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RESEARCH ARTICLE

# OPERATION ON GRAPHS AND GRAPH FOLDINGS

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**ABSTRACT**

In this paper we examining the relation between folding of a given pair of graphs and folding of new graphs generating from these given pair of graphs by some known operations like union, intersection, join, cartesian product, composition, normal product and tensor product of two graphs.



## 1. INTRODUCTION:

By a simple graph $G$, we mean that a graph with no loops or multiple edges. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be simple graphs. Then

(1) The simple graph $G = (V, E)$, where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ is called the union of $G_1$ and $G_2$, and is denoted by $G_1 \cup G_2$, [2]. When $G_1$ and $G_2$ are vertex disjoint, $G_1 \cup G_2$, is denoted by, $G_1 + G_2$, and is called the sum of the graphs $G_1$ and $G_2$.

(2) If $V_1 \cap V_2 \neq \phi$, the graph $G = (V, E)$, where $V = V_1 \cap V_2$, and $E = E_1 \cap E_2$ is called the intersection of $G_1$ and $G_2$ and is written as $G_1 \cap G_2$, [7,4].

(3) If $G_1$ and $G_2$ be vertex-disjoint graphs. Then the join, $G_1 \vee G_2$, is the super-graph of $G_1 + G_2$, in which each vertex of $G_1$ is adjacent to every vertex of $G_2$, [7,4 ].

(4) The cartesian product, $G_1 \times G_2$, is the simple graph with vertex set $V(G_1 \times G_2) = V_1 \times V_2$ and edge set $E(G_1 \times G_2) = [\,(E_1 \times V_2) \cup V_1 \times E_2\,)\,]$ such that two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent in $G_1 \times G_2$ if, and only if, either

(i) $u_1 = v_1$ and $u_2$ is adjacent to $v_2$ in $G_2$, or

(ii) $u_1$ is adjacent to $v_1$ in $G_1$ and $u_2 = v_2$, [7,3].

(5) The composition, or lexicographic product, $G_1[G_2]$, is the simple graph with $V_1 \times V_2$ as the vertex set in which the vertices $u = (u_1, u_2)$, $v = (v_1, v_2)$ are adjacent if either $u_1$ is adjacent to $v_1$ or $u_1 = v_1$ and $u_2$ is adjacent to $v_2$.

The graph $G_1[G_2]$ need not to be isomorphic to $G_2[G_1]$, [2].

(6) The normal product, or the strong product, $G_1 \circ G_2$, is the simple graph with $V(G_1 \circ G_2) = V_1 \times V_2$ where in $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent in $G_1 \circ G_2$ if, and only

if, either

(i) $u1 = v1$ and $u2$ is adjacent to $v2$ , or

(ii) $u1$ is adjacent to $v1$ and $u2 = v2$ , or

(iii) $u1$ is adjacent to $v1$ and $u2$ is adjacent to $v2$ , [6,5].

(7) The tensor product, or Kronecher product, $G1 \otimes G2$ , is the simple graph with $V(G1 \otimes G2) = V1 \times V2$ and where in $(u1, u2)$ and $(v1, v2)$ are adjacent in $G1 \otimes G2$ if, and only if, $u1$ is adjacent to $v1$ in $G1$ and $u2$ is adjacent to $v2$ in $G2$ .

Note that $G1 \circ G2 = (G1 \times G2) \bigcup (G1 \otimes G2)$ , [6].

From now on by a graph, we mean a simple graph.

**2- Graph Folding:**

Let $G1$ and $G2$ be graphs and $f : G1 \to G2$ , be a continuous function. Then $f$ is called a graph map, if

i) for each vertex $v \in V(G_1)$, $f(v)$ is a vertex in $V(G_2)$,

ii) for each edge $e \in E(G_1)$, $\dim(f(e)) \leq \dim(e)$,[1,2]

A graph map , is a graph folding if, and only if maps vertices to vertices and edges to edges, i.e., for each and for each $v \in V(G_1)$, ) $f(v) \in V(G_2)$ $f(v) \in V(G_2)$ $e \in E(G_1)$, $f(e) \in E(G_2)$ We denote the set of graph foldings between graphs G1 and G2 by $\eta$ ( G1, G2 ) and the set of graph foldings of G1 into itself by $\eta$ ( G1 ) ,[8,10] .

3. The Relation between Folding a Pair of Graphs and Folding them after Some Operations:

In mathematics, one always tries to get new structures from given ones. This also applies to the realm of graphs, where one can generate many new graphs from a given set of graphs.

In the following we consider folding of a given pair of graphs, then examining the relation between this folding and folding of new graphs generating from the given pair of graphs by some known operations like union, intersection, join, cartesian product, composition, normal product and tensor product, of two graphs.

**3.1. Definitions :**

Let $G1 = (V1, E1)$ , $G2 = (V2, E2)$ , $G3 = (V3, E3)$ and $G4 = (V4, E4)$ be graphs. Let $f : G1 \to G3$ and $g : G2 \to G4$ be graph maps. Then,

**(1)** By a union map of graph maps $f$ and $g$, $f \cup g$ , we mean a graph map from the graph $G1 \bigcup G2$ to the graph $G3 \bigcup G4$ ,

$f \cup g : G1 \bigcup G2 \to G3 \bigcup G4$ such that

$f(v) = g(v)$, for all $v \in V_1 \cap V_2$ and $f(e) = g(e)$, for all $e \in E_1 \cap E_2$ .

This map is defined by

$$\text{(i)} \forall\ v \in V_1 \cup V_2\ ,\ (f \cup g)(v) = \begin{cases} f(v), & \text{if } v \in V_1 \\ g(v) & \text{if } v \in V_2 \end{cases}$$

$$\text{(ii)} \forall\ v \in E_1 \cup E_2\ ,\ (f \cup g)(e) = \begin{cases} f(e), & \text{if } e \in V_1 \\ g(e) & \text{if } e \in V_2 \end{cases}$$

(2) If $f$ and $g$ agree on $V_1 \cap V_2$ and $E_1 \cap E_2$. Then by the intersection graph map, $f \cap g$, we mean a graph map $f \cap g$, we mean a graph map $f \cap g$: $G1 \vee G2 \rightarrow G3 \vee G4$ defined by

(i)$\forall\ v \in V_1 \cup V_2$ , $(f \cap g)\{v\} =$ $f\{v\}$, or $g\{v\}$

(ii)$\forall\ e \in E_1 \cup E_2$ , $(f \cap g)(e) =$ $f(e)$, or $g\{e\}$.

(3) By a join map of two graph maps, or a join graph map, $f \vee g$ , we mean a graph map $f \vee g : G1 \vee G2 \rightarrow G3 \vee G4$ defined by

(i) $\forall\ v \in V_1 \cup V_2$ , $(f \cup g)(v) = \begin{cases} f(v), & \text{if } v \in V_1 \\ g(v) & \text{if } v \in V_2 \end{cases}$

(ii) $\forall\ e = (v_1, v_2), v_1 \in V_1, v_2 \in V_2$ ,

$(f \vee g)\{e\} = (f \vee g)\{(v_1, v_2)\} = \{(f\{v_1\}, g\{v_2\})\} \in E_3 \cup E_4$,

(iii) if $e = (u_2, v_2), u_1 \in E_2$, then

$(f \vee g)\{e\} = (f \vee g)\{(u_1, u_1)\} = \{(f\{u_1\}, f\{u_1\})\}$

Also, if $e = (u_2, v_2) \in E_2$, then

$(f \vee g)\{e\} = (f \vee g)\{(u_2, v_2)\} = \{(g\{u_1\}, g\{v_2\})\}$

Note that if $f\{u_1\} = f\{v_1\}$, then the image of the join graph map $(f \vee g)\{e\}$ will be a vertex of, $G_3 \vee G_4$, otherwise it will be an edge of $G_3 \vee G_4$.

(4) By the Cartesian product map of the two maps $f$ and $g$ denoted by , $f \times g$ , we mean the map $f \times g$ : $G_1 \times G_2 \rightarrow G_3 \times G_4$ defined by

(i) if $v = (v_1, v_2) \in V_1 \times V_2$ , $v_1 \in V_1$, $v_2 \in V_2$ . Then

$f \times g\{(v_1, v_2)\} = \{(f\{v_1\}, g\{v_2\})\} \in V_3 \times V_4$,

(ii) if $e = \{(\{v_1\}_i, \{v_2\}_j),(\{v_1\}_i, \{v_2\}_k)\}, \{v_1\}_i, V(G_1)$ and $\{v_2\}_j\}, \{v_1\}_k \in V(G_2)$, then,

$f \times g\{(v_1\}_i, \{v_2\}_j), (\{v_1\}_k, \{v_1\}_j)\} = \{f\{v_1\}_i, \{v_2\}_j), (f(\{v_1\}_k, \{v_2\}_j)\}$.

Note that if $g\{v_2\}_j = g\{v_2\}_k$, o $f\{v_1\}_i = f\{v_1\}_k$, the image of the edge e will be a vertex.

(5) By the composition map of the two maps f and g, denoted by, f[g] , we mean the map f[g] : $G_1[G_2] \rightarrow G_3[G_4]$ defined by

(i) if $v = (v_1, v_2) \in V_1 \times V_2$ , $v_1 \in V_1$, $v_2 \in V_2$ . Then

$f[g]\{(v_1, v_2)\} = \{(f\{v_1\}, g\{v_2\})\} \in V(G_3[G_4])$,

(ii) let $e = \{(\{v_1\}_i, \{v_2\}_j),(\{v_1\}_k, \{v_2\}_l)\}$. If $\{v_1\}_i$ is adjacent to $\{v_1\}_k$, then

$f[g]\{e\} = \{(\{v_1\}_i, g\{v_2\}_j),(\{v_1\}_i, g\{v_2\}_l)\}$.

Note that $f\{v_1\}_i = f\{v_1\}_k$ and $g\{v_2\}_j = g\{v_2\}_l$, then, f[g]{e} will be a vertex, also if $g\{v_2\}_j = g\{v_2\}_l$, then f[g]{e} will be a vertex.

(6) By the normal product map of two graph maps f and g, $f \circ g$, we mean the map $f \circ g$: $G_1[G_2] \rightarrow G_3[G_4]$ defined by

(i) For any vertex $v = (v_1, v_2) \in V(G_1 \circ G_2) = V_1 \times V_2$ . then

$f \circ g\{(v_1, v_2)\} = \{(f\{v_2\}, g\{v_2\})\} \in V(G_3 \circ G_4)$,

(ii) For any $e = \{(\{v_1\}_i, \{v_2\}_j),(\{v_1\}_k, \{v_2\}_l)\}$, where $\{v_1\}_i$ is adjacent to $\{v_2\}_k\}$ and $\{v_1\}_j$ is adjacent to $\{v_2\}_l)\}$, then

$f \otimes g\{e\}\{(v_1, v_2)\} = f\{(\{v_1\}_i, \{v_2\}_k)\} \otimes g\{(\{v_2\}_j, \{v_2\}_l)\}$ i.e.,

$f \otimes g(G_1 \otimes G_2) = f(G_1) \otimes g(G_2)$.

Lemma (1):

Consider the graphs $G_1, G_2, G_3, G_4$. Let f: $G_1 \rightarrow G_3$ and g : $G_2 \rightarrow G_4$ be graph maps.

Then the union map, $f \cup g$, is a graph folding iff f, g graph folding, i.e.,

$(f \cup g) \in \eta(G_1 \cup G_2, G_3 \cup G_4)$ iff $f \in \eta(G_1 \cup G_3)$ and $g \in \eta(G_2 \cup G_4)$.

Proof:

Since both f and g are graph foldings, then each maps vertices to vertices and edges to edges. Thus

(i) $\forall$ v $\in$ V $(G_1) \cup V(G_2)$, $(f \cup g)$(v) = $\begin{cases} f(v), & if\ v \in V(G_1) \\ g(v), & if\ v \in V(G_2) \end{cases}$,

i.e., $(f \cup g)$(v) $\in$ V$(G_3 \cup G_4)$, also

(ii) $\forall$ v $\in$ E $(G_1) \cup E(G_2)$, $(f \cup g)$(e) = $\begin{cases} f(e), & if\ v \in E(G_1) \\ g(e), & if\ v \in E(G_2) \end{cases}$,

and in both cases the image is an edge of $G_3 \cup G_4$. Thus the union map is a graph folding.

To prove the converse, let $f \cup g$: $G_1 \cup G_2 \rightarrow G_3 \cup G_4$ be a graph folding. Then for all v $\in$ V$(G_1) \cup V(G_2)$

$(f \cup g)$(v) = $\begin{cases} f(v), & if\ v \in V(G_1) \\ g(v), & if\ v \in V(G_2) \end{cases}$.

But $(f \cup g)$(v) $\in$ V$(G_3 \cup G_4)$ = V$(G_3)$ $\cup$ V$(G_4)$. Thus each of f and g must map vertices to vertices.

Now, let e $\in$ E$(G_1)$ $\cup$ E$(G_2)$, then

$(f \cup g)$(e) = $\begin{cases} f(v), & if\ e \in V(G_1) \\ g(v), & if\ e \in V(G_2) \end{cases}$

Suppose e = $(u_1, v_1)$ $\in$ E $(G_1)$, $(f \cup g)$(e) = f{$(u_1, v_1)$}. Now, if f{$(u_1, v_1)$} $\in$ V $(G_3)$ then $(f \cup g)$ ( e) $\in$ V $(G_3)$ which contradi the assumption that $(f \cup g)$ is a graph folding. Thus f and g must maps edges to edges. Hence each of f and g is a graph folding.

Lemma2:

Let $G_1, G_2, G_3, G_4$ be graphs. If f: $G_1 \rightarrow G_3$ and g : $G_2 \rightarrow G_4$ be graph folding, then the intersection map, f $\cap$ g, is a graph folding.

Proof:

The proof is obious since, f $\cap$ g, on $G_1 \cap G_2$ will send vertices to vertices and also edges to edges as each of *f* and *g* do.

Examples(1):

Let $G_1$= $(V_1, E_1)$, where $V_1$= { $v_1, v_2, v_3, v_4, v_5, v_6$} and $E_1$= { $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$}
Let $G_2$= $(V_2, E_2)$, where $V_2$= { $v_1, v_5, v_6, v_7$ } and $E_1$= { $e_5, e_8, e_9, e_{10}, e_{11}$}, see Fig.(1).

Let f : $G_1 \rightarrow G_2$ be a graph folding defined by f{ $v_1, v_2$} ={ $v_5, v_4$} and $f$ { $e_1, e_2, e_7, e_8$} = $f$ { $e_4, e_3, e_6, e_5$}.

Let g : $G_2 \rightarrow G_2$ be a graph folding defined by g{ $v_1$} = g{ $v_5$} and $f$ { $e_8, e_9$} and $g$ { $e_8, e_9$} = {$e_5, e_{11}$}.

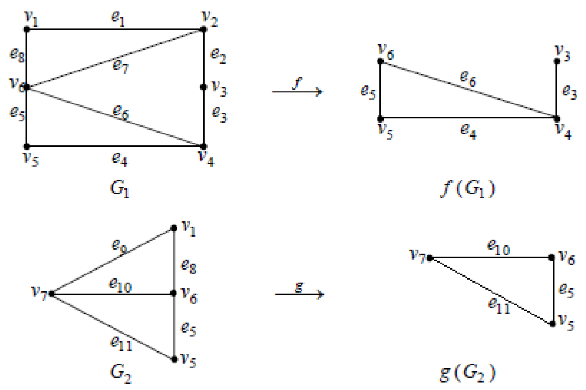



Fig. (1)

Then the union map $f \cup g: G_1 \cup G_2 \rightarrow G_1 \cup G_2$ defined by $f \cup g\{v_1, v_2\} = \{v_5, v_4\}$ and $f \cup g\ \{e_1, e_2, e_7, e_8, e_9\} = \{e_4, e_3, e_6, e_5, e_{11}\}$ is a graph folding, see Fig(2) (a).

also, the intersection $f \cap G \rightarrow: G_1 \cup G_2 \rightarrow G_1 \cup G_2$ defined by $f \cap g\ \{v_1\} = \{v_5\}$ and $f \cap g\ \{e_8\} = \{e_5\}$, is a graph folding, see Fig. (2) (b).

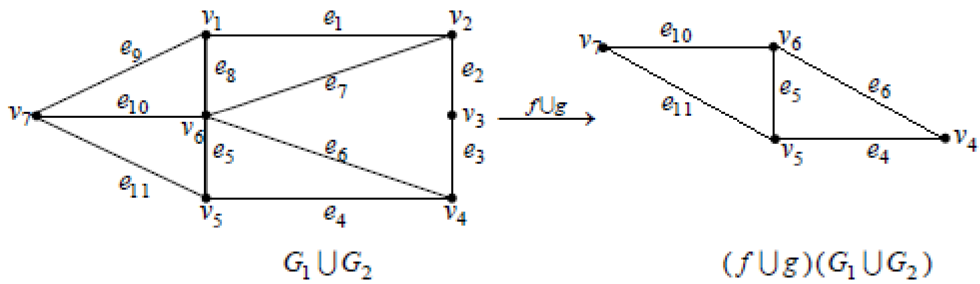



Fig. (2) (a)

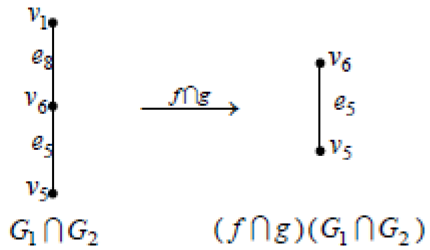



Fig (2)(b)

Theorem(1):

Let $G_1, G_2, G_3, G_4$ be graphs. If f: $G_1 \rightarrow G_3$ and g : $G_2 \rightarrow G_4$ be graph folding. Then the $f \vee g$ : $\eta(G_1 \vee G_2 \vee G_3 \vee G_4)$ iff $f, g$ are graph folding.

Proof:

Let $f$ and $g$ be graph folding. Then

$(f \vee g)\{V_1 \cup V_2\} = f\{V1\} \cup g\{V_2\}\}$. But $f\{V1\} \in V(G_3)$, $g\{V_2\} \in G_4$.

Thus $\{f\{V_1\} \cup g\ \{V_2\}\} \in V(G_3 \vee G_4)$ i.e., $f \vee g$ maps vertices to vertices.

Now, let e ϵ $E(G_1 \vee G_2)$. Then either e ϵ $E(G_1)$, or e e ϵ $E(G_2)$, or e is an edge joining a vertex of $G_1$ to a vetex of $G_2$. In the first two cases and since each of $f$ and g is a graph folding, $(f \vee g)$ (e) ϵ E $(G_3 \vee G_4)$. Now, if e = $(v_1, v_2)$, $v_1 \in G_1$ , $v_2 \in G_2$, then

$$(f \vee g)\{e\} \in (f \vee g)\ \{(v_1, v_2)\} = \{f\{v_1\}, g\ \{v_2\})\}$$

$$= \{(v_3, v_4)\} \in E(G_3, G_4).$$

Thus $(f \vee g)$ maps edges to edges and hence the join graph map, $f \vee g$, is a graph folding.

The converse is guaranteed by the definition of the join map.

Theorem(2):

Let $G_1, G_2, G_3, G_4$ be graphs. If f: $G_1 \rightarrow G_3$ and g : $G_2 \rightarrow G_4$ be graph folding. Then the $f \times g$ : $\eta(G_1 \times G_2,\ G_3 \times G_4$ iff $f \in \eta\ (G_1, G_3)$ and $g \in \eta \in (G_2, G_4)$.

Proof:

Let $f, g$ be graph folding, then

(i)For each vertex $(v_1, v_2)$ ϵ G $(G_1, G_2) = (V_1, V_2)$, where $v_1 \in V_1$ , and $v_2 \in V_2$, we have

$f \times g\{(v_1, v_2)\} = \{f\{v_1\}, g\{v_2\}\} = \{(v_3, v_4)\} \in (G_3 \times G_4) = V_3 \times V_4$ ,

(ii) Let e ={$(\{v_1\}_i, \{v_2\}_j)$,$(\{v_1\}_i, \{v_2\}_k)$}, where $v_1 \in V(G_1)$ and $\{v_2\}_j$,$\{v_1\}_k \in V(G_1)$,

Then

$f \times g\{e\} = f \times g$={$(\{v_1\}_i, \{v_2\}_j)$,$(\{v_1\}_i, \{v_2\}_k)$}

= = {$(\{v_1\}_i$, g $\{v_2\}_j)$,$(\{v_1\}_i$, g $\{v_2\}_k)$}.

Since $\{v_2\}_j$ is a adjacent $\{v_2\}_k$ and g is a graph folding, then g$\{v_2\}_k$. Thus $f \times g\{e\} \in E(G_3 \times G_4)$, i.e., ( $f \times g$) maps edges to edges and hence ( $f \times g$) is a graph folding.

To prove the converse suppose that ( $f \times g$) is a graph folding and f, or g, is not a graph folding. In this case f, or g, will map an edge a vertex, say $f\{(u_1, v_1)\} = \{u_3\}$, $u_3 \in V(G_3)$. Then

$f \times g$ {$(\{u_1\}, \{v_2\}_j)$,$(\{v_1\}, \{v_2\}_j)$}= {$(f\{u_1\}_i$, g$\{v_2\}_j)$,$(f\{v_1\}_i$, g$\{v_2\}_l)$}.

= {$(\{u_3\}, \{v_2\}_j)$,$(\{u_3\}, \{v_2\}_j)$}$\in V(G_3 \times G_4)$

This contradicts the assumption and thus each of f and g must be a graph foldings

Theorem(3):

Let $G_1, G_2, G_3, G_4$ be graphs. If f: $G_1 \rightarrow G_3$and g : $G_2 \rightarrow G_4$ be graph folding. Then the $f\ [g] \in \eta(G_1[G_2],\ G_3\ [G_4]$ if f, g are graph folding.

Proof:

Let f and g be graph folding, then

(i) for each vertex $v = (v_1, v_2) \in V(G_1[G_2] = V_1 \times V_2$.

f[g]{ $(v_1, v_2)$}= {(f{ $v_1$}, g$\{v_2\}$)} $\in V(G_3[G_4\}) = V_1 \times V_2$,

(ii) let e ={$(\{v_1\}_i, \{v_2\}_j)$,$(\{v_1\}_k, \{v_2\}_l)$}. If { $v_1\}_i$ and suppose that $\{v_1\}_i$, is adjacent to $\{v_1\}_k$. Then there exists an edge {$(\{v_1\}_i, \{v_2\}_k\})$ $\epsilon$ $E_1$,

f[g]{e} ={(f$\{u_1\}_i$, g$\{v_2\}_j$ }),(f$\{v_1\}_i$, g$\{v_2\}_l$)}. Since f is a graph folding and edge {$(\{u_1\}_i$, $\{v_2\}_k\})$ $\epsilon$ $E_1$. Then

f[g] {$(\{u_1\}, \{v_2\}_i)$,$(\{v_1\}, \{v_2\}_j)$}={(f$\{u_1\}_i$, g$\{v_2\}_i$}),(f$\{v_1\}$, g$\{v_2\}_j$)}

= {$(\{u_3\}_i$, g$\{v_2\}_i$}),$(u_3\}_i$, g$\{v_2\}_j$)}

Which is an edge of ( $G_3[G_4]$.

Theorem(4):

Let $G_1, G_2, G_3, G_4$ be graphs. If f: $G_1 \rightarrow G_3$and g : $G_2 \rightarrow G_4$ be graph folding. Then the $f \circ g:\ \eta(G_1 \circ G_2,\ G_3 \circ G_4$ if $f$ and g are graph folding.

Proof:

Let f,g be graph folding, then

(i) for each vertex $v = (v_1, v_2) \in V(G_1 \circ G_2] = V_1 \times V_2$.

f∘g]{ $(v_1, v_2)$}= {(f{ $v_1$}, g$\{v_2\}$)} $\in V(G_3 \circ G_4\})$ ,

(ii) let e ={$(\{v_1\}_i, \{v_2\}_l)$,$(\{v_1\}_k, \{v_2\}_l)$}. and suppose that $\{v_1\}_i = \{v_1\}_k$ and $\{v_1\}_j$ is adjacent to $\{v_1\}_l$. Then there exists an edge {$(\{v_1\}_j, \{v_2\}_l\})$ $\epsilon$ $E_2$, such that

{ (g$\{v_2\}_i$, g$\{v_2\}_l$)} $\epsilon$ $E_4$ because g is a graph folding and edge g$\{v_1\}_j \neq \{v_2\}_l$) . Then

f ∘g {e}= f ∘g {$(\{v_1\}_i, \{v_2\}_j)$,$(\{v_1\}_k, \{v_2\}_l)$}

= {(f$\{v_1\}_i$, g$\{v_2\}_i$}),(f$\{v_1\}_k$, g$\{v_2\}_l$)}

= {$(\{u_1\}_i$, g$\{v_2\}_j$}),$(u_1\}_j$, g$\{v_2\}_l$)},

Where f$\{v_1\}_i$ = f$\{v_1\}_k = u_1$. Thus f ∘g {e} $\epsilon$ $E_4(G_3 \circ G_4)$.

Similarly we can prove that if $\{v_1\}_i$ is adjacent to $\{v_1\}_k$ and $\{v_1\}_j = \{v_1\}_l$ . Then f ∘g {e} is an edge of the graph $G_3 \circ G_4$.

In general if $\{v_1\}_i$ is a adjacent to $\{v_1\}_k$ and $\{v_1\}_j$ is adjacent of $\{v_1\}_l$, f ∘g {e} is an edge of $G_3 \circ G_4$.

The converse is not true since if f ∘g is a graph folding and f, or g, is not a graph folding. In this case

f{$( u_1 , v_2)$}= { $( u_3, u_3)$}, $u_3 \in V( G_3)$, then

f ∘g $\{( u_1 , v_2) \}_j$ , $\{( v_1 , v_2) \}_k$ = f$\{( u_1 , g (v_2) \}_j$ , $( v_1 , g (v_2) \}_k$ ,

= f$\{( u_1 ), g (v_2) \}_j$ , f( $\{v_1 \}, g (v_2) \}_k$ ,

= $\{( u_3 , g (v_2) \}_j$ , $( u_3 , g (v_2) \}_k$ ,

Which is an edge of $(G_3 \circ G_4)$.

Theorem(3):

Let $G_1, G_2, G_3, G_4$ be graphs. If f: $G_1$→ $G_3$and g : $G_2$ → $G_4$ be graph folding. Then the

$f \otimes g$ € η($G_1 \otimes G_2$], $G_3 \otimes G_4$] iff f, g are graph folding.

Proof:

Lef f, g be graph foldings, then

(i)for each vertex $v$ = {$( v_1 , v_2)$ € $(G_1 \otimes G_2)$ = $V_1 \times V_2$

$( f \otimes g)$ {$( v_1 , v_2)$} ={(f{$v_1$} , $g$ {$v_2$})} € $V(G_3 \otimes G_4$

(ii)let e = {({$v_1$}$_i$, {$v_2$}$_j$}),({$v_1$}$_k$, {$v_2$}$_l$)}, and suppose that $\{v_1\}_i$ is adjacent to $\{v_1\}_k$ and {({$v_1$}$_j$ is adjacent to {({$v_1$}$_l$. Then there exist edge e = {({$v_1$}$_i$, {({$v_1$}$_k$)} ∈ $E_1$ and

$e_1$= {({$v_1$}$_j$, {({$v_1$}$_l$)} ∈ $E_2$

such that f{$e_1$}∈ $E_3$ and g{$e_2$}∈ $E_4$ This isdue the fact that each of f and g is a graph foldings,i.e., f$\{v_1\}_i \neq$ f$\{v_1\}_k$and g$\{v_2\}_j \neq$ g$\{v_2\}_l$.

Now,

$( f \otimes g)$ {e}= ( $f$ {$e_1$}⊗ $g${ $e_2$} = f{{ $v_1$}$_i$, { $v_1$}$_k$)} ⊗ g{{ $v_2$}$_j$, { $v_2$}$_l$)}

= f{{ $v_1$}$_i$, g{ $v_2$}$_j$)} , f{{ $v_1$}$_k$, g{ $v_2$}$_l$)}

But since f{{ $v_1$}$_i$ is adjacent to f{ $v_1$}$_k$ and g{{ $v_2$}$_j$ is adjacent to g{ $v_2$}$_l$, then thee exist an edge of $E(G_3 \otimes G_4)$ joining these two vertices, i.e.

$( f \otimes g)$ {e} $E(G_3 \otimes G_4$. Hence, f ⊗ g, is a gaph folding.

To prove the converse suppose that (f ⊗ g) is a graph folding and f, or g, is not a graph folding. In this case f, or g, will map an edge to a vertex, say f{$( u_1 , v_1)$= {$( u_3 , u_3)$} ,

$u_3$ € $V(G_3)$. then

(f ⊗ $g$) {e}= f{$( u_1 , v_1)$ ⊗g{{ $v_2$}$_j$, { $v_2$}$_l$)}

= f{{ $u_1$}, g{ $v_2$}$_j$)} , f{{ $v_1$}, g{ $v_2$}$_l$)}

= {($u_3$, g{ $v_2$}$_j$, $u_3$, g{ $v_2$}$_l$)}

Which is not an edge of (f ⊗ $g$) then f, g must be graph foldings.

Examples2):

Let $G_1$ = ($V_1$, $E_1$), $G_2$ = ($V_2$, $E_2$) be graphs where $V_1$= ($u_1, v_1, w_1, x_1$},

$E_1$ = { $( v_1, v_2)$, $( v_1, w_1)$, $( w_1, x_1)$, ]{ $( x_1, u_1)$}, $V_2$= ($u_2, v_2, w_2$}, and

$E_2$ = { $( u_2, v_2)$, $( v_2, w_2)$, Fig( 3).

Now, let f: $G_1$→ $G_2$ be the graph folding defined by

g{ $u_2$} = {$w_2$} and g{$w_2$, $v_2$)}.

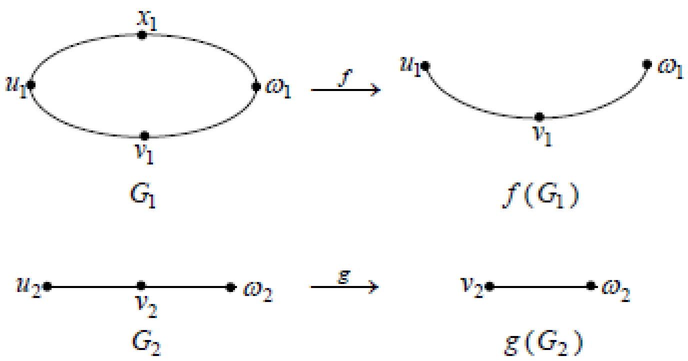



Fig (3)

Then,

1.The join graph map $(f \vee g) : G_1 \vee G_2$ is a graph folding. This map is defined as follows:

(i) $(f \vee g)\ (x_1, u_2\} = \{ v_1, w_2\}$,

(ii) $\forall$ e $\in E(G_1)$, e $\in$ E($G_2$), then

$(f \vee g)$ {e} $\in$ E($G_1$), E($G_2$), i.e., $(f \vee g)$ {e} $\in$ E($G_1 \vee G_2$),

(iii)let e = $\{(x_1, u_2)\}$, $x_1 \in V_1$, $u_2 \in V_2$. Then

$(f \vee g)\{e\} = (f \vee g)\ \{(x_1, u_2)\} = \{(f\ \{x_1\},\ g\ \{u_2\}\ )\}$

$= = \{(v_1, w_2)\} \in$ E [ $(f \vee g)\ (G_1 \vee G_2)$ ] = E($G_1 \vee G_2$),

If e = $\{(x_1, u_2)\}$, $x_1, u_2 \in V_1$. Then

$(f \vee g)\{e\} = (f \vee g)\ \{(x_1, u_1)\} = \{(f\ \{x_1\},\ g\ \{u_1\}\ )\}$

$= \{(v_1, u_1)\} \in$ E($G_1 \vee G_2$) and so on see Fig (4).

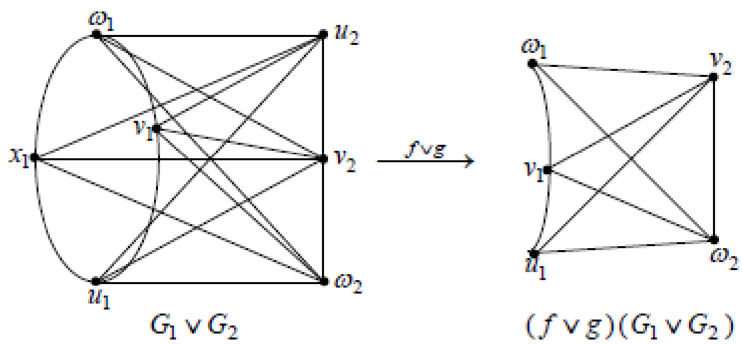



Fig (4)

2. The Cartesian product map f × g : $G_1 \times G_2 \to G_1 \times G_2$ is a graph folding. This map is defined as

$(f \times g)\{\ \{(x_1, w_2)\} = \{(\ f\ \{x_1\},\ g\ \{w_2\})\} - \{(x_1, w_2)\}$,

$(f \times g)\{\ \{(x_1, v_2)\} = \{(\ f\ \{x_1\},\ g\ \{v_2\})\} - \{(v_1, v_2)\}$,

And so on

Also,

$(f \times g)\{\ \{(w_1, u_2), \{(u_1, u_2\} = \{(\ f\ \{w_1\},\ g\ \{u_2\})\}\ , (f\ \{x_1\},\ g\ \{\ u_2\}\}$,

$= \{(w_1, w_2), \{(v_1, w_2\}$,

$(f \times g)\{\ \{(x_1, v_2), \{(x_1, u_2\} = \{(\ f\ \{x_1\},\ g\ \{v_2\})\}\ , (f\ \{x_1\},\ g\ \{\ u_2\}\}$,

$= \{(v_1, v), \{(v_1, w_2\}$,

And so on see fig. (5)

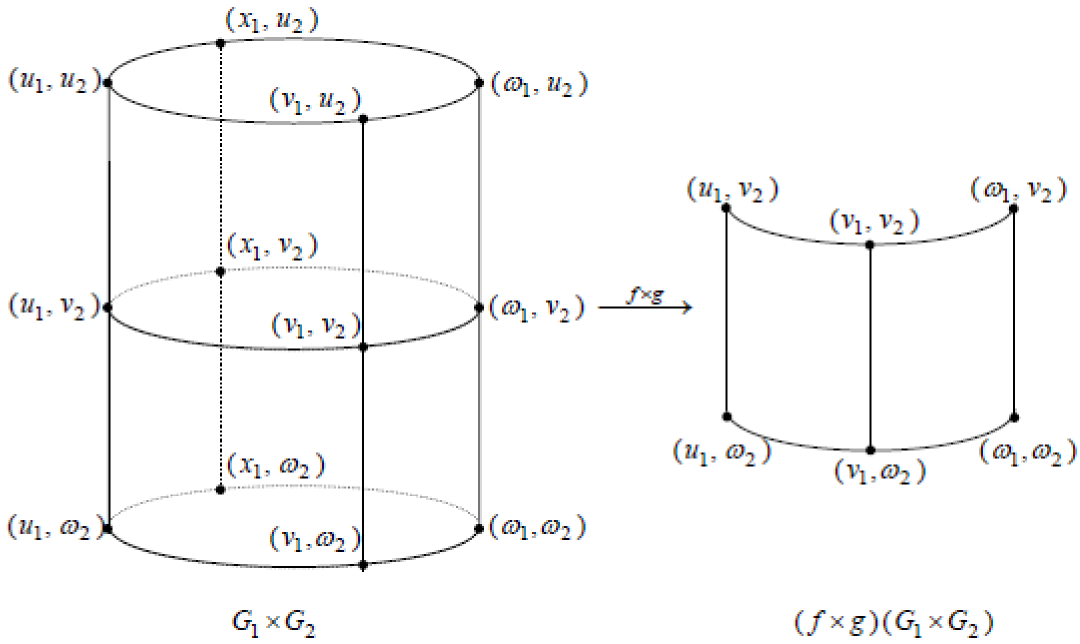



Fig .5

3. The composition f [g ]: $G_1$ $[G_2] \to G_1$ [ $G_2$ ]is a graph folding. This map is defined as

$(f$ [g] $\{\{(u_1, u_2)\} = \{(f\{u_1\},\ g\{u_2\})\} - \{(u_1, w_2)\}$,

$(f$ [ $g]\{\{(x_1, v_2)\} = \{(f\{x_1\},\ g\{v_2\})\} - \{(v_1, v_2)\}$,

And so on

Also,

$(f\ [g]\{\{(u_1, u_2), \{(u_1, v_2\} = \{(f\{u_1\},\ g\{u_2\})\}\ , (f\{u_1\},\ g\{v_2\})\}$,

$= \{(u_1, w_2), \{(u_1, w_2\}$,

$(f$ [ g ])$\{\{(u_1, v_2), \{(x_1, v_2\} = \{(f\{u_1\},\ g\{v_2\})\}\ , (f\{x_1\},\ g\{v_2\})\}$,

$= \{(u_1, v), \{(v_1, v_2\}$,

And so on see fig. (6)

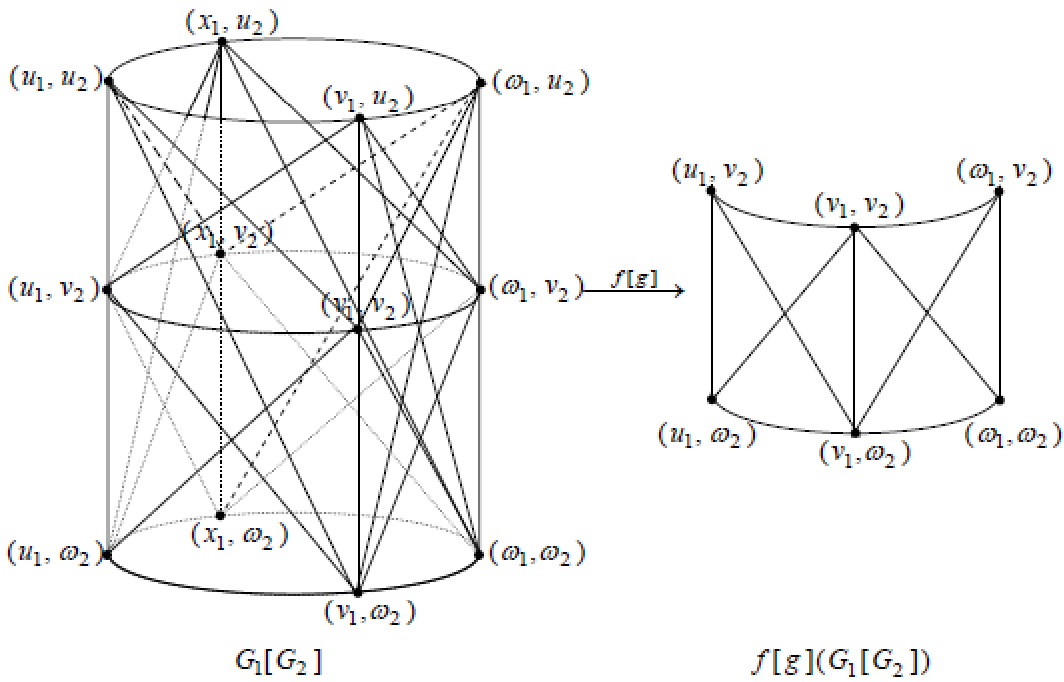



Fig (6)

And, the composition f [g ]: $G_2$ $[G_1] \to G_2$ [ $G_1$ ]is a graph folding. This map is defined as

$g$ [f] { $\{(u_2, u_1)\}$ = {( $g$ $\{u_2\}$, $f\{u_1\}$)} - $\{(w_1, u_1)\}$,

$g$ [ f] { $\{(w_2, x_1)\}$ = {( $g$ $\{w_2\}$, $f$ $x_1$})} - $\{(w_1, v_1)\}$,

and so on

also,

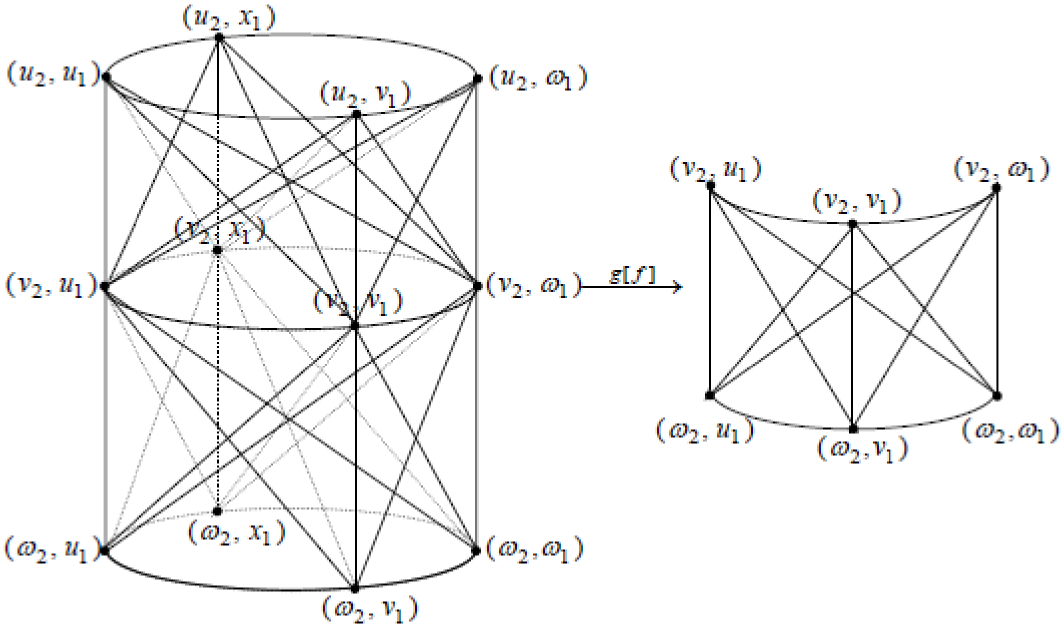
*g[f]* { $\{(u_2, u_1)$ $\{(v_2, v_1\}$ = {( $g$ $\{u_2\}$, $f$ $\{u_1\}$)} , ($g$ $\{v_2\}$, $f\{$ $v_1$)},

= $\{(w_1, u_1)$, $\{(v_2, v_1)\}$

*g[f]* { $\{(v_2, x_1)$ $\{(w_2, x_1\}$ = {( $g$ $\{v_2\}$, $f$ $\{x_1\}$)} , ($g$ $\{w_2\}$, $f\{$ $x_1$)},

= $\{(v_2, v_1)$, $\{(w_2, v_1)\}$

And so on see fig. (7)




$G_2[G_1]$ $g[f](G_2[G_1])$

Fig. (7)

4. The normal product map $f \circ g$ ]: $G_1$ $\circ$ $G_2] \to G_1$ $\circ$ $G_2$ ]is a graph folding. This map is defined as

*(f* $\circ$g] { $\{(u_1, u_2)\}$ = {( $f$ $\{u_1\}$, $g$ $\{u_2\}$)} - $\{(u_1, w_2)\}$,

*(f* $\circ$ *g]* { $\{(x_1, u_2)\}$ = {( $f$ $\{x_1\}$, $g$ $\{u_2\}$)} - $\{(v_1, w_2)\}$,

And so on

Also,

*(f* $\circ$ *g]* { $\{(u_1, u_2)$, $\{(v_1, v_2\}$ = {( $f$ $\{u_1\}$, $g$ $\{u_2\}$)} , ($f$ $\{v_1\}$, $g$ { $v_2$)},

= $f$ $\{(u_1, w_2)$, $\{(v_1, v_2\}$,

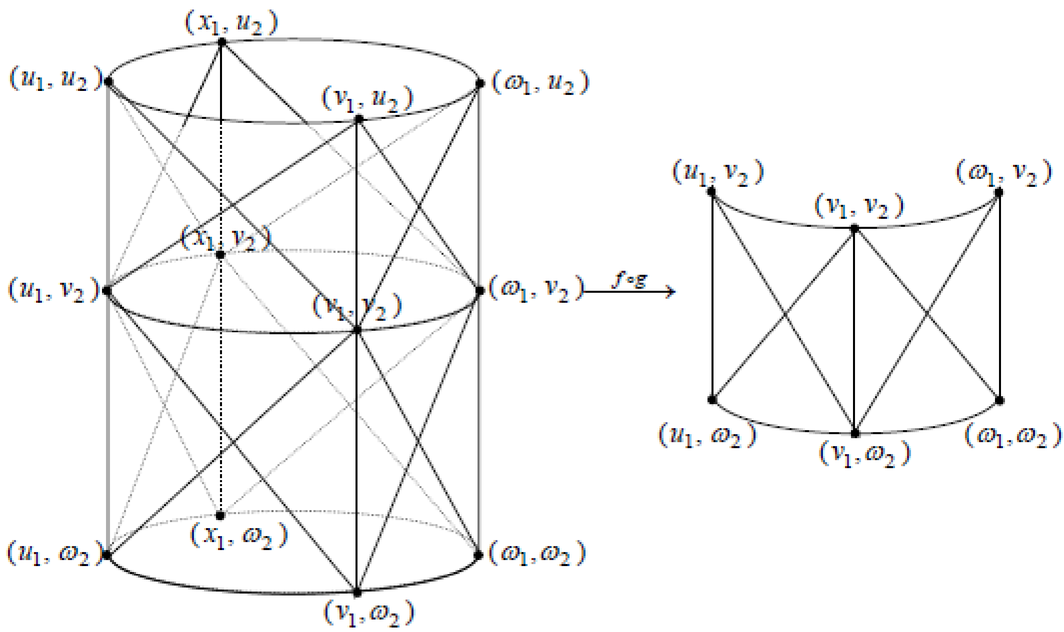
= {( $f$ $\{u_1\}$, $w_2$) , ($f$ $\{v_1\}$, $v_2$) } = $\{(u_1, w_2)$,$(v_1, v_2)\}$,

*(f* $\circ$ *g]* { $(x_1, v_2)$, $\{(w_1, w_2\}$ = {( $f$ ($x_1$, $g$ $\{u_2\}$) , ( $w_1$, $g$ { $w_2$)},

= $f$ $\{(x_1, v_2)$, $\{(w_1, w_2)\}$

= {( $f$ $\{x_1\}$, $v_2$) , ($f$ $\{w_1\}$, $w_2$) } = $\{(v_1,$ $_2)$,$(w_1, w_2)\}$,

And so on see fig. (8)




$G_1 \circ G_2$ $(f \circ g)(G_1 \circ G_2)$

Fig .8

5. The tensor product map $f \otimes g : G_1 \otimes G_2 \rightarrow G_1 \otimes G_2$ ]is a graph folding. This map is defined as

$(f \otimes g] \{\{(u_1, u_2)\} = \{(f\{u_1\},\ g\{u_2\})\} - \{(u_1, w_2)\},$

$(f \otimes g] \{\{(x_1, u_2)\} = \{(f\{x_1\},\ g\{u_2\})\} - \{(v_1, w_2)\},$

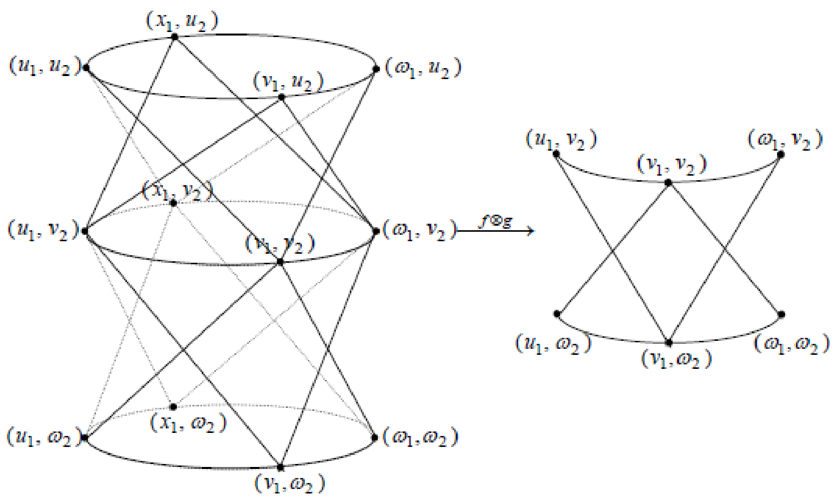
And so on

Also,

$(f \otimes g] \{\{(u_1, u_2), \{(v_1, v_2\} = \{(f\{(u_1, u_2)\} \otimes g\{(u_2, v_2)\}$
$= f\{(u_1\}, g\{u_2\}, (f\{v_1\}, g\{v_2\})\}$
$= \{(u_1, w_2), (v_1, v_2)\},$

$(f \otimes g] \{\{(x_1, u_2), \{(u_1, v_2\} = \{(f\{(x_1, u_1)\} \otimes g\{(u_2, v_2)\}$
$= f\{(x_1\}, g\{u_2\}, (f\{u_1\}, g\{v_2\})\}$
$= \{(u_1, w_2), (u_1, v_2)\},$

And so on see fig. (9)




$G_1 \otimes G_2$ $(f \otimes g)(G_1 \otimes G_2)$

Fig (9)

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