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RESEARCH ARTICLE

OPTIMAL CONVEX COMBINATION BOUNDS OF THE CLASSICAL HERANLAN AND QUADRATIC MEANS FOR NEUMAN-SÁNDOR MEAN

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ABSTRACT

In this paper, we present the least value a and the greatest value b such that the double inequality

$$aH_e(a,b)+(1-a)Q(a,b)<M(a,b)<bH_e(a,b)+(1-b)Q(a,b)$$

holds for all $a,b>0$ with $a\neq b$. Here $H_e(a,b)$, $Q(a,b)$ and $M(a,b)$ denote the classical Heronian, quadratic and Neuman-Sándor means of two positive numbers a and b, respectively.

Keywords: Inequality, Neuman-Sándor mean, Heronian mean, quadratic mean.

1. INTRODUCTION

For $a,b>0$ with $a\neq b$, the Neuman-Sándor mean $M(a,b)$ [1] was defined by

$$M(a,b)=\frac{a-b}{2\sinh^{-1}[(a-b)/(a+b)]}, \tag{1.1}$$

where $\sinh^{-1}x=\log(x+\sqrt{1+x^2})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the subject of intensive research. In particular, many remarkable inequalities for $M(a,b)$ can be found in the literature [1, 2]. Let $H(a,b)=(2ab)/(a+b)$, $G(a,b)=\sqrt{ab}$, $L(a,b)=(a-b)/(\log a-\log b)$, $N(a,b)=\left[(\sqrt{a}+\sqrt{b})/2\right]^2$, $P(a,b)=(a-b)\big/(4\tan^{-1}\sqrt{a/b}-\pi)$, $H_e(a,b)=1/3\cdot(a+\sqrt{ab}+b)$, $I(a,b)=1/e\cdot\left(b^b/a^a\right)^{1/(b-a)}$, $T(a,b)=(a-b)\big/[2\tan^{-1}(a-b)/(a+b)]$, $A(a,b)=(a+b)/2$, $\overline{C}(a,b)=2/3\cdot(a^2+ab+b^2)\big/(a+b)$, $Q(a,b)=\sqrt{(a^2+b^2)/2}$ and $C(a,b)=(a^2+b^2)\big/(a+b)$ be the harmonic, geometric, logarithmic, square-root, first Seiffert, classical

Heronian, identric, second Seiffert, arithmetic, centroidal, quadratic and contra-harmonic mean of a and b, respectively. Then

$$\begin{aligned}&\min\{a,b\} < H(a,b) < G(a,b) < L(a,b) < N(a,b) < P(a,b) < H_e(a,b) < I(a,b)\\ &< A(a,b) < M(a,b) < T(a,b) < \overline{C}(a,b) < Q(a,b) < C(a,b) < \max\{a,b\}\end{aligned} \tag{1.2}$$

hold for all $a,b > 0$ with $a \neq b$.

In [3], Neuman proved that the double inequalities

$$\alpha Q(a,b) + (1-\alpha)A(a,b) < M(a,b) < \beta Q(a,b) + (1-\beta)A(a,b) \tag{1.3}$$

and

$$\lambda C(a,b) + (1-\lambda)A(a,b) < M(a,b) < \mu C(a,b) + (1-\mu)A(a,b) \tag{1.4}$$

hold for all $a,b > 0$ with $a \neq b$ if and only if

$\alpha \le [1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})] = 0.3249\ldots$, $\beta \ge$

$1/3$, $\lambda \le [1-\log(1+\sqrt{2})]/\log(1+\sqrt{2})$ and $\mu \ge 1/6$.

In [4], Li etc showed that the double inequality

$$L_{p_0}(a,b) < M(a,b) < L_2(a,b) \tag{1.5}$$

holds for all $a,b > 0$ with $a \neq b$, where

$L_p(a,b) = [(a^{p+1} - b^{p+1})/(p+1)(a-b)]^{1/p}$ $(p \neq -1, 0)$, $L_0(a,b)$

$= 1/e[(a^a)/b^b]^{1/(a-b)}$ and $L_{-1}(a,b) = (a-b)/(\log a - \log b)$ is the p-th generalized logarithmic mean of a and b, and $p_0 = 1.843\ldots$ is the unique solution of the equation $(p+1)^{1/p} = \log(1+\sqrt{2})$.

In [5], Chu etc proved that the double inequalities

$$\alpha_1 L(a,b) + (1-\alpha_1)Q(a,b) < M(a,b) < \beta_1 L(a,b) + (1-\beta_1)Q(a,b) \tag{1.6}$$

and

$$\alpha_2 L(a,b) + (1-\alpha_2)C(a,b) < M(a,b) < \beta_2 L(a,b) + (1-\beta_2)C(a,b) \tag{1.7}$$

hold for all $a,b > 0$ with $a \neq b$ if and only if

$\alpha_1 \ge 2/5$, $\beta_1 \le 1 - 1/[\sqrt{2}\log(1+\sqrt{2})] = 0.1977\ldots$, $\alpha_2 \ge 5/8$

and $\beta_2 \le 1 - 1/[2\log(1+\sqrt{2})] = 0.4327\ldots$.

The main purpose of this paper is to found the least value α and the greatest value β such that the double inequality

$$\alpha H_e(a,b) + (1-\alpha)Q(a,b) < M(a,b) < \beta H_e(a,b) + (1-\beta)Q(a,b)$$

holds for all $a,b > 0$ with $a \neq b$.

2. MAIN RESULTS

THEOREM 2.1. The double inequality

$$\alpha H_e(a,b) + (1-\alpha)Q(a,b) < M(a,b) < \beta H_e(a,b) + (1-\beta)Q(a,b)$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \ge 1/2$ and

$\beta \le \left[3(3+\sqrt{2})\right]/7 \cdot \left[1 - 1/\left(\sqrt{2}\log(1+\sqrt{2})\right)\right]$

$= 0.37405\ldots$.

Proof. Without loss of generality, we assume that $a > b > 0$. Let $x = (a-b)/(a+b) \in (0,1)$, $\lambda =$

$\left[3(3+\sqrt{2})\right]/7\cdot\left[1-1/\left(\sqrt{2}\log(1+\sqrt{2})\right)\right]$ and $p\in\{1/2,\lambda\}$. Then

$$\frac{H_e(a,b)}{A(a,b)}=\frac{1}{3}(2+\sqrt{1-x^2}),\quad \frac{M(a,b)}{A(a,b)}=\frac{x}{\sinh^{-1}x},\quad \frac{Q(a,b)}{A(a,b)}=\sqrt{1+x^2}. \tag{2.1}$$

Firstly we prove that

$$\frac{1}{2}[H_e(a,b)+Q(a,b)]<M(a,b) \tag{2.2}$$

and

$$M(a,b)<\lambda H_e(a,b)+(1-\lambda)Q(a,b). \tag{2.3}$$

From (2.1) we have

$$\frac{pH_e(a,b)+(1-p)Q(a,b)-M(a,b)}{A(a,b)}=\frac{p(2+\sqrt{1-x^2})+3(1-p)\sqrt{1+x^2}}{3\log(x+\sqrt{1+x^2})}D_p(x), \tag{2.4}$$

where

$$D_p(x)=\log(x+\sqrt{1+x^2})-\frac{3x}{p(2+\sqrt{1-x^2})+3(1-p)\sqrt{1+x^2}}. \tag{2.5}$$

Equation (2.5) leads to

$$\lim_{x\to 0^+}D_p(x)=0, \tag{2.6}$$

$$\lim_{x\to 1^-}D_p(x)=\log(1+\sqrt{2})-\frac{3}{2p+3(1-p)\sqrt{2}}, \tag{2.7}$$

and

$$D_p'(x)=\frac{1}{\left[p(2+\sqrt{1-x^2})+3(1-p)\sqrt{1+x^2}\right]^2}F_p(x), \tag{2.8}$$

where

$$\begin{aligned}F_p(x)=&\frac{(8p^2-18p+9)x^2+p(14p-9)}{\sqrt{1+x^2}}+\frac{4p^2(1-x^2)}{\sqrt{1-x^4}}\\&+\frac{6p(p-1)x^2+3p(1-2p)}{\sqrt{1-x^2}}+6p(1-2p).\end{aligned} \tag{2.9}$$

Let $x=\sqrt{t}$, $t\in(0,1)$, then

$$\begin{aligned}F_p(x)=&\frac{(8p^2-18p+9)t+p(14p-9)}{\sqrt{1+t}}+\frac{4p^2(1-t)}{\sqrt{1-t^2}}\\&+\frac{6p(p-1)t+3p(1-2p)}{\sqrt{1-t}}+6p(1-2p)\\=&G_p(t).\end{aligned} \tag{2.10}$$

Computations for $G_p(t)$ yield

$$\lim_{t\to 0^+}G_p(t)=0, \tag{2.11}$$

$$\lim_{t\to 1^-} G_p(t) = -\infty, \tag{2.12}$$

and

$$G_p'(t) = \frac{1}{2(1-t^2)^{3/2}} L_p(t), \tag{2.13}$$

where

$$\begin{aligned} L_p(t) = {} & [(8p^2 - 18p + 9)t + (2p^2 - 27p + 18)](1-t)^{3/2} \\ & + [6p(1-p)t + 3p(2p-3)](1+t)^{3/2} + 8p^2(t-1). \end{aligned} \tag{2.14}$$

Now we distinguish between two cases:

Case 1. $p = 1/2$. (2.14) leads to

$$L_{1/2}(t) = (2t+5)(1-t)^{3/2} - 3(1-\frac{t}{2})(1+t)^{3/2} + 2(t-1). \tag{2.15}$$

The Taylor series of the functions $\sqrt{1-t}$ and $\sqrt{1+t}$ are

$$\sqrt{1-t} = 1 - \frac{1}{2}t - \sum_{n=2}^{\infty} \frac{(2n-3)!!}{(2n)!!} t^n, \; t \in (-1,1) \tag{2.16}$$

and

$$\sqrt{1+t} = 1 + \frac{1}{2}t + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{(2n-3)!!}{(2n)!!} t^n, \; t \in (-1,1), \tag{2.17}$$

respectively, so inequalities

$$\sqrt{1-t} < 1 - \frac{1}{2}t - \frac{1}{8}t^2 \tag{2.18}$$

and

$$\sqrt{1+t} > 1 + \frac{1}{2}t - \frac{1}{8}t^2 \tag{2.19}$$

hold for all $t \in (0,1)$. Making use of the inequalities (2.18) and (2.19) for (2.15) cause the conclusion that

$$\begin{aligned} L_{1/2}(t) & < (2t+5)(1-t)(1-\frac{1}{2}t - \frac{1}{8}t^2) - 3(1-\frac{t}{2})(1+t)(1+\frac{1}{2}t - \frac{1}{8}t^2) + 2(t-1) \\ & = \frac{t}{16}(t^3 + 37t^2 - 104) \\ & < 0 \end{aligned} \tag{2.20}$$

for all $t \in (0,1)$. (2.13) and (2.20) imply that $G_{1/2}'(t) < 0$ for $t \in (0,1)$, hence $G_{1/2}(t)$ is strictly decreasing in $(0,1)$. It follows from (2.8), (2.10) and (2.11) together with the monotonicity of $G_{1/2}(t)$ that $D_{1/2}'(x) < 0$ for $x \in (0,1)$, thus $D_{1/2}(x)$ is strictly decreasing in $(0,1)$. Therefore inequality (2.2) follows from (2.4) and (2.6) together with the monotonicity of $D_{1/2}(x)$.

Case 2. $p = \lambda$. (2.14) result in

$$\begin{aligned} L_\lambda(t) = {} & [(8\lambda^2 - 18\lambda + 9)t + (2\lambda^2 - 27\lambda + 18)](1-t)^{3/2} \\ & + [6\lambda(1-\lambda)t + 3\lambda(2\lambda - 3)](1+t)^{3/2} + 8\lambda^2(t-1). \end{aligned} \tag{2.21}$$

Simple calculations yield

$$\lim_{t\to 0^+} L_\lambda(t)(t) = 18(1-2\lambda) > 0, \tag{2.22}$$

$$\lim_{t\to 1^-} L_\lambda(t)(t) = -6\sqrt{2}\lambda < 0, \tag{2.23}$$

$$\begin{aligned} L_\lambda'(t) = {} & \left[5\left(4\lambda^2 - 9\lambda + \frac{9}{2}\right)t + \left(5\lambda^2 + \frac{45}{2}\lambda - 18\right)\right]\sqrt{1-t} \\ & + \left[15\lambda(1-\lambda)t + 3\lambda\left(\lambda - \frac{5}{2}\right)\right]\sqrt{1+t} + 8\lambda^2, \end{aligned} \tag{2.24}$$

$$\lim_{t\to 0^+} L_\lambda'(t) = 16\lambda^2 + 15\lambda - 18 < 0, \tag{2.25}$$

$$\lim_{t\to 1^-} L_\lambda'(t) = \frac{\lambda}{2}[15\sqrt{2} + 8(2-3\sqrt{2})\lambda] > 0, \tag{2.26}$$

$$L_\lambda''(t) = \frac{90\lambda(1-\lambda)t + 9\lambda(5-6\lambda)}{4\sqrt{1+t}} + \frac{15(8\lambda^2 - 18\lambda + 9)t - 9(10\lambda^2 - 15\lambda + 6)}{4\sqrt{1-t}}, \tag{2.27}$$

$$\lim_{t\to 0^+} L_\lambda''(t) = -\frac{9}{2}(3-4\lambda)(1-2\lambda) < 0, \tag{2.28}$$

$$\lim_{t\to 1^-} L_\lambda''(t) = +\infty, \tag{2.29}$$

and

$$L_\lambda'''(t) = \frac{90\lambda(1-\lambda)t + 9\lambda(15-14\lambda)}{8(1+t)^{3/2}} + \frac{3(50\lambda^2 - 135\lambda + 72) - 15(8\lambda^2 - 18\lambda + 9)t}{8(1-t)^{3/2}} > 0. \tag{2.30}$$

for all $t \in (0,1)$. From (2.30) we clearly see that $L_\lambda''(t)$ is strictly increasing in $(0,1)$.

It follows from (2.28) and (2.29) together with the monotonicity of $L_\lambda''(t)$ that there exists $t_0 \in (0,1)$ such that $L_\lambda''(t) < 0$ for $t \in (0,t_0)$ and $L_\lambda''(t) > 0$ for $t \in (t_0,1)$, so $L_\lambda'(t)$ is strictly decreasing in $(0,t_0)$ and strictly increasing in $(t_0,1)$. From (2.25) and (2.26) together with the monotonicity of $L_\lambda'(t)$ we know that there exists $t_1 \in (t_0,1)$ such that $L_\lambda'(t) < 0$ for $t \in (0,t_1)$ and $L_\lambda'(t) > 0$ for $t \in (t_1,1)$, hence $L_\lambda(t)$ is strictly decreasing in $(0,t_1)$ and strictly increasing in $(t_1,1)$. From (2.22), (2.23) and (2.13) together with the monotonicity of $L_\lambda(t)$ we affirm that there exists $t_2 \in (0,t_1)$ such that $G_\lambda'(t) > 0$ for $t \in (0,t_2)$ and $G_\lambda'(t) < 0$ for $t \in (t_2,1)$, thus $G_\lambda(t)$ is strictly increasing in $(0,t_2)$ and strictly decreasing in $(t_2,1)$. (2.11) and (2.12) together with the monotonicity of $G_\lambda(t)$ cause the conclusion that there exists $t_3 \in (t_2,1)$ such that $G_\lambda(t) > 0$ for $t \in (0,t_3)$ and $G_\lambda(t) < 0$ for $t \in (t_3,1)$, this fact together with (2.8) and (2.10) imply that $D_\lambda'(x) > 0$ for $x \in (0,x_0)$ and $D_\lambda'(x) < 0$ for $x \in (x_0,1)$, where $x_0 = \sqrt{t_3}$, thereby $D_\lambda(x)$ is strictly increasing in $(0,x_0)$ and strictly decreasing in $(x_0,1)$.

Notice that (2.7) becomes

$$\lim_{x\to 1^-} D_\lambda(x) = 0. \tag{2.31}$$

Therefore inequality (2.3) follows from (2.4), (2.6) and (2.31) together with the monotonicity of $D_\lambda(x)$.

Finally we prove that $\frac{1}{2}[H_e(a,b)+Q(a,b)]$ is the best possible lower convex combination bound and $\lambda H_e(a,b)+(1-\lambda)Q(a,b)$ is the best possible upper convex combination bound of the classical Heronian and quadratic means for the Nueman-Sándor mean.

(2.1) leads to

$$\frac{Q(a,b)-M(a,b)}{Q(a,b)-H_e(a,b)} = \frac{\sqrt{1+x^2}-\dfrac{x}{\sinh^{-1}x}}{\sqrt{1+x^2}-\dfrac{1}{3}(2+\sqrt{1-x^2})} = B(x). \tag{2.32}$$

From (2.32) one has

$$\lim_{x\to 0^+} B(x) = \frac{1}{2}, \tag{2.33}$$

and

$$\lim_{x\to 1^-} B(x) = \lambda. \tag{2.34}$$

If $\alpha < 1/2$, then (2.32) and (2.33) lead to the conclusion that there exists $0<\delta_1<1$ such that $\alpha H_e(a,b)+(1-\alpha)Q(a,b) > M(a,b)$ for all $a,b>0$ with $(a-b)/(a+b)\in(0,\delta_1)$.

If $\beta > \lambda$, then (2.32) and (2.34) lead to the conclusion that there exists $0<\delta_2<1$ such that $M(a,b) > \beta H_e(a,b)+(1-\beta)Q(a,b)$ for all $a,b>0$ with $(a-b)/(a+b)\in(1-\delta_2,1)$.

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