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RESEARCH ARTICLE

A COMMON RANDOM FIXED POINT THEOREM IN HILBERT SPACE

NEERAJ MALVIYA[1], GEETA AGRAWAL[2]

[1]NRI Institute of Information Science and Technology, Bhopal, India

maths.neeraj@ gmail.com

[2]Department of Mathematics, Govt. Motilal Vigyan Mahavidyalaya, Bhopal (M.P.)

agrawalgeeta1961@gmail.com

NEERAJ MALVIYA

GEETA AGRAWAL

ABSTRACT

The object of this paper is to obtain a common random fixed point theorems for a pair of non –commuting continuous random operators defined on a non empty closed subset of a separable Hilbert space.

Keywords: Separable Hilbert space, random operators, common random fixed point.

Mathematics Subject classification: 47H10, 54H25

1. INTRODUCTION

The study of random fixed points have attracted much attention, some of the recent literatures in random fixed point may be noted in [1 to 9]. In this paper we construct a sequence of measurable functions and consider its convergence to the common random fixed point of a pair of non –commuting continuous random operators defined on a non empty closed subset of a separable Hilbert space. For the purpose of obtaining the random fixed point of the random operators we have used a inequality [Theorem 2.1 from 11] and the parallelogram law.

Throughout this paper, (Ω, Σ) denotes a measurable space consisting of a set Ω and sigma algebra Σ of subsets of Ω, H stands for a separable Hilbert space, and *C* is a nonempty closed subset of *H.*

2. Preliminaries

Definition 2.1: A function $f : \Omega \to C$ is said to be measurable if $f^{-1}(B \cap C) \in \Sigma$ for every Borel subset B of H.

Definition 2.2: A function $F:\Omega\times C\to C$ is said to be a random operator if $F(.,x):\Omega\to C$ is measurable for every $x\in C$.

Definition 2.3: A measurable function $g:\Omega\to C$ is said to be a random fixed point of the random operator $F:\Omega\times C\to C$ if $F(t,g(t))=g(t)\ for\ all\ t\in\Omega$

Definition 2.4: A random operator $F:\Omega\times C\to C$ is said to be continuous if for fixed $t\in\Omega,\ F(t,.):C\to C$ is continuous.

Condition (A)—Two mappings $S,T:C\to C$ where C is a non-empty subset of a Hilbert space *H* is said to satisfy condition A if

$$\|STx-TSy\|^2\le a\max\{\|x-y\|^2,\|x-STx\|^2,\|y-TSy\|^2,\frac{[\|x-TSy\|^2+\|y-STx\|^2]}{2}\}$$
$$+b\max\{\|x-STx\|^2,\|y-TSy\|^2\}+c[\|x-TSy\|^2+\|y-STx\|^2]$$

for each $x,y\in C\ Where\ a,b,c\ge 0\ and\ a+b+c<1/4$

3 Main Result:

Theorem 3. 1—Let C be a closed non empty subset of a separable Hilbert space *H* .Let S and *T* be two non commuting continuous random operators defined on *C* such that for $t\in\Omega, S(t,.),T(t,.):C\to C$ satisfy condition A. Then *ST* and *TS* have a common unique random fixed point in C.

Proof-- We construct a sequence of functions $\{g_n\}$ as $g_0:\Omega\to C$ is arbitrary measurable function. For $t\in\Omega$, and n = 0,1,2,3---------

$$g_{2n+1}(t)=ST(t,g_{2n}(t)),\qquad g_{2n+2}(t)=TS(t,g_{2n+1}(t))------(3.1)$$

If $g_{2n}(t)=g_{2n+1}(t)=g_{2n+2}(t)$ for $t\in\Omega$, for some *n* then we see that $g_{2n}(t)$ is a random fixed point of *ST and TS*. Therefore we suppose that no two consecutive terms of sequence $\{g_n\}$ are equal.----------(3.2)

Now consider for $t\in\Omega$,

$$\|g_{2n+1}(t)-g_{2n+2}(t)\|^2=\|ST(t,g_{2n}(t))-TS(t,g_{2n+1}(t))\|^2$$
$$\le a\max\{\|g_{2n}(t)-g_{2n+1}(t)\|^2,\|g_{2n}(t)-ST(t,g_{2n}(t))\|^2,\|g_{2n+1}(t)-TS(t,g_{2n+1}(t))\|^2$$
$$,\frac{[\|g_{2n}(t)-TS(t,g_{2n+1}(t))\|^2+\|g_{2n+1}(t)-ST(t,g_{2n}(t))\|^2]}{2}\}$$
$$+\ b\max\{\|g_{2n}(t)-ST(t,g_{2n}(t))\|^2,\|g_{2n+1}(t)-TS(t,g_{2n+1}(t))\|^2\}$$
$$+c[\|g_{2n}(t)-TS(t,g_{2n+1}(t))\|^2+\|g_{2n+1}(t)-ST(t,g_{2n}(t))\|^2]$$
$$=a\max\{\|g_{2n}(t)-g_{2n+1}(t)\|^2,\|g_{2n}(t)-g_{2n+1}(t)\|^2,\|g_{2n+1}(t)-g_{2n+2}(t)\|^2$$
$$,\frac{[\|g_{2n}(t)-g_{2n+2}(t)\|^2+\|g_{2n+1}(t)-g_{2n+1}(t)\|^2]}{2}\}$$
$$+b\max\{\|g_{2n}(t)-g_{2n+1}(t)\|^2,\|g_{2n+1}(t)-g_{2n+2}(t)\|^2\}$$
$$+c[\|g_{2n}(t)-g_{2n+2}(t)\|^2+\|g_{2n+1}(t)-g_{2n+1}(t)\|^2]$$
$$=a\max\{\|g_{2n}(t)-g_{2n+1}(t)\|^2,\|g_{2n+1}(t)-g_{2n+2}(t)\|^2,\frac{[\|g_{2n}(t)-g_{2n+2}(t)\|^2]}{2}\}$$
$$+b\max\{\|g_{2n}(t)-g_{2n+1}(t)\|^2,\|g_{2n+1}(t)-g_{2n+2}(t)\|^2\}+c[\|g_{2n}(t)-g_{2n+2}(t)\|^2]$$
$$\le(a+b+c)\max\{\|g_{2n}(t)-g_{2n+1}(t)\|^2,\|g_{2n+1}(t)-g_{2n+2}(t)\|^2\ ,\|g_{2n}(t)-g_{2n+2}(t)\|^2\}$$

Case I

$\Rightarrow \|g_{2n+1}(t)-g_{2n+2}(t)\|^2 \le (a+b+c)\|g_{2n}(t)-g_{2n+1}(t)\|^2$

$\Rightarrow \|g_{2n+1}(t)-g_{2n+2}(t)\| \le (a+b+c)^{1/2}\|g_{2n}(t)-g_{2n+1}(t)\|$

$\Rightarrow \|g_{2n+1}(t)-g_{2n+2}(t)\| \le k\|g_{2n}(t)-g_{2n+1}(t)\| \, where \; k=(a+b+c)^{1/2} < 1/4 < 1$

$\Rightarrow \|g_n(t)-g_{n+1}(t)\| \le \; k\|g_{n-1}(t)-g_n(t)\|$

$\Rightarrow \|g_n(t)-g_{n+1}(t)\| \le k^n\|g_0(t)-g_1(t)\| \; for \; all \; t\in\Omega$.............(3.3)

Case II

$\Rightarrow \|g_{2n+1}(t)-g_{2n+2}(t)\|^2 \le (a+b+c)\|g_{2n+1}(t)-g_{2n+2}(t)\|^2$

$\Rightarrow (1-a-b-c)\|g_{2n+1}(t)-g_{2n+2}(t)\|^2 \le 0$

$\Rightarrow \|g_{2n+1}(t)-g_{2n+2}(t)\|^2 = 0 \; [as \; (a+b+c) < 1/4 < 1]$

$\Rightarrow g_{2n+1}(t)=g_{2n+2}(t)$

In general

$g_n(t)=g_{n+1}(t) \; for \; all \; t\in\Omega$

Which Contradicts the fact (3.8) and in this case $g_{2n}(t)$ is the fixed point of $ST \; and \; TS$.

Case III

$\Rightarrow \|g_{2n+1}(t)-g_{2n+2}(t)\|^2 \le (a+b+c)\|g_{2n}(t)-g_{2n+2}(t)\|^2$

$\Rightarrow \|g_{2n+1}(t)-g_{2n+2}(t)\|^2 \le (a+b+c)[2\|g_{2n}(t)-g_{2n+1}(t)\|^2+2\|g_{2n+1}(t)-g_{2n+2}(t)\|^2]$

(by parallelogram law $\|x+y\|^2 \le 2\|x\|^2+2\|y\|^2$)

$\Rightarrow [1-2(a+b+c)]\|g_{2n+1}(t)-g_{2n+2}(t)\|^2 \le 2(a+b+c)\|g_{2n}(t)-g_{2n+1}(t)\|^2$

$\Rightarrow \|g_{2n+1}(t)-g_{2n+2}(t)\|^2 \le \frac{2(a+b+c)}{[1-2(a+b+c)]}\|g_{2n}(t)-g_{2n+1}(t)\|^2$

In general

$\Rightarrow \|g_n(t)-g_{n+1}(t)\| \le k\|g_{n-1}(t)-g_n(t)\|$

$$\text{where } k = \left[\frac{2(a+b+c)}{1-2(a+b+c)}\right]^{1/2} < 1 \; (as \; 4(a+b+c)<1)$$

$\Rightarrow \|g_n(t)-g_{n+1}(t)\| \le \; k^n\|g_0(t)-g_1(t)\| \; for \; all \; t\in\Omega$ ---------- (3.4)

Now we shall prove that for $t\in\Omega$, $\{g_n(t)\}$ is a Cauchy sequence for the Case I. and Case III

For this for every positive integer *p* we have, for $t\in\Omega$

$$\begin{aligned}\|g_n(t)-g_{n+p}(t)\| &= \|g_n(t)-g_{n+1}(t)+g_{n+1}(t)-.........+g_{n+p-1}(t)-g_{n+p}(t)\| \\ &\le \|g_n(t)-g_{n+1}(t)\|+\|g_{n+1}(t)-g_{n+2}(t)\|+..........+\|g_{n+p-1}(t)-g_{n+p}(t)\| \\ &\le (k^n+k^{n+1}+k^{n+2}+.............+k^{n+p-1})\|g_0(t)-g_1(t)\| (by \; 3.2) \\ &= k^n(1+k+k^2+.....+k^{p-1})\|g_0(t)-g_1(t)\| \\ &\le \frac{k^n}{1-k}\|g_0(t)-g_1(t)\| \; for \; all \; t\in\Omega\end{aligned}$$

as $n\to\infty, \|g_n(t)-g_{n+p}(t)\|\to 0$,it follows that for $t\in\Omega, \{g_n(t)\}$ is a Cauchy sequence and hence is convergent in Hilbert space *H*.

For $t \in \Omega$,let $\{g_n(t)\} \to g(t)$ as $n \to \infty$,-----------(3.5)

Since *C* is closed, g is a function from *C to C*.

Existence of random fixed point. For $t \in \Omega$,

$$\|g(t) - TS(t, g(t))\|^2 = \|g(t) - g_{2n+1}(t) + g_{2n+1}(t) - TS(t, g(t))\|^2$$

$$\leq 2\|g(t) - g_{2n+1}(t)\|^2 + 2\|g_{2n+1}(t) - TS(t, g(t))\|^2$$

$$[by\ parallelogram\ law \|x+y\|^2 \leq 2\|x\|^2 + 2\|y\|^2]$$

$$=2\|g(t) - g_{2n+1}(t)\|^2 + 2\|ST(t, g_{2n}(t)) - TS(t, g(t))\|^2$$

$$\leq 2\|g(t) - g_{2n+1}(t)\|^2 + 2a\max\{\|g_{2n}(t) - g(t)\|^2, \|g_{2n}(t) - ST(t, g_{2n}(t))\|^2$$

$$, \|g(t) - TS(t, g(t))\|^2, \frac{[\|g_{2n}(t) - TS(t, g(t))\|^2 + \|g(t) - ST(t, g_{2n}(t))\|^2]}{2}\}$$

$$+ 2b\max\{\|g_{2n}(t) - ST(t, g_{2n}(t))\|^2, \|g(t) - TS(t, g(t))\|^2\}$$

$$+ 2c[\|g_{2n}(t) - TS(t, g(t))\|^2 + \|g(t) - ST(t, g_{2n}(t))\|^2]$$

$$= 2\|g(t) - g_{2n+1}(t)\|^2 + 2a\max\{\|g_{2n}(t) - g(t)\|^2, \|g_{2n}(t) - g_{2n+1}(t)\|^2$$

$$, \|g(t) - TS(t, g(t))\|^2, \frac{[\|g_{2n}(t) - TS(t, g(t))\|^2 + \|g(t) - g_{2n+1}(t)\|^2]}{2}\}$$

$$+ 2b\max\{\|g_{2n}(t) - g_{2n+1}(t)\|^2, \|g(t) - TS(t, g(t))\|^2\}$$

$$+ 2c[\|g_{2n}(t) - TS(t, g(t))\|^2 + \|g(t) - g_{2n+1}(t)\|^2]$$

As $\{g_{2n}(t)\}, \{g_{2n+1}(t)\}$ are subsequences of $\{g_n(t)\}$,

as $n \to \infty, \{g_{2n}(t)\} \to g(t), \{g_{2n+1}(t)\} \to g(t)$ for all $t \in \Omega$,

Therefore

$$\|g(t) - TS(t, g(t))\|^2 \leq 2\|g(t) - g(t)\|^2 + 2a\max\{\|g(t) - g(t)\|^2, \|g(t) - g(t)\|^2$$

$$, \|g(t) - TS(t, g(t))\|^2, \frac{[\|g(t) - TS(t, g(t))\|^2 + \|g(t) - g(t)\|^2]}{2}\}$$

$$+2b\max\{\|g(t) - g(t)\|^2, \|g(t) - TS(t, g(t))\|^2\} + 2c[\|g(t) - TS(t, g(t))\|^2 + \|g(t) - g(t)\|^2]$$

.

$$\Rightarrow \|g(t) - TS(t, g(t))\|^2 \leq 2(a+b+c)\|g(t) - TS(t, g(t))\|^2$$

$$\Rightarrow [1 - 2(a+b+c)]\|g(t) - TS(t, g(t))\|^2 \leq 0$$

$$\Rightarrow \|g(t) - TS(t, g(t))\|^2 = 0\ [as\ (a+b+c) < 1/4]$$

$$\Rightarrow TS(t, g(t)) = g(t) \quad \forall t \in \Omega - - - - - (3.6)$$

In an exactly similar way we can prove that for all $t \in \Omega$, $ST(t, g(t)) = g(t) - - - - - - - - - - (3.7)$

Again if $A:\Omega\times C\rightarrow C$ is a continuous random operator on a non-empty subset C of a separable Hilbert space H, then for any measurable function $f:\Omega\rightarrow C$,the function $h(t)=A(t,f(t))$ is also measurable[10]---------------------------------(3.8)

It follows from the construction of $\{g_n\}$ by (3.1) and the above consideration that $\{g_n\}$ is a sequence of measurable functions. From (3.8) it follows that *g* is also a measurable function. This fact along with (3.6 & 3.7) shows that $g:\Omega\rightarrow C$ is a common random fixed point of *ST &TS*.

Uniqueness—

Let $h:\Omega\rightarrow C$ *be another random fixed point common to ST & TS, that is for* $t\in\Omega,$

$$ST(t,h(t))=h(t)$$

$$TS(t,h(t))=h(t)$$

Then For $t\in\Omega$

$$\|g(t)-h(t)\|^2=\|ST(t,g(t))-TS(t,h(t))\|^2$$

$$\leq a\max\{\|g(t)-h(t)\|^2,\|g(t)-ST(t,g(t))\|^2,\|h(t)-TS(t,h(t))\|^2,\frac{[\|g(t)-TS(t,h(t))\|^2+\|h(t)-ST(t,g(t))\|^2]}{2}\}$$

$$+b\max\{\|g(t)-ST(t,g(t))\|^2,\|h(t)-TS(t,h(t))\|^2\}$$

$$+c[\|g(t)-TS(t,h(t))\|^2+\|h(t)-ST(t,g(t))\|^2]$$

$$=a\max\{\|g(t)-h(t)\|^2,\|g(t)-g(t)\|^2,\|h(t)-h(t)\|^2,\frac{[\|g(t)-h(t)\|^2+\|h(t)-g(t)\|^2]}{2}\}$$

$$+b\max\{\|g(t)-g(t)\|^2,\|h(t)-h(t)\|^2\}+c[\|g(t)-h(t)\|^2+\|h(t)-g(t)\|^2]$$

$$\Rightarrow(1\text{- }a\text{ - }2c)\|g(t)-h(t)\|^2\leq 0$$

$$\Rightarrow\|g(t)-h(t)\|^2=0\ (as\ a+b+c<\frac{1}{4}\ and\ hence\ a+2c<1)$$

$$\Rightarrow g(t)=h(t)\ for\ all\ t\in\Omega$$

This completes the proof of the theorem 3.1

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