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RESEARCH ARTICLE

CHARACTERIZATIONS OF FINITE BOOLEAN LATTICES

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ABSTRACT

This paper deals with characterizations of finite Boolean lattices. In this paper, we discuss some lemmas and some important theorems such as "Let L be a finite lattice if and only if each congruence relation θ on L can be associated a congruence relation K on G_L, where xKy if and only if $x\theta y$, then L is a BOOLEAN lattice".

Key Words: Binary relation, HASSE diagram, sublattice, distributive lattice, congruence relation, finite Boolean lattice.

1.INTRODUCTION

A binary, reflexive, symmetric and transitive relation K on an undirected graph G=(V,E) is called a congruence relation on G, if xKy only when $\Gamma x K \Gamma y$. The purpose of this brief paper is to show that a finite lattice is BOOLEAN iff each lattice congruence θ on L is a congruence relation K on the H-ASSE diagram graph G_L of L.

The definitions of terms of graph and lattice theories not given here can be found in monographs [4] and [5] of ORE and SZA'SZ respectively, to which the reader is referred.

Suppose L is a finite BOOLEANlattice and $G_L=(V_L, E_L)$ the HASSE diagram graph associated with L. Consider G_L an undirected graph, where $V_L=L$ and $(a, b)\in E_L$ whenever, a covers b or b covers a in L, a, $b\in L$.

As well known, G_L is a graph without loops, multiple edges and isolated vertices. If $x\in V_L$, then Γx means the set of all vertices y for which $(x,y)\in E_L$. Suppose R is a binary, symmetric and reflexive relation on the vertex set V_L. ZELINKA [6] calls R a tolerance relation on G_L, if R satisfies the condition: xRy only if $\Gamma x R \Gamma y$. That is $\exists$, $\forall$ $u\in\Gamma x$, an element $z\in\Gamma y$ such that uRz and vice versa, $\forall$ $w\in\Gamma y$ there is an element $t\in\Gamma x$ such that wRt. If K is a tolerance relation on G_L and if it is also transitive. That is xKw and $wKz\Rightarrow xKz$, K is called a congruence relation on G_L. In [1] CHAJDA and

ZELINKA consider tolerance relations defined by means of meet and join operations on lattices and in [2] and [3] the reader can find some properties of congruence relations on graphs.

2.BASIC PART

In a finite BOOLEAN lattice L each congruence relation θ on L is uniquely determined by its kernel I_θ, where $I_\theta = (a] = \{ x : x \leq a, x \in L\}$ according to the finity of L. Thus $x\theta y \Leftrightarrow x \vee a = y \vee a$, hence each θ on L is determined by a specified translation $s_a(x)=a\vee x$ as follows:

$x\theta y$ iff $s_a(x)=s_a(y)$. Conversely one can easily shown that, each translation φ on L is determined by a congruence relation θ the kernel of which is (b] whenever dJ_φ= [b). Thus we can write

Lemma 2.1: In a finite BOOLEANlattice L each congruence relation θ, I_θ=(a], is determined by the translation s_a:$x\theta y$ iff $s_a(x) = s_a(y)$, and conversely, each translation φ on L, dJ_φ= [b) is determined by the congruence relation θ:I_θ=(b], $\varphi(x) = x\vee b$ and $\varphi(x) = \varphi(y)$ iff $x\theta y$.

Lemma 2.2: Suppose L is a finite lattice and φ a translation on L. Then φ determines a congruence relation K on G_L as follows: xKy iff $\varphi(x) = \varphi(y)$.

Proof: The relation K is evidently reflexive, symmetric and transitive. So it remains to show that xKy only when $\Gamma x K \Gamma y$. Assume $x\neq y$; the case x = y is trivial.

Suppose $z\in\Gamma x$ and assume z covers x i.e.; z>x. Then $\varphi(z)\geq\varphi(x)$ and thus we have two case to consider: (i) $\varphi(x) = \varphi(z)$ and (ii) $\varphi(z)>\varphi(x)$. The case x>z can be proved analogously.

(i) $\varphi(z) = \varphi(x) = \varphi(y)$, $\varphi(x\vee y) = \varphi(x)\vee\varphi(y) = \varphi(y)$ and suppose that $x\vee y>y$. Thus $\forall$ $w\in[y, x\vee y] = \{v:y\leq v\leq x\vee y, v\in L\}$, $\varphi(w) = \varphi(y)$. Since L is finite, there is also an element $w\in[y, x\vee y]$ such that w>y, and so $w\in\Gamma y$ such that zKw. If $y = x\vee y$, then x<y and $w\in[x,y]$, w<y such that $\varphi(w) = \varphi(y) = \varphi(z)$. Thus wKz. The part vice versa can be proved similarly.

(ii) $\varphi(z)>\varphi(x)$. Now, $\varphi(x)\vee z>\varphi(x)$, as in the other case $z\leq\varphi(x)$, from which it follows the contradiction: $\varphi(z)\leq\varphi(\varphi(x)) = \varphi(x)$. Since L is distributive and z>x, $\varphi(x)\vee z>\varphi(x)$. On the other hand, $\varphi(\varphi(x)\vee z)=\varphi(x)\vee\varphi(z)=\varphi(z)=\varphi(x\vee z)$ and thus $\varphi(z)>\varphi(x)$. Now, we must show that there is an element w>y in L such that $\varphi(w) = \varphi(z)$. If $y = \varphi(x)$, the case is trivial : $w = \varphi(z)$. Therefore we assume that $y \neq\varphi(x)$, hence $\varphi(x)\in[y, \varphi(z)]$. The convex sublattice $[y, \varphi(z)]$ of L is complemented and hence there is an element $u\in[y, \varphi(z)]$ such that $u\vee\varphi(x) = \varphi(z)$ and $u\wedge\varphi(x) = y$. Since $\varphi(z)>\varphi(x)$ and $[y, \varphi(z)]$ is distributive u>y and since $u\in[y,\varphi(z)]$, $u\leq\varphi(u)\leq\varphi(z)$. Suppose $\varphi(u)<\varphi(z)$. From $u\wedge\varphi(x) = y$ we have $\varphi(u\wedge\varphi(x)) = \varphi(y) = \varphi(u)\wedge\varphi(x)$ and hence $\varphi(u)\geq\varphi(y)$. Now, $\varphi(z)>\varphi(y)$ and $\varphi(z)>\varphi(u)\geq\varphi(x) = \varphi(y)$; consequently $\varphi(u) = \varphi(y)$. Then $\varphi(u\vee\varphi(x)) = \varphi(u)\vee\varphi(x) = \varphi(x)\neq\varphi(z)$, which is a contradiction. Therefore $\varphi(u) = \varphi(z)$.

So, if xKy then $\forall$ $z\in\Gamma x$ there is an element $u\in\Gamma y$ such that zKu. The vice versa part can be prove similarly. Thus the proof is complete.

Lemma 2.3: Let L be a finite lattice. If each congruence relation θ on L can be associated a congruence relation K on G_L, where xKy iff $x\theta y$, then L is a BOOLEANlattice.

Proof: First we show that L must be distributive where after the complementedness of L can be proved.

Consider the lattices L_1 and L_2 of Fig. 1. In the lattice L_1, {b, 0}, {a, 1} and {c} are congruence classes module θ_{a1}. Thus if bK0, the relation $\Gamma b K \Gamma 0$ does not hold, since there is no $u\in\Gamma b$ such that $u\theta_{a1}c$, $c\in\Gamma 0$. Furthermore, in the lattice L_2, {a, 1}, {b} and {c, 0} are the congruence classes module θ_{a1}. When 0Kc, the relation $\Gamma 0 K \Gamma c$ does not hold since $b\in\Gamma 0$ and there are no element $u\in\Gamma c$ such that uKb. Thus L can not contain L_1 or L_2 as a sublattice from which the distributivity of L follows:

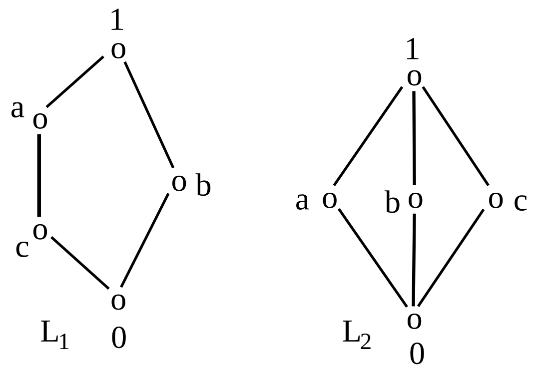


Fig. 2.1

In a distributive lattice L each ideal (a] determines a congruence relation $\theta[(a]]$, where $0\in L$, $i\theta[(a]]\,0$ iff $i\in(a]$. In particular, $\forall$ dual atom $z\in L$, i.e., z<l, there is at least one element $k\in L$ such that k>0, $k\notin(z]$. Indeed, in all other cases all the elements in $\Gamma 0\in(z]$ and since $1\in\Gamma z$ and $1\theta[(z]]z$, the relation yKx iff $x\theta y$ would not be a congruence relation on G_L. Thus for each dual atom z of L, k is the complement of z in L.

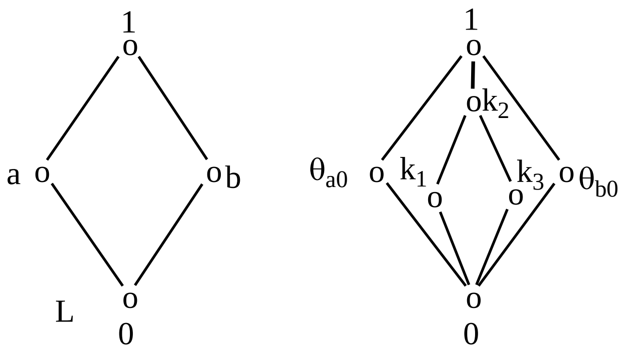
The complement is unique, since L is distributive. We can show as above that, if w<z<1, there is an element k>0 such that $k\notin(w]$ and $k\wedge w=0$. Since k is the unique complement of an element $y<1, w\vee k=r<1$. If r′ denotes the complement of r in L, $w\wedge r'=(r\wedge w)\wedge r'=0$ and so $(k\vee r')$ is the complement of w in L. It is unique according to the distributivity of L. Since L is finite, we can construct by this process a complement for each element of L. Thus the proof is complete.

By combining the results of Lemmas 1, 2 and 3 we have our characterization.

Theorem 2.1:A finite lattice L is a BOOLEAN lattice iff for each congruence relation θ on L the relation K, $xKy\Leftrightarrow x\theta y$, is a congruence relation on the graph G_L.

We shall finally make some remarks on the congruence relations K on the graph G_L when L is a finite BOOLEAN lattice. As well known, the congruence relations K on G_L form a lattice $K(G_L)$ w.r.to the meet and join operations defined as follows: If K, $H\in K(G_L)$, then $x(K\wedge H)y$ iff xKy and xHy and $x(K\vee H)y$ iff there is in V_L a sequence u_1, u_2, u_3 --------, u_n of elements, $x=u_1$ and $y=u_n$ such that for each value of i at least one of the relations u_iHu_{i+1}, u_iKu_{i+1} holds, i=1, 2-------, n-1.

The lattice $K(G_L)$ need not be even modular since one can see by means of the lattice L of Fig. 2. The only non-trivial lattice congruence on L are θ_{a0} and θ_{b0}. The congruence relations K on G_L that are not simultaneously lattice congruence on L are K_1,K_2 and K_3. The classes modulo K_1 are {1} and {0}. Those modulo K_2: {1, 0} and {a, b} and those modulo K_3:{1,0}, {a} and {b}. The lattice $K(G_L)$ is given in Fig. 2.



$K(G_L)$

Fig. 2.2

According to the definitions of join and meet operations in $K(G_L)$ and in the lattice $\theta(L)$ of all lattice congruences on L, we can write

3.CONCLUSION

If L is a finite BOOLEAN lattice, $\theta(L)$ is a BOOLEAN sublattice of the lattice $K(G_L)$.

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