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RESEARCH ARTICLE

A Peer Reviewed International Research Journal



CHARACTERIZATIONS OF FINITE BOOLEAN LATTICES

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ABSTRACT

This paper deals with characterizations of finite Boolean lattices. In this paper, we discuss some lemmas and some important theorems such as “Let L be a finite lattice if and only if each congruence relation θ on L can be associated a congruence relation K on G_L , where xKy if and only if $x\theta y$, then L is a BOOLEAN lattice”.

Key Words: Binary relation, HASSE diagram, sublattice, distributive lattice, congruence relation, finite Boolean lattice.

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1.INTRODUCTION

A binary, reflexive, symmetric and transitive relation K on an undirected graph $G=(V,E)$ is called a congruence relation on G , if xKy only when $\Gamma xK\Gamma y$. The purpose of this brief paper is to show that a finite lattice is BOOLEAN iff each lattice congruence θ on L is a congruence relation K on the HASSE diagram graph G_L of L .

The definitions of terms of graph and lattice theories not given here can be found in monographs [4] and [5] of ORE and SZA'SZ respectively, to which the reader is referred.

Suppose L is a finite BOOLEAN lattice and $G_L=(V_L, E_L)$ the HASSE diagram graph associated with L . Consider G_L an undirected graph, where $V_L=L$ and $(a, b)\in E_L$ whenever, a covers b or b covers a in L , $a, b\in L$.

As well known, G_L is a graph without loops, multiple edges and isolated vertices. If $x\in V_L$, then Γx means the set of all vertices y for which $(x,y)\in E_L$. Suppose R is a binary, symmetric and reflexive relation on the vertex set V_L . ZELINKA [6] calls R a tolerance relation on G_L , if R satisfies the condition: xRy only if $\Gamma xR\Gamma y$. That is $\exists, \forall u\in \Gamma x$, an element $z\in \Gamma y$ such that uRz and vice versa, $\forall w\in \Gamma y$ there is an element $t\in \Gamma x$ such that wRt . If K is a tolerance relation on G_L and if it is also transitive. That is xKw and $wKz\Rightarrow xKz$, K is called a congruence relation on G_L . In [1] CHAJDA and

ZELINKA consider tolerance relations defined by means of meet and join operations on lattices and in [2] and [3] the reader can find some properties of congruence relations on graphs.

2. BASIC PART

In a finite BOOLEAN lattice L each congruence relation θ on L is uniquely determined by its kernel I_θ , where $I_\theta = (a) = \{x : x \leq a, x \in L\}$ according to the finity of L . Thus $x\theta y \Leftrightarrow x \vee a = y \vee a$, hence each θ on L is determined by a specified translation $s_a(x) = a \vee x$ as follows:

$x\theta y$ iff $s_a(x) = s_a(y)$. Conversely one can easily shown that, each translation φ on L is determined by a congruence relation θ the kernel of which is (b) whenever $dJ_\varphi = (b)$. Thus we can write

Lemma 2.1: In a finite BOOLEAN lattice L each congruence relation θ , $I_\theta = (a)$, is determined by the translation $s_a: x\theta y$ iff $s_a(x) = s_a(y)$, and conversely, each translation φ on L , $dJ_\varphi = (b)$ is determined by the congruence relation $\theta: I_\theta = (b)$, $\varphi(x) = x \vee b$ and $\varphi(x) = \varphi(y)$ iff $x\theta y$.

Lemma 2.2: Suppose L is a finite lattice and φ a translation on L . Then φ determines a congruence relation K on G_L as follows: xKy iff $\varphi(x) = \varphi(y)$.

Proof: The relation K is evidently reflexive, symmetric and transitive. So it remains to show that xKy only when $\Gamma x K \Gamma y$. Assume $x \neq y$; the case $x = y$ is trivial.

Suppose $z \in \Gamma x$ and assume z covers x i.e.; $z > x$. Then $\varphi(z) \geq \varphi(x)$ and thus we have two case to consider: (i) $\varphi(x) = \varphi(z)$ and (ii) $\varphi(z) > \varphi(x)$. The case $x > z$ can be proved analogously.

(i) $\varphi(z) = \varphi(x) = \varphi(y)$, $\varphi(x \vee y) = \varphi(x) \vee \varphi(y) = \varphi(y)$ and suppose that $x \vee y > y$. Thus $\forall w \in [y, x \vee y] = \{v: y \leq v \leq x \vee y, v \in L\}$, $\varphi(w) = \varphi(y)$. Since L is finite, there is also an element $w \in [y, x \vee y]$ such that $w > y$, and so $w \in \Gamma y$ such that zKw . If $y = x \vee y$, then $x < y$ and $w \in [x, y]$, $w < y$ such that $\varphi(w) = \varphi(y) = \varphi(z)$. Thus wKz . The part vice versa can be proved similarly.

(ii) $\varphi(z) > \varphi(x)$. Now, $\varphi(x) \vee z > \varphi(x)$, as in the other case $z \leq \varphi(x)$, from which it follows the contradiction: $\varphi(z) \leq \varphi(\varphi(x)) = \varphi(x)$. Since L is distributive and $z > x$, $\varphi(x) \vee z > \varphi(x)$. On the other hand, $\varphi(\varphi(x) \vee z) = \varphi(x) \vee \varphi(z) = \varphi(z) = \varphi(x \vee z)$ and thus $\varphi(z) > \varphi(x)$. Now, we must show that there is an element $w > y$ in L such that $\varphi(w) = \varphi(z)$. If $y = \varphi(x)$, the case is trivial: $w = \varphi(z)$. Therefore we assume that $y \neq \varphi(x)$, hence $\varphi(x) \in [y, \varphi(z)]$. The convex sublattice $[y, \varphi(z)]$ of L is complemented and hence there is an element $u \in [y, \varphi(z)]$ such that $u \vee \varphi(x) = \varphi(z)$ and $u \wedge \varphi(x) = y$. Since $\varphi(z) > \varphi(x)$ and $[y, \varphi(z)]$ is distributive $u > y$ and since $u \in [y, \varphi(z)]$, $u \leq \varphi(u) \leq \varphi(z)$. Suppose $\varphi(u) < \varphi(z)$. From $u \wedge \varphi(x) = y$ we have $\varphi(u \wedge \varphi(x)) = \varphi(y) = \varphi(u) \wedge \varphi(x)$ and hence $\varphi(u) \geq \varphi(y)$. Now, $\varphi(z) > \varphi(y)$ and $\varphi(z) > \varphi(u) \geq \varphi(x) = \varphi(y)$; consequently $\varphi(u) = \varphi(y)$. Then $\varphi(u \vee \varphi(x)) = \varphi(u) \vee \varphi(x) = \varphi(x) \neq \varphi(z)$, which is a contradiction. Therefore $\varphi(u) = \varphi(z)$.

So, if xKy then $\forall z \in \Gamma x$ there is an element $u \in \Gamma y$ such that zKu . The vice versa part can be prove similarly. Thus the proof is complete.

Lemma 2.3: Let L be a finite lattice. If each congruence relation θ on L can be associated a congruence relation K on G_L , where xKy iff $x\theta y$, then L is a BOOLEAN lattice.

Proof: First we show that L must be distributive where after the complementedness of L can be proved.

Consider the lattices L_1 and L_2 of Fig. 1. In the lattice L_1 , $\{b, 0\}$, $\{a, 1\}$ and $\{c\}$ are congruence classes module θ_{a1} . Thus if $bK0$, the relation $\Gamma b K \Gamma 0$ does not hold, since there is no $u \in \Gamma b$ such that $u\theta_{a1}c$, $c \in \Gamma 0$. Furthermore, in the lattice L_2 , $\{a, 1\}$, $\{b\}$ and $\{c, 0\}$ are the congruence classes module θ_{a1} . When $0Kc$, the relation $\Gamma 0 K \Gamma c$ does not hold since $b \in \Gamma 0$ and there are no element $u \in \Gamma c$ such that uKb . Thus L can not contain L_1 or L_2 as a sublattice from which the distributivity of L follows:

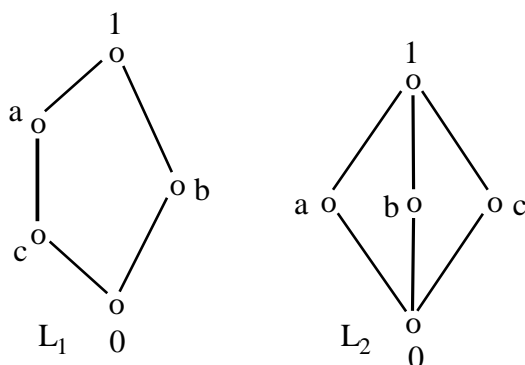


Fig. 2.1

In a distributive lattice L each ideal (a) determines a congruence relation $\theta[(a)]$, where $0 \in L$, $i\theta[(a)] 0$ iff $i \in (a)$. In particular, \forall dual atom $z \in L$, i.e., $z < 1$, there is at least one element $k \in L$ such that $k > 0$, $k \notin (z)$. Indeed, in all other cases all the elements in $\Gamma 0 \in (z)$ and since $1 \in \Gamma z$ and $1\theta[(z)]z$, the relation yKx iff $x\theta y$ would not be a congruence relation on G_L . Thus for each dual atom z of L , k is the complement of z in L .

The complement is unique, since L is distributive. We can show as above that, if $w < z < 1$, there is an element $k > 0$ such that $k \notin (w)$ and $k \wedge w = 0$. Since k is the unique complement of an element $y < 1$, $w \vee k = r < 1$. If r' denotes the complement of r in L , $w \wedge r' = (r \wedge w) \wedge r' = 0$ and so $(k \vee r')$ is the complement of w in L . It is unique according to the distributivity of L . Since L is finite, we can construct by this process a complement for each element of L . Thus the proof is complete.

By combining the results of Lemmas 1, 2 and 3 we have our characterization.

Theorem 2.1: A finite lattice L is a BOOLEAN lattice iff for each congruence relation θ on L the relation $K, xKy \Leftrightarrow x\theta y$, is a congruence relation on the graph G_L .

We shall finally make some remarks on the congruence relations K on the graph G_L when L is a finite BOOLEAN lattice. As well known, the congruence relations K on G_L form a lattice $K(G_L)$ w.r.to the meet and join operations defined as follows: If $K, H \in K(G_L)$, then $x(K \wedge H)y$ iff xKy and xHy and $x(K \vee H)y$ iff there is in V_L a sequence $u_1, u_2, u_3, \dots, u_n$ of elements, $x = u_1$ and $y = u_n$ such that for each value of i at least one of the relations $u_i H u_{i+1}$, $u_i K u_{i+1}$ holds, $i = 1, 2, \dots, n-1$.

The lattice $K(G_L)$ need not be even modular one can see by means of the lattice L of Fig. 2. The only non-trivial lattice congruence on L are θ_{a0} and θ_{b0} . The congruence relations K on G_L that are not simultaneously lattice congruence on L are K_1, K_2 and K_3 . The classes modulo K_1 are $\{1\}$ and $\{0\}$. Those modulo K_2 : $\{1, 0\}$ and $\{a, b\}$ and those modulo K_3 : $\{1, 0\}$, $\{a\}$ and $\{b\}$. The lattice $K(G_L)$ is given in Fig. 2.

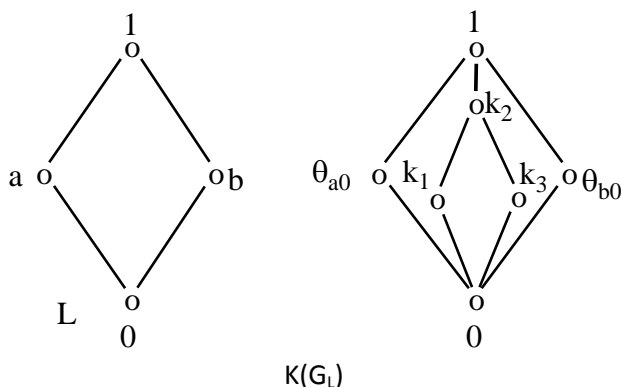


Fig. 2.2

According to the definitions of join and meet operations in $K(G_L)$ and in the lattice $\theta(L)$ of all lattice congruences on L , we can write

3. CONCLUSION

If L is a finite BOOLEAN lattice, $\theta(L)$ is a BOOLEAN sublattice of the lattice $K(G_L)$.

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