is contained in $\Im$. $\Im_\theta$-compactness and $\Im_\theta$-connectedness of subsets of X have been studied. It has been shown that the class of all H-continua is closed under formation of sum, product and continuous image. Relations among connectedness, θ-connectedness and $\Im_\theta$-connectedness have been discussed.

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Keywords : $\Im_\theta$**-compact,** $\Im_\theta$-connected, H-closed, H-continuum, Sum.

1.INTRODUCTION

The concept of a θ-open set was introduced by *Veličko*[5].We have established the arbitrary union and finite intersection property of θ-open sets. Clay and Joseph [3] introduced the notion of θ-connectivity as generalization of ordinary connectivity and established some properties of θ- connectedness. Ganguly and Bandayopadhayay [2] defined an H-continuum as a generalization of continuum by using a generalized concept of compactness called H-closedness. Sums of topological spaces were studied by Majumdar and Asaduzzaman [4]. We show that connectivity implies θ-connectivity and then $\Im_\theta$- counnectivity. Here it has been shown that the sums of two θ-connected spaces is θ-connected. A compact set is also $\Im_\theta$-compact. We have also proved that if X and Y are H-continua, then so is X×Y, if moreover $X \cap Y \neq \Phi$ then $X \cap Y$ and X+Y (when it exists) are H-continua too. We have shown that none of a subspace, a continuous image, and an open image of an H-continuum need be an H-continuum; however the image of an H-continuum under a map which is both continuous and open is an H-continuum. In particular, every identification space of an H-continuum is an H-continuum.

2. $\Im_\theta$-Topology.

In this section we shall prove that the θ-open sets in a topological space (X, $\Im$) form a topology $\Im_\theta$ on X. We shall also prove a few results on (X, $\Im_\theta$).

Let X be a topological space. We recall [1] that for a subset A of X the θ-closure of A, written cl_θ (A), is defined as cl_θ (A)= {x∈X | ∀ open sets G in X with x∈ G, $\overline{G}$ ∩A≠ Φ }.

A is said to be θ-closed if A=cl_θ (A).A is called θ-open if X-A is θ-closed. Thus, **A is θ-open ⇔ [∀ x∈X |(∀ open sets G in X with x∈ G, $\overline{G}$ ∩(X-A)≠ Φ) ⇔ $x \in X - A$].**

Lemma 2.1: Union of θ-open sets is θ-open.

Proof:

Let {V_α}be a non-empty collection ofθ-open sets in X.Let W_0 be an open set in X with x∈W such that $\overline{W_0}$ ∩(X- $\bigcup_\alpha$ V_α)≠ Φ ………..(1)

Then $\overline{W_0}$ ∩(X- $\bigcup_\alpha$ V_α)≠ Φ and so, $\cap$ ($\overline{W_0}$ ∩(X- $\bigcup_\alpha$ V_α)) ≠ Φ .

Hence, $\overline{W_0}$ ∩(X- $\bigcup_\alpha$ V_α)≠ Φ

Thus x∈W_0& $\overline{W_0}$ ∩(X-V_α)≠ Φ , for each α.

By Lemma-2.1 x∉ V_α, for each α, since each V_α is θ-open.

Therefore x ∉ $\bigcup_\alpha$ V_α and so, (1) implies that $\bigcup_\alpha$ V_α is θ-open in X.

Lemma-2.2: The intersection of a finite number of θ-open sets is θ-open.

Proof:

Let V_1, V_2,… ,V_n be θ-open sets in X.

Then let x∈X and W be an open set in X with x∈W and

$\overline{W} \cap (X - (V_1 \cap V_2 \cap \dots \cap V_n) \neq \Phi$

$\Rightarrow \overline{W}$ ∩($\bigcup_{i=1}^{n}$ (X-V_i))≠ Φ

$\Rightarrow \overline{W}$ ∩ (X-V_i)≠ Φ for at least one $1 \leq i \leq n$

$\Rightarrow$ x∉ V_i for at least one $1 \leq i \leq n$ [since $V_1, V_2, \dots, V_n$ are θ-open in X]

$\Rightarrow$ x∉ $V_1 \cap V_2 \cap \dots \cap V_n$

$\Rightarrow V_1 \cap V_2 \cap \dots \cap V_n$ is θ-open in X.

Since obviously both X and Φ are θ-closed, both are θ-open as well. Lemma 2.1 & Lemma 2.2 therefore yield:

Theorem 2.1: The θ-open sets in X form a topology on X.

If $\Im$ is a topology on X, we denote by $\Im_\theta$ the topology on X consisting of theθ-open sets.

Theorem-2.2: $\Im_\theta \subseteq \Im$

Proof:

Let A be a subsets of X.

Then, A$\subseteq \overline{A} \subseteq$ cl_θ A.

Hence, A is θ-closed $\Rightarrow$ cl_θ A=A

$\Rightarrow \overline{A}$=A

$\Rightarrow$ A is closed.

Hence, V $\in \Im_\theta \Rightarrow$ X-V is θ-closed

$\Rightarrow$ X-V is closed

$\Rightarrow$ V $\in \Im$.

Remark 2.1: It is easily seen that if X and Y are two topological spaces and G $\subseteq$ X, H $\subseteq$ Y, then $\overline{G \times H}$ $= \overline{G} \times \overline{H}$

Theorem-2.3: Product of two θ-closed sets in two different topological spaces is θ-closed in their product space.

Proof:

Let (X, $\Im_1$) and (X, $\Im_2$) be two Hausdorff spaces and let A and B be two θ-closed subsets of X and Y respectively.

Since A and B are θ-closed A= cl_θ A, B= cl_θ A,

$$\left.\begin{array}{l}\text{i.e.}, A = \{x \in X \mid \forall \text{ open sets G in X with } x \in G, \overline{G} \cap A \neq \Phi\} \\ \text{and } B = \{y \in Y \mid \forall \text{ open sets H in Y with } y \in H, \overline{H} \cap B \neq \Phi\}\end{array}\right\} \ldots\ldots\ldots (2)$$

Let (x, y) $\in cl_\theta$ (A$\times B$). Then (x, y) $\in$ X×Y is such that for each open set W in X×Y with (x, y) $\in$W, $\overline{W}$ $\cap$(A$\times B$) $\neq \Phi$. In particular, for each open sets G in X with x$\in$ G and for each open set H in Y such that (x, y) $\in$ G$\times H$ and $\overline{G \times H}$ $\cap$(A$\times B$) $\neq \Phi$, i.e., ($\overline{G} \times \overline{H}$)$\cap$(A$\times B$) $\neq \Phi$ by Remark 2.1. (2) implies, (x, y) $\in$ A$\times B$. So, A$\times B$= cl_θ (A$\times B$), i.e., A$\times B$is θ-closed.

Corollary -2.1: Product of two θ-open subsets in two different topological spaces is θ-open in their product space.

Proof:

Let (X, $\Im_1$) and (X, $\Im_2$) be two topological spaces and let A and B be two θ-open subsets of X and Y respectively. Then X-A and Y-B are θ-closed in X and Y respectively. Now (X×Y)-(A$\times B$)=[(X-A) ×Y] ∪[X×(Y-B)].

Since X-A and Yare θ-closed,(X-A) ×Y is θ-closed. Similarly, X×(Y-B) is also θ-closed.

Hence, (A$\times B$) is θ-open in X×Y.

Corollary-2.2: If $\Im$ and $\overline{\Im}$ denote the product topologies on (X, $\Im^1$) ×(Y, $\Im^2$) and (X, $\Im_\theta^1$) ×(Y, $\Im_\theta^2$) respectively then $\overline{\Im} \subseteq \Im_\theta$ or briefly, ($\Im_\theta^1 \times \Im_\theta^2$)$\subseteq$($\Im^1 \times \Im^2$)$_\theta$.

Defination-2.1. $\Im_\theta$-compactness

We call a subset A of X $\Im_\theta$-compact ($\Im_\theta$-connected) if A is compact (connected) in (X, $\Im_\theta$**)**.

Since $\Im_\theta \subseteq \Im$, X is compact $\Rightarrow$ X is $\Im_\theta$-compact.

3.θ- connectedness

We recollect the definition of θ- connectedness defined in [2].

A pair (P, Q) of non-empty subsets of X is called **θ-separation** [2] relative to X if (P$\cap cl_\theta Q$) $\cup(Q \cap cl_\theta P) = \Phi$.

A subset A of X is called **θ-connected** [2] if A ≠ P$\cup Q$, where (P, Q) is a θ-separation relative to X.

Here we prove some results on θ-connectedness and $\Im_\theta$-connectedness.

Theorem-3.1: X is connected $\Rightarrow$ X is θ-connected $\Rightarrow$ X is $\Im_\theta$-connected.

Proof:

Suppose X is connected. If possible, let X be θ-disconnected. Then X = P∪Q, where P, Q are non-empty and $P \cap cl_\theta Q = \Phi$. Clearly $P \cap Q = \Phi$.

Let x ∈ P. Then $x \notin cl_\theta Q$, and so, there exists an open set G in X such that x ∈ G and $Q \cap \overline{G} = \Phi$. Then $\overline{G} \subseteq X-Q=P$ and so $G \subseteq P$. Hence P is open.

Similarly, we can show that Q is open. Therefore X is disconnected. The contradiction proves that X is θ-connected.

Next let X be θ-connected. Suppose X is not $\Im_\theta$-connected. Then, X = P∪Q for disjoint non-empty θ-open sets P and Q. Since X is θ-connected, either $P \cap cl_\theta Q \neq \Phi$ or $Q \cap cl_\theta P \neq \Phi$. i.e., either $cl_\theta Q \not\subset X-P=Q$ or $cl_\theta P \not\subset X-Q=P$

i.e., either Q is not θ-closed or P is not θ-closed.

But this is a contradiction to the hypothesis. Hence X is $\Im_\theta$-connected.

Comment 3.1: While connectedness implies θ-connectedness, the converse is not true. This was proved by Clay and Joseph [3]. They provided an example of a θ-connected space which is not connected. Regularity of a space of course compels the two properties to coincide. We give below the above –mentioned example of a θ-connected space which is not connected.

Example ([3], p. 270).

Let I be the unit interval [0, 1] and Y=I× {0} and let X=I×I with the topology generated by the following base for the open sets:

(1) The relative open sets from the plane X-Y

and (2) for x∈ Y, sets of the form (V∩(X-Y)) ∪{x}where V is open in the plane with x∈V.

Y is discrete in the relative topology from X and hence Y is not connected.

Suppose that (P, Q) is a θ-separation relative to X and that Y=P∪Q. Choose (r, 0) ∈P without loss of generality, assume that there is an s∈ I with r<s and (s, 0) ∈Q. Let c = sup {r∈ I:r<s and (r, 0) ∈ P}. We see easily that $(c, 0) \in cl_\theta Q$, we obtain a contradiction and Y is θ-connected relative to X.

Majumdar & Asaduzzaman [4] defined sum of two topological space. Let X and Y be two topological spaces with topologies $\Im_1$ and $\Im_2$ respectively. Let either X∩Y be empty or be a subspace of both X and Y. Then X and Y are said to be **compatible** with each other and X∪Y made into a topological space by imposing on it the topology $\Im$ generated by $\Im_1 \cup \Im_2$ is called the **sum** of X and Y. We denote it by X+Y.

Our next result is about θ-connectivity of the sum of two spaces when it exits.

Theorem-3.2: Sum of two θ-connected spaces is also θ-connected.

Proof:

Let X and Y be two θ-connected spaces. If possible suppose X+Y is not θ-connected. Then there exists two non-empty subsets P, Q of X+Y such that X+Y=P∪Q with

$P \cap cl_\theta Q = \Phi$ ………….(α)

$Q \cap cl_\theta P = \Phi$ ………….(β)

Let $P_1=P\cap X$, $Q_1=Q\cap X$…………(1)

$P_2=P\cap Y$, $Q_2=Q\cap Y$…………(2)

Then $X=P_1\cup Q_1$, $Y=P_2\cup Q_2$.

Since X and Y are θ-connected,

(i) $P_1 \cap cl_\theta Q_1 \neq \Phi$ or $Q_1 \cap cl_\theta P_1 \neq \Phi$

(ii) $P_2 \cap cl_\theta Q_2 \neq \Phi$ or $Q_2 \cap cl_\theta P_2 \neq \Phi$

Now $P \cap cl_\theta Q = (P_1 \cup P_2) \cap cl_\theta (Q_1 \cup Q_2)$

$= (P_1 \cup P_2) \cap (cl_\theta Q_1 \cup cl_\theta Q_2)$

$= (P_1 \cap cl_\theta Q_1) \cup (P_1 \cap cl_\theta Q_2) \cup (P_2 \cap cl_\theta Q_1) \cup (P_2 \cap cl_\theta Q_2)$

By (α)$(P_1 \cap cl_\theta Q_1) = \Phi$ and $(P_2 \cap cl_\theta Q_2) = \Phi$

So from (i) and (ii)

$Q_1 \cap cl_\theta P_1 \neq \Phi$ and $Q_2 \cap cl_\theta P_2 \neq \Phi$

Hence, $Q \cap cl_\theta P = (Q_1 \cap cl_\theta P_1) \cup (Q_1 \cap cl_\theta P_2) \cup (Q_2 \cap cl_\theta P_1) \cup (Q_2 \cap cl_\theta P_2) \neq \Phi$, by (4).

This contradicts (β).

Hence, X+Y is θ-connected.

4. H-continuum

Velicko [5] defined a spaceX to be**H-closed** if every open cover $\{V_\alpha\}$ of X has a finite sub collection $V_{\alpha_1}, \ldots, V_{\alpha_n}$ such that $\overline{V}_{\alpha_1} \cup \ldots \cup \overline{V}_{\alpha_n} = X$

Ganguly and Bandyopadhyaya [2] defined and studied **H-continua.** An**H-continuum** is a topological space which is both connected and H-closed.

As for compact spaces, we have

Theorem-4.1: The product of two H-closed spaces is H-closed.

Proof:

Let X and Y be two H-closed spaces and W be an open cover of X×Y. Without loss of generality we may assume that each member of W is of the form $W_{\alpha\beta} = U_\alpha \times V_\beta$ where U_α and V_β are open sets in X and Y respectively. Then $\{U_\alpha\}$ is an open cover of X and $\{V_\beta\}$ is an open cover of Y. Since X and Y are H-closed, there exist $\{U_{\alpha_1}, \ldots, U_{\alpha_m}\}$ and $\{V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_n}\}$ such that $\overline{U_{\alpha_1}} \cup \ldots \cup \overline{U_{\alpha_m}} = X$ and $\overline{V_{\beta_1}} \cup \ldots \cup \overline{V_{\beta_n}} = Y$. Then $\bigcup_{\substack{i=1 \\ j=1}}^{\substack{n \\ m}} \overline{W}_{\alpha_i \beta_j} = X \times Y$. So, X×Y is an H-closed space.

It is known that if both X and Y are Hausdorff or connected then X×Y too is Hausdorffor connected respectively. We therefore have from Theorem-4.1:

Theorem-4.2: Product of two H-continua spaces is also H-continuum.

Majumdar and Asaduzzaman have established the fact that connectivity of each compatible spaces X and Y implies the same of X+Y iff $X \cap Y \neq \Phi$.

Lemma-4.1: If X and Y are compatible H-closed spaces then X+Y is H-closed.

Proof:

Let $\{W_\alpha\}$ be an open cover ofX+Y. Then each $W_\alpha = U_\alpha \cup V_\alpha$, for some U_α, V_α open in X and Y respectively. Then $\{U_\alpha\}$ and $\{V_\alpha\}$ are open covers of X and Y respectively. Since X and Y are closed, $X = \overline{U}_{\alpha_1} \cup \overline{U}_{\alpha_2} \cup \ldots\ldots \cup \overline{U}_{\alpha_m}$ and $Y = \overline{V}_{\beta_1} \cup \overline{V}_{\beta_2} \cup \ldots\ldots \cup \overline{V}_{\beta_n}$ for some $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$.

Then $X+Y = \overline{W}_{\alpha_1} \cup \ldots\ldots \cup \overline{W}_{\alpha_m} \cup \overline{W}_{\beta_1} \cup \ldots\ldots \cup \overline{W}_{\beta_n}$. Hence X+Y is H-closed.

So we have,

Theorem-4.3: If X and Y are H-continua, then X+Y will be an H-continuum iff $X \cap Y \neq \Phi$.

Comment: A subspace of an H-continuum space need not be so. For [0, 1] is an H-continuum, but the subspace {0, 1} is not H-continuum as it is not connected. The property of being an H-continuum does not hold for intersection. If $C = \{(x, y) \mid x^2+y^2=1\}$, $C_1 = \{(x, y) \in C \mid x \leq 0\}$ and $C_2 = \{(x, y) \in C \mid x \geq 0\}$, then $C_1 \cap C_2 = \{(0, 1), (0, -1)\}$ is not H-continuum as it is not connected.

Theorem-4.4: Let X be an H-continuum and Y a topological space and let f : X→Y be both continuous and open. Then f(X) is an H-continuum.

Proof:

X is an H-continuum and so, X is H-closed,Hausdorff and connected. Let $\{V_\alpha\}$be an open cover off(X). Since f is continuous, $\{f^{-1}(V_\alpha)\}$ is an open cover of X. As X is H-closed, there exist $\{f^{-1}(V_{\alpha_1}),\ldots,f^{-1}(V_{\alpha_n})\}$ such that $\overline{f^{-1}(V_{\alpha_1})} \cup \ldots \cup \overline{f^{-1}(V_{\alpha_n})} = X$. So, $\overline{V_{\alpha_1}} \cup \ldots \cup \overline{V_{\alpha_n}} = f(X)$. Thus f(X) is H-closed. As f is continuous, f(X) is connected. Hence f(X) is an H-continuum.

Comments: If f is only open or only continuous, then f(X) need not be an H-continuum.

For, if $(X, \Im)$ is a continuum and X has at least two elements and $f : (X, \Im) \to (X, D)$ is the identity map on X where D is the discrete topology, then f is open but f(X) is not an H-continuum because it is disconnected.

If for the above continuum $(X, \Im)$, $f : (X, \Im) \to (X, \Im_0)$is the identity map on X where $\Im_0$ denotes the indiscrete topology, then f is continuous, but f(X)=X is not an H-continuum since f(X) is not Hausdorff.

(1) If X is an H-continuum and R an equivalence relation on X, then the identification space X/R is an H-continuum as the projection map X → X/R is onto and both continuous and open.

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