Vol.3.Issue.4.2015 (Oct-Dec.)

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com

RESEARCH ARTICLE

ISSN:2348-0580

FIXED POINT THEOREM FOR A SEQUENCE OF MAPPINGS IN DISLOCATED QUASI-METRIC SPACE

Dr. S.S. PAGEY[1], NEENA GUPTA[2]

[1]Institute for excellence in Higher Education, Bhopal, Retired Professor, Department of Mathematics, Institute for excellence in Higher Education, Bhopal

[2]Department of Mathematics, Career College, Bhopal,University of Barkatullah, Bhopal

[1]pagedrss@rediffmail.com, [2]gneena33@gmail.com

NEENA GUPTA

ABSTRACT

In this Paper we have proved Fixed Point Theorems in dislocated quasi-metric space for sequence of mappings using rational inequality.

Key-words: Dislocated metric space, dislocated quasi-metric space, fixed point, dq- Cauchy sequence, dq limit.

1 INTRODUCTION

In 1922, S. Banach [8] proved a fixed point theorem for contraction mapping in metric space. Since then a number of fixed point theorems have been proved by different authors and many generalizations of this theorem have been established. In 2000, P. Hitzler and A.K. Seda [5,7] introduced the notion of dislocated metric space in which self distance of a point need not be equal to zero. They also generalized the famous Banach contraction principle in this space. Dislocated metric space plays a very important role not only in topology but also in other branches of science involving mathematics especially in logic programming and electronic engineering [6]. D.S Jaggi [3] proved fixed point theorem using rational type of contractive condition which generalize the Banach contradiction principle in complete metric space. Zeyada et. Al. [4] initiated the concept of dislocated quasi metric space and generalized the result of Hitzler and Seda [7] in dislocated quasi metric space .C.T. Aage and J.N. Salunke [2], A Isufati [1] established some important fixed point theorems in single and pair of mappings in dislocated metric space. In this paper we established a fixed point theorem in the context of dislocated quasi metric space.

2 Preliminaries

We introduce below necessary notions and present a few results in dislocated quasi-metric space that will be used throughout the paper.

Definition 2.1 [4,5] Let X be a non-empty set d: X × X→ R^+ be a function, called a distance function if for all x,y,z∈X, satisfies:

d_1 : $d(x,x) = 0$

d_2 : $d(x,y) = d(y,x) = 0 \Rightarrow x = y$

d_3 : $d(x,y) = d(y,x)$

d_4 : $d(x,y) \le d(x,z) + d(z,y)$

If d satisfies the condition $d_1 - d_4$ then d is called a metric on X.

If it satisfies the condition $d_1,\ d_2$ and d_4, it is called quasi-metric space.

If d satisfies condition $d_2,\ d_3$ and d_4, it is called dislocated metric (or simply d-metric)

If d satisfies only d_2 and d_4, then d is called a dislocated quasi-metric (or simply dq-metric) on X.

Definition 2.2 [4,5] A sequence $(x_n)_{n\in N}$ in dq-metric space (X,d) is called Cauchy if for all $\varepsilon > 0, \exists n_0 \in N$ such that $\forall\ m,n \ge n_0,\ d(x_{m,}x_n) < \varepsilon$ or $d(x_n, x_m) < \varepsilon$

Definition 2.3 [4] A sequence $(x_n)_{n\in N}$dislocated quasi converges or dq-converges to x if $\lim_{n\to\infty} d(x_n,x) = \lim_{x\to\infty} d(x,x_n) = 0$

In this case x is called a dq- limit of $(x_n)_{n\in N}$ and we write $x_n \to x$

Definition 2.4 [4,5] A dq-metric space (X, d) is complete if every Cauchy sequence in it is dq-convergent.

Lemma 2.5 [4] Every subsequence of dq-convergent sequence to a point x_0 is dq-convergent to x_0

Definition 2.6 [4,5] Let (X,d) be a dq-metric space. A mapping f:$X \to X$ is called contraction if there exists $0 \le \lambda < 1$ such that:

$d(fx, fy) \le \lambda d(x,y)$ for all $x, y \in X$.

Lemma 2.7 [4,5] dq-limits in a dq-metric space are unique.

Further some theorems [5] give common fixed points for continuous contraction mapping satisfying contractive type condition and rational inequality in dislocated and dislocated quasi-metric space. Our theorem prove the result for sequence of mappings.

3 Main Result

We Prove the following theorem.

Theorem- Let (X,d) be a complete dislocated quasi-metric space. Let <T_k> be a sequence of self mappings on X satisfies the condition :

$$d(T_i x, T_j y) \le \alpha \frac{d(x,T_j y)d(y,T_j y)}{d(T_{ix},x)+d(x,T_j y)} + \beta[d(x,T_j y) + d(y,T_j y)] + \gamma d(x,y) \quad \ldots\ldots\ldots\ldots\ldots (3.1)$$

For all $x, y \in X, \alpha, \beta, \gamma$ are non negative with $0 \le 2\alpha + 3\beta + \gamma < 1$

Then <T_k> have a unique common fixed point.

Proof- Let $x_0 \varepsilon X$. We define a sequence <x_n> in X such that $T_i x_{n-1} = x_n$ and $T_j x_n = x_{n+1}$, for n=1,2,3……..

Then $d(x_{n,}x_{n+1}) = d(T_i x_{n-1,} T_j x_n)$

$$\le \alpha \frac{d(x_{n-1,}T_j x_n)d(x_n,T_j x_n)}{d(T_i x_{n-1,}x_{n-1})+d(x_{n-1}\,,T_j x_n)} + \beta[d(x_{n-1,}T_j x_n) + d(x_{n,}T_j x_n)] + \gamma d(x_{n-1,}x_n)$$

by (3.1)

$$= \alpha \frac{d(x_{n-1,}x_{n+1})\,d(x_n,x_{n+1})}{d(x_{n,}x_{n-1})+d(x_{n-1,}x_{n+1})} + \beta[d(x_{n-1,}x_{n+1}) + d(x_n, x_{n+1}) + \gamma d(x_{n-1,}x_n)$$

$$\le \alpha d(x_{n-1}x_{n+1}) + \beta[d(x_{n-1,}x_{n+1}) + d(x_{n,}x_{n+1}) + \gamma d(x_{n-1,}x_n)$$

∵ By d_4

$$d(x_n, x_{n+1}) \le d(x_n, x_{n-1}) + d(x_{n-1}, x_{n+1})$$

$$\Rightarrow \frac{d(x_n,x_{n+1})}{d(x_n,x_{n-1})+d(x_{n-1},x_{n+1})} \le 1$$

$= (\alpha+\beta)d(x_{n-1},x_{n+1}) + \beta[d(x_n,x_{n+1})] + \gamma d(x_{n-1},x_n)$

$\le (\alpha+\beta)[d(x_{n-1},x_n) + d(x_n,x_{n+1})] + \beta[d(x_n,x_{n+1})] + \gamma d(x_{n-1},x_n)$

$\Rightarrow d(x_n,x_{n+1}) \le \alpha d(x_{n-1},x_n) + 2\beta d(x_n,x_{n+1}) + \beta d(x_{n-1},x_n) + \alpha d(x_n,x_{n+1}) + \gamma d(x_{n-1},x_n)$

$= \alpha d(x_{n-1},x_n) + (\alpha+2\beta)d(x_n,x_{n+1}) + (\beta+\gamma)d(x_{n-1},x_n)$

$\Rightarrow d(x_n,x_{n+1}) - (\alpha+2\beta)d(x_n,x_{n+1}) \le (\alpha+\beta+\gamma)d(x_{n-1},x_n)$

$\Rightarrow (1-\alpha-2\beta)d(x_n,x_{n+1}) \le (\alpha+\beta+\gamma)d(x_{n-1},x_n)$

$$d(x_n,x_{n+1}) \le \frac{\alpha+\beta+\gamma}{1-\alpha-2\beta} d(x_{n-1},x_n)$$

Where $\lambda = \frac{\alpha+\beta+\gamma}{1-\alpha-2\beta}$, $0 \le \lambda < 1$

$$\Rightarrow \frac{\alpha+\beta+\gamma}{1-\alpha-2\beta} < 1$$

$\Rightarrow \alpha+\beta+\gamma < 1-\alpha-2\beta$

$\Rightarrow 2\alpha+3\beta+\gamma < 1$

Hence $d(x_n,x_{n+1}) \le \lambda d(x_{n-1},x_n)$ ………………… (3.2)

Similarly $d(x_{n-1},x_n) \le \lambda d(x_{n-2},x_{n-1})$

$\Rightarrow d(x_n,x_{n+1}) \le \lambda.\lambda d(x_{n-2},x_{n-1})$ by using 3.2

$\Rightarrow d(x_n,x_{n+1}) \le \lambda^2 d(x_{n-2},x_{n-1})$

and $d(x_n,x_{n+1}) \le \lambda^3 d(x_{n-3},x_{n-2})$

$d(x_n,x_{n+1}) \le \lambda^4 d(x_{n-4},x_{n-3})$

Continuing in this way. We have

$d(x_n,x_{n+1}) \le \lambda^n d(x_0,x_1)$

$\because 0 \le \lambda < 1$

and as $n \to \infty$ $d(x_n,x_{n+1}) \to 0$

Similarly we show that $d(x_{n+1},x_n) \to 0$

Hence $<x_n>$ is a Cauchy sequence in a complete dislocated quasi-metric space (X,d) So there exist $u \in X$ such that $< x_n >$ converges to u in dislocated quasi-metric space.

i.e. $\lim_{n\to\infty} x_n = u$ ……………………….(3.3)

Now $d(u,T_j u) \le d(u,x_n) + d(x_n,T_j u)$ by d_4

$= d(u,x_n) + d(T_i x_{n-1},T_j u)$

$\le d(u,x_n) + \alpha \frac{(x_{n-1}T_j u).d(u,T_j u)}{d(T_i x_{n-1},x_{n-1})+d(x_{n-1}T_j u)} + \beta[d(x_{n-1},T_j u) + d(u,T_j u)] + \gamma d(x_{n-1},u)$ … (3.1)

$\to 0$ as $n \to \infty$ by (3.3)

$\Rightarrow$ d(u, $T_j u$) $\to 0$ as $n \to \infty$

$\Rightarrow$ u is a fixed point of T_j

Similarly we can prove that u is a fixed point of $T_{i.}$

$d(T_i u,u) \to 0$ as $n \to \infty$

Hence we have proved that u is a common fixed point of $T_{i.}$ and $T_{j.}$

Uniqueness – Let u and v are fixed point of $T_{i.}$ and $T_{j.}$ Such that $T_i u = u$ and $T_j v = v$

Then $d(u,u) = d(T_i u,T_i u)$

$$\le \alpha \frac{d(u,T_i u)d(u,T_i u)}{d(T_i u,u)+d(u,T_i u)} + \beta[d(u,T_i u) + d(u,T_i u)] + \gamma d(u,u)$$

$= \alpha \frac{d(u,u)d(u,u)}{d(u,u)+d(u,u)} + \beta[d(u,u) + d(u,u)] + \gamma d(u,u)$

$= \alpha \frac{[d(u,u)]^2}{2d(u,u)} + 2\beta d(u,u) + \gamma d(u,u)$

$\leq \alpha d(u,u) + 2\beta d(u,u) + \gamma d(u,u)$

$= (\alpha + 2\beta + \gamma)d(u,u)$

$\Rightarrow d(u,u) - (\alpha + 2\beta + \gamma)d(u,u) \leq 0$

d(u, u)[1-$(\alpha + 2\beta + \gamma)$] ≤ 0

$\Rightarrow d(u,u) = 0$ ……………………(3.4)

Thus d(u, u)=0 for a fixed point u of T_i.

Similarly we get $d(v,v) = 0$ for a fixed point v of T_j ……………….(3.5)

Now

$d(u,v) = d(T_i, T_j v)$

$\leq \alpha \frac{d(u,T_j v)d(v,T_j v)}{d(T_i u,u)+d(u,T_j v)} + \beta\left[d\left(u, T_j v\right) + d\left(v, T_j v\right)\right] + \gamma d(u,v)$

$= \alpha \frac{d(u,v)d(v,v)}{d(u,u) + d(u,v)} + \beta[d(u,v) + d(v,v)] + \gamma d(u,v)$

$\leq \alpha d(v,v) + \beta[d(u,v)] + \gamma d(u,v) + \beta d(v,v)$

By using d_4

$d(u,v) \leq d(u,u) + d(u,v)$

$= (\alpha + \beta)d(v,v) + (\beta + \gamma)d(u,v)$

$\Rightarrow [1 - (\beta + \gamma)d(u,v) \leq (\alpha + \beta)d(v,v)$

$d(u,v) \leq \frac{\alpha + \beta}{1 - (\beta + \gamma)} d(v,v)$

$d(u,v) \leq 0$ …………(3.6) By (3.5) and $\because \frac{\alpha+\beta}{1-(\beta+\gamma)} < 1$

Similarly

$d(v,u) \leq \frac{\alpha + \beta}{1 - (\beta + \gamma)} d(u,u)$

$\Rightarrow d(v,u) \leq 0$ ………… (3.7) By (3.5)

Hence $|d(u,v) - d(v,u)| \leq \left|\frac{\alpha+\beta}{1-(\beta+\gamma)}\right| |d(v,v) - d(u,u)| \leq 0$

By 3.4 and $\because \frac{\alpha+\beta}{1-(\beta+\gamma)} < 1$

$\Rightarrow |d(u,v) - d(v,u)| = 0$ $\because$ Modulus is not negative

$d(u,v) = d(v,u)$ ……………..(3.8)

From (3.6), (3.7) and (3.8)

$d(u,v) = d(v,u) = 0$

Then $u = v$ by d_2

Hence fixed point is unique

Reference :

[1] A. Isufati Fixed point Theorem in Dislocated Quasi-Metric Space, Applied Math, Sci., 4(5), 217-223, (2010).

[2] C.T. Aage and J.N. Salunke, The Result on fixed point Theorem in Dislocated and Dislocated-Metric Space, Applied Math, Sci, 2(59) 2941, 2948, (2008).

[3] D.S. Jaggi. Some Unique fixed point theorems. Indian J. Pure Appl. Math, 8(2): 223-230, (1977).

[4] F. M. Zeyada, G.H. Hassan and M.A. Ahmed, A Generalization of a fixed point theorem due to Hitzler and seda in dislocated quasi-metric spaces. The Arabian J. Sci. Engg., 31(1A) 111-114, (2006).

[5] Kastriot Zoto, Elida Hoxha and Arben Isufati, Some New Results in Dislocated and Dislocated Quasi-Metric Spacs, Applied Mathematical Science,6(71), 3519-3526 (2012).

[6] P. Hitzler, Generalized Metric and Topology in Logic Programming Semantics, Ph.D. Thesis, National University of Ireland, (University College, Cork), (2001).

[7] P. Hitzler and A.K. Seda, Dislocated Topplogies, J. Electr. Engg., 51(12/s), 3-7, (2000).

[8] S. Banach, "Sur les operations dans les ensembles abstraits et leur applications aux equations integrals," Fundamental Mathematicae,3(7), pp. 133–181, (1922).