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RESEARCH ARTICLE

SOME COMMON FIXED POINT THEOREMS FOR SINGLE-VALUED AND SET-VALUED MAPPINGS

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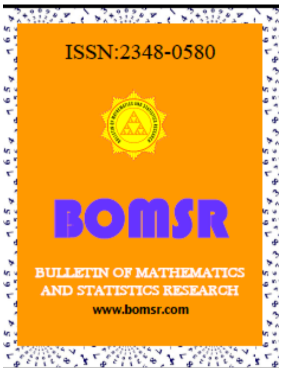

ABSTRACT

The aim of this paper is to establish some common fixed point theorems for single-valued and set-valued mappings under Ciric-type strict contractive conditions with no compacity and without using the continuity of maps. Our results extend and unify the result due to Ahmed [1] and others.

Keywords: δ-compatible and weakly compatible mappings; D-maps; Single and set-valued mappings; common fixed point.

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1 INTRODUCTION

It is well-known that the theory of fixed points is one of the traditional theories in mathematics that has a broad set of applications. In 1922, Polish mathematician Stephan Banach published his famous contraction principle. Since then, this principle has been extended and generalized in several ways either by using the contractive conditions or by imposing some additional conditions on the ambient spaces. In recent years several fixed point theorems for single and set valued mappings for pairs of mappings are proved and have numerous applications and by now there exists an extensive considerable and rich literature in this domain. Note that common fixed point theorems for single-valued and set-valued mappings are interesting and play a major role in many areas.

In this work, we establish some common fixed point theorems for single and set-valued mappings under Ciric-type strict contractive condition. To prove our results we use minimal type commutativity without using the continuity of maps and compacity requirement.

2 PRELIMINARIES

Throughout this paper, (X,d) denotes a metric space and $B(X)$ is the set of all nonempty bounded subsets of X. As in [4,5], we define the functions $\delta(A,B)$ and $D(A,B)$ as follows:

$D(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$
$\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\},$
for all A, B in $B(X)$. If A contains a single point a, we write $\delta(A,B) = \delta(a,B)$. Also, if B contains a single point b, it yields $\delta(A,B) = d(a,b)$.

The definition of the function $\delta(A,B)$ yields the following:

$$\delta(A,B) = \delta(B,A),$$
$$\delta(A,B) \le \delta(A,C) + \delta(C,B),$$
$$\delta(A,B) = 0 \text{ if and only if } A = B = \{a\},$$
$$\delta(A,A) = diam\ A,$$

for all A, B, C in $B(X)$.

Definition 2.1. [4] A sequence $\{A_n\}$ of subsets of X is said to be convergent to a subset A of X if

i. Given $a \in A$, there is a sequence $\{a_n\}$ in X such that $a_n \in A_n$ for $n \in N^*$ and $\{a_n\}$ converges to a.
ii. Given $\varepsilon > 0$, there exists a positive integer N such that $A_n \subseteq A_\varepsilon$ for $n > N$ where A_ε is the union of all open spheres with centers in A and radius ε.

Lemma 2.1. [4,5] If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B in $B(X)$ respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A,B)$.

Lemma 2.2. [5] Let $\{A_n\}$ be a sequence in $B(X)$ and y be a point in X such that $\delta(A_n, y) \to 0$ as $n \to \infty$. Then, the sequence $\{A_n\}$ converges to the set $\{y_n\}$ in $B(X)$.

Definition 2.2. [5] A set-valued mapping F of X into $B(X)$ is said to be continuous at $x \in X$ if the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx whenever $\{x_n\}$ is a sequence in X converging to x in X. F is said to be continuous on X if it is continuous at every point in X .

Lemma 2.3. [5] Let $\{A_n\}$ be a sequence of nonempty subsets of X and z in X such that
$\lim_{n\to\infty} a_n = z,$
where z independent of the particular choices of each $a_n \in A_n$. If a self-map I of X is continuous, then $\{Iz\}$ is the limit of the sequence $\{IA_n\}$.

In year, Sessa [11] introduced the concept of weakly commuting mappings as a generalization of commuting mappings. Later on, Jungck [6] enlarged the class of non-commuting mappings by compatible mappings. Further generalizations of compatible mappings are given by Jungck et al. [7], Pathak and Khan [10], Cho et al. [2].

Recently, Li-Shan [9] introduced the following definition:

Definition 2.3. [9] The mappings $F: X \to B(X)$ and $f: X \to X$ are δ-compatible if
$$\lim_{n\to\infty} \delta(Ffx_n, fFx_n) = 0$$
whenever $\{x_n\}$ is a sequence in X such that $fFx_n \in B(X), Fx_n \to \{t\}$ and $fx_n \to t$ for some $t \in X$.

As a generalization of the above definition Jungck and Rhoades [8] gave the following definition:

Definition 2.4. [5] The mappings $F: X \to B(X)$ and $f: X \to X$ are weakly compatible if they commutes at coincidence points, that is
$\{t \in X: Ft = \{ft\}\} \subseteq \{t \in X: Fft = fFt\}.$

It can be seen that δ-compatible mappings are weakly compatible but the converse is not true. Example supporting this fact can be found in [8].

Recently, M. A. Ahmed has proved in his paper [1] the following theorem.

Theorem 2.1. [1] Let I, J be functions of a compact metric space (X, d) into itself and $F, G: X \to B(X)$ be two set-valued functions with

(1) $\cup F(X) \subseteq J(X)$ and $\cup G(X) \subseteq I(X)$.

(2) Suppose that the inequality

$$\delta(Fx, Gy) < \alpha\, max\{d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy)\}$$
$$+(1-\alpha)[aD(Ix, Gy) + bD(Jy, Fx)], \quad (2.1)$$

for all $x, y \in X$, where $0 \le \alpha < 1, a \ge 0, b \ge 0, a \le \frac{1}{2}, b < \frac{1}{2}, \alpha|a - b| < 1 - (a + b)$, holds whenever the right hand side of (2.1) is positive. If the pair $\{F, I\}$ and $\{G, J\}$ are compatible and if the functions F and I are continuous, then there is a unique point u in X such that

$Fu = Gu = \{u\} = \{Iu\} = \{Ju\}$.

The main purpose of this paper is to extend and unify Theorem 2.1 above under Ciric-type strict contractive condition by dropping the hypothesis of compacity and without using the continuity of mappings with $a, b, c \in [0,1)$ on condition that their sum is strictly less than 1, also we use a new concept of mapping called D-mappings.

Definition 2.5. [3] The mappings $F: X \to B(X)$ and $I: X \to X$ are said to be D-mappings if there exists a sequence $\{x_n\}$ in X such that, $\lim_{n\to\infty} Ix_n = t$ and $\lim_{n\to\infty} Fx_n = \{t\}$ for some $t \in X$.

Example 2.1. [3] Let $X = [0, \infty)$. Define $F: X \to B(X)$ and $I: X \to X$ by $Fx = [0,2x]$ and $Ix = 3x$ for all $x \in X$. Consider the sequence $x_n = \frac{1}{4n}$ for all $n \in N^*$.

Obviously, $\lim_{n\to\infty} Ix_n = 0$ and $\lim_{n\to\infty} Fx_n = \{0\}$. Then F and I are D-mappings.

3 RESULTS

Theorem 3.1. Let (X, d) be a metric space, let $F, G: X \to B(X)$ and $I, J: X \to X$ be set-valued and single-valued mappings respectively satisfying the conditions

(1) $F(X) \subseteq J(X)$ and $G(X) \subseteq I(X)$.

(2) Satisfying the inequality

$$\delta(Fx, Gy) < ad(Ix, Jy) + b\, max\{\delta(Ix, Fx), \delta(Jy, Gy)\}$$
$$+c[D(Ix, Gy) + D(Jy, Fx)] \quad (3.1)$$

for all $x, y \in X$, where $a, b, c \ge 0$ with $a + b + 2c < 1$, whenever the right hand side of (3.1) is positive. If either

(3) F, I are weakly compatible D-mappings, G, J are weakly compatible and $F(X)$ or $J(X)$ is closed

or

(4) G, J are weakly compatible D-mappings, F, I are weakly compatible and $G(X)$ or $I(X)$ is closed.

Then there is a unique common fixed point t in X such that $Ft = Gt = \{t\} = \{It\} = \{Jt\}$.

Proof. Let F and I are D-mappings, then there exists a sequence $\{x_n\}$ in X such that, $\lim_{n\to\infty} Ix_n = t$ and $\lim_{n\to\infty} Fx_n = \{t\}$ for some $t \in X$. Since $F(X)$ is closed and $F(X) \subseteq J(X)$, then there exists a point u in X such that $Ju = t$. Then from inequality (3.1), we have

$$\delta(Fx_n, Gu) < ad(Ix_n, Ju) + b\, max\{\delta(Ix_n, Fx_n), \delta(Ju, Gu)\}$$

$$+c[D(Ix_n, Gu) + D(Ju, Fx_n)]$$

Taking limit as $n \to \infty$ and using Lemma 2.1, we have

$\delta(Ju, Gu) \leq (b+c)\delta(Ju, Gu)$

Since $b + c < 1$, then the above contradiction demands that $Gu = \{Ju\}$.

Again since G and J are weakly compatible, $Gu = \{Ju\}$ implies that $GJu = JGu$ and hence $GGu = GJu = JGu = \{JJu\}$.

Using (3.1) again, we have

$$\delta(Fx_n, GGu) < ad(Ix_n, JGu) + b\ max\{\delta(Ix_n, Fx_n), \delta(JGu, GGu)\}$$
$$+c[D(Ix_n, GGu) + D(JGu, Fx_n)]$$

Taking limit as $n \to \infty$ and in view of Lemma 2.1, we obtain

$$\delta(Ju, GGu) \leq ad(Ju, GGu) + 2cD(Ju, GGu)$$
$$\leq (a+2c)\delta(Ju, GGu)$$

which implies that $GGu = \{Ju\}$ because $a + 2c < 1$. Hence $\{Ju\} = GGu = JGu$, i.e. $Gu = GGu = JGu$ and Gu is a common fixed point of G and J. Since $G(X) \subseteq I(X)$, then there is a point $v \in X$ such that $\{Iv\} = Gu$.

Moreover, from (3.1), we have

$$\delta(Fv, Gu) < ad(Iv, Ju) + b\ max\{\delta(Iv, Fv), \delta(Ju, Gu)\}$$
$$+c[D(Iv, Gu) + D(Ju, Fv)]$$
$$= b\delta(Gu, Fv) + cD(Gu, Fv)$$
$$\leq (b+c)\delta(Gu, Fv).$$

It is easy to see that $Fv = Gu = \{Iv\}$ as $b + c < 1$.

Since $Fv = \{Iv\}$, by the weak compatibility of F and I, we get $FIv = IFv$ and hence $FFv = FIv = IFv = \{IIv\}$.

Moreover, from (3.1), we have

$$\delta(FFv, Gu) < ad(IFv, Ju) + b\ max\{\delta(IFv, FFv), \delta(Ju, Gu)\}$$
$$+c[D(IFv, Gu) + D(Ju, FFv)]$$
$$= ad(IFv, Ju) + 2cD(IFv, Gu)$$
$$= ad(FFv, Gu) + 2cD(FFv, Gu)$$
$$\leq (a+2c)\delta(FFv, Gu) < \delta(FFv, Gu)$$

which is a contradiction, thus $FFv = Gu$ i.e. $FGu = Gu = IGu$ and Gu is also a common fixed point of F and I. Since $Gu = \{t\}$, then $Ft = Gt = \{t\} = \{It\} = \{Jt\}$.

Similarly one can obtain this conclusion using (4) instead of (3).

Finally, we shall show that t is unique. Suppose that t' be another common fixed point of maps I, J, F and G such that $t \neq t'$. Then by (3.1), we have

$$d(t, t') = \delta(Ft, Gt')$$
$$< a\ d(It, Jt') + b\ max\{\delta(It, Ft), \delta(Jt', Gt')\} + c[D(It, Gt') + D(Jt', Ft)]$$
$$= a\ d(t, t') + 2cD(t, t')$$
$$\leq (a+2c)d(t, t') < d(t, t'),$$

which is a contradiction. Thus $t = t'$ i.e. t is the unique common fixed point of I, J, F and G.

This completes the proof of the theorem.

If $F = G$ and $I = J$, then we get the following result as a corollary.

Corollary 3.1. Let (X, d) be a metric space, let $F: X \to B(X)$ and $I: X \to X$ be set-valued and single-valued mappings respectively satisfying the conditions

(1) $F(X) \subseteq I(X)$.

(2) Satisfying the inequality

$$\delta(Fx, Fy) < ad(Ix, Iy) + b\, max\{\delta(Ix, Fx), \delta(Iy, Fy)\} + c[D(Ix, Fy) + D(Iy, Fx)] \quad (3.2)$$

for all $x, y \in X$, where $a, b, c \geq 0$ with $a + b + 2c < 1$, whenever the right hand side of (3.2) is positive. If F and I are weakly compatible D-mappings and $F(X)$ or $I(X)$ is closed, then F and I have a unique common fixed point t in X.

For three mappings, we have the following result as a corollary.

Corollary 3.2. Let $I: X \to X$ be a self map of a metric space (X, d) and $F, G: X \to B(X)$ be two set-valued mappings such that

(1) $F(X) \subseteq I(X)$ and $G(X) \subseteq I(X)$.

(2) Satisfying the inequality

$$\delta(Fx, Gy) < ad(Ix, Iy) + b\, max\{\delta(Ix, Fx), \delta(Iy, Gy)\} + c[D(Ix, Gy) + D(Iy, Fx)] \quad (3.3)$$

for all $x, y \in X$, where $a, b, c \geq 0$ with $a + b + 2c < 1$, whenever the right hand side of (3.3) is positive. Further, if either

(3) F, I are weakly compatible D-mappings, G, I are weakly compatible and $F(X)$ or $I(X)$ is closed or

(4) G, I are weakly compatible D-mappings, F, I are weakly compatible and $G(X)$ or $I(X)$ is closed.

Then F, G and I have a unique common fixed point in X.

Next, we give our second result which is a generalization of the above result.

Theorem 3.2. Let $I, J: X \to X$ be self mappings of a metric space (X, d) and $F_i: X \to CB(X),\ i \in N^*$ be set-valued mappings such that

(1) $F_i \subseteq J(X)$ and $F_{i+1} \subseteq I(X)$.

(2) the inequality

$$\delta(F_i x, F_{i+1} y) < ad(Ix, Jy) + b\, max\{\delta(Ix, F_i x), \delta(Jy, F_{i+1} y)\} + c[D(Ix, F_{i+1} y) + D(Jy, F_i x)] \quad (3.4)$$

holds for all $x, y \in X$ and $\forall\ i \in N^*$, where $a, b, c \geq 0$ with $a + b + 2c < 1$, whenever the right hand side of (3.4) is positive. Further, if either

(3) F_i, I are weakly compatible D-mappings, F_{i+1}, I are weakly compatible and $F_i(X)$ or $J(X)$ is closed or

(4) F_{i+1}, I are weakly compatible D-mappings, F_i, I are weakly compatible and $F_{i+1}(X)$ or $I(X)$ is closed.

Then there is a unique common fixed point $t \in X$ such that

$F_i t = \{It\} = \{Jt\} = \{t\}, \forall i \in N^*$.

Proof. The proof follows from Theorem 3.1.

Now, we a generalization of Theorem 2.1 of [1].

Theorem 3.3. Let (X,d) be a metric space, let $I,J: X \to X$ be single-valued mappings and $F_i: X \to CB(X),\ i \in N^*$ be set-valued mappings such that

(1) $\cup F_i \subseteq J(X)$ and $\cup F_{i+1} \subseteq I(X)$.

(2) the inequality

$$\delta(F_i x, F_{i+1} y) < ad(Ix,Jy) + b\, max\{\delta(Ix,F_i x), \delta(Jy,F_{i+1}y)\} \\ +c[D(Ix,F_{i+1}y) + D(Jy,F_i x)] \quad (3.5)$$

holds for all $x,y \in X$ and $\forall\, i \in N^*$, where $a,b,c \geq 0$ with $a+b+2c < 1$, whenever the right hand side of (3.4) is positive. Suppose that one of $I(X)$ or $J(X)$ is complete. If both pairs $\{F_i, I\}$ and $\{F_{i+1}, J\}$ are weakly compatible, then there exists $z \in X$ such that

$F_i t = \{It\} = \{Jt\} = \{t\}, \forall i \in N^*.$

Proof. Letting $i = 1$, we get the hypothesis of Theorem 2.1 of [1] for the maps I,J,F_1 and F_2 with the unique common fixed point t. Now, t is a unique common fixed point of I,J,F_1 and of I,J,F_2. Otherwise, if t' is a second distinct fixed point of I,J,F_1, then by inequality (3.5), we get

$$\begin{aligned} d(t,t') &= \delta(F_1 t, F_2 t') \\ &< a\, d(It,Jt') + b\, max\{\delta(It,F_1 t) + \delta(Jt',F_2 t')\} \\ &\quad +c[D(It,F_2 t') + D(Jt',F_1 t)] \\ &< (a+2c) d(t,t') \end{aligned}$$

Since $(a+2c) < 1$, hence $t = t'$.

Similarly, one can show that t is a unique common fixed point of the mappings I,J,F_2.

Again, by letting $i = 2$, we get the hypothesis of Theorem 2.1 of Ahmed [1] for maps I,J,F_2 and F_3 and consequently, they have a unique common fixed point t'. Analogously, t' is the unique common fixed point of I,J,F_2 and I,J,F_3. Thus $t = t'$. Continuing in this way, we see that t is the required point.

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