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RESEARCH ARTICLE

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ON CURVATURE BOUNDS OF THE SOLUTION SURFACE TO A HESSIAN EQUATION IN THE BALL

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ABSTRACT

Curvature bounds of the solution surface to a Hessian equation is considered in this Note. It is proved that some power of the smooth admissible solution to the Hessian equation is strictly convex in the ball. Upper and lower bound curvature estimates are also given.

Key words and phrases: admissible solution; Hessian equation; power convexity; strict convexity.

Math Subject Classification: 35J20, 35J60.

1. INTRODUCTION

Convexity of solutions to partial differential equations is an interesting issue and has been investigated for a long time. One interesting question in the study of convexity is the following: Is there a monotone real function f, such that the function of solution $f(u(x))$ is concave or convex. If f is a power function, we call the solution u has power convexity property. For this question, A typical example is that in 1971, Makar-Limanov [1] considered the following elliptic boundary value problem

$$\begin{cases} \Delta u=-1 & in \quad \Omega, \\ u=0 & on \quad \partial\Omega. \end{cases}$$

in a bounded and convex planar domain Ω. By an ingenious argument involving the maximum principle, he proved that the square root $u^{\frac{1}{2}}$ of the solution u is strictly concave. Another well-known example is that in 1976, Brascamp-Lieb [2] used a probabilistic approach to establish the log-concavity of the fundamental solution of diffusion equation with convex potential in a bounded and convex domain in R^n. Consequently, they proved the log-concavity of the first eigenfunction of the Laplacian operator in convex domain. Similar to linear equations, the same kind phenomena appears for the fully nonlinear operators. One typical result is that Ma-Xu [3] considered the smooth admissible solution u of the following Hessian equation

$$\begin{cases} \sigma_2(D^2u)=1 & in \quad \Omega\subset R^3, \\ u=0 & on \quad \partial\Omega, \end{cases}$$

where and in the following, D^2u means the Hessian of u and $\sigma_2(D^2u)$ denotes the Hessian operator which is exactly the second elementary symmetric function of the eigenvalues of D^2u. Under the assumption $\Omega\subset R^3$ be a bounded and strictly convex domain, they proved that the function $v=-(-u)^{\frac{1}{2}}$ is strictly convex and then they gave an example to illustrate the sharpness of the convexity index $\frac{1}{2}$. Another interesting result is that in 2010, Liu-Ma-Xu [4] considered the following eigenvalue problem for the Hessian operator in a bounded and strictly convex domain

$$\begin{cases} \sigma_2(D^2u)=\lambda(-u)^2 & in \quad \Omega\subset R^3, \\ u=0 & on \quad \partial\Omega. \end{cases}$$

They obtained the strict logarithmic concavity of the eigenfunction. As an application, they get Brunn-Minkowski inequality for the Hessian eigenvalue and characterize the equality case. In addition to these works, Salani [5] reconsidered power convexity for the Hessian operator in $3-$ dimensional convex domain and he gave a unified proof by using a macroscopic technique. Ye [6] also has some generalizations of power convexity concerning a general class of Hessian equations in $3-$ dimensional convex domain. However, all these results are done when the domains are in three dimensions.

A question naturally ask whether the phenomena of power convexity holds for the general Hessian operator σ_k and for the domain being in general dimensions. It is also asked whether there exist quantitative convexity estimates for the solutions of these Hessian equations. To study problems of these kinds are interesting and important but difficult. In this paper, we consider these problems for a special simple case–a Hessian equation when the domain is $n-$ dimensional ball.

We consider convexity estimates for the admissible solution to the following Hessian equation in the ball $B_R(o)\subset R^n$:

$$\begin{cases} \sigma_k(D^2u)=C_n^k & in \quad B_R(o)\subset R^n, \\ u<0 & in \quad B_R(o), \\ u=0 & on \quad \partial B_R(o), \end{cases} \qquad (1)$$

where $1\le k\le n$.

For the above equation (1), we first derive an explicit expression for the admissible solution.

Proposition 1. Let $B_R(o)$ be the ball in R^n with radius $R>0$ centering at o and $0\le k\le n$. If u is the admissible solution of equation (1) in $B_R(o)$, then the admissible solution has the form $u(x)=\frac{1}{2}(|x|^2-R^2)$, where $|x|=\sqrt{x_1^2+x_2^2+\cdots+x_n^2}$.

Convexity estimates and curvature estimates are interesting problems for nonlinear partial differential equations. Ma-Shi-Ye [7] considered convexity estimates for the solutions of two elliptic equations involving the Laplacian operator and obtain lower bound Gaussian estimates for the corresponding function of the solution. Recently, Shi-Ye [8] considered convexity estimates for a class of semi-linear elliptic equations. We are more interested in upper and lower bound curvature estimates for the solutions of the fully-nonlinear elliptic equations. There are many difficulties in the study of general fully-nonlinear case. However, since we have an explicit expression for the solution to problem (1), we can compute the principal curvatures and the Gaussian curvature of the functions of the solution and therefore obtain the following Theorem concerning the quantitative convexity estimates.

Theorem 1. Let u be the smooth admissible solution of problem (1). Then for the graph corresponding to the function $v=-(-u)^{\frac{1}{2}}$, we denote $\kappa(x)$ and $K_G(x)$ the principal curvature and the Gaussian curvature of the graph v at x respectively. Then we have

(1) Upper bound and lower bound principal curvature estimates

$$\frac{1}{\sqrt{2}R}\le\kappa(x)\le\frac{2}{R}.$$

(2) Upper bound and lower bound Gaussian curvature estimates

$$2^{-\frac{n}{2}}R^{-n}\le K_G(x)\le 2R^{-n}.$$

The key point in the proof of Theorem 1 is to derive formulas of the principal curvatures and the Gaussian curvatures of the graph v and then derive the desired curvature estimates.

Using lower bound principal curvature estimates in Theorem 1, we also get the following corollary which is a quantitative version of the strict power convexity of problem (1).

Corollary 1. Let u be the smooth admissible solution to problem (1), then the function $v=-(-u)^{\frac{1}{2}}$ is strictly convex in the ball $B_R(o)$.

The paper is organized as follows. In section 2, we give the radially symmetric form of the Hessian operator and then derive the specific solution to the Hessian equation (1) in the ball. In section 3, we calculate and then give the upper and lower bound estimates for the principal curvatures and the Gaussian curvature of the surface corresponding to the function $v=-(-u)^{\frac{1}{2}}$. In section 4, we give some final remarks concerning our further studies.

We will use the definitions of elementary symmetric function and curvature formulas for the surfaces during the proving process. These definitions and formulas are standard, the readers can consult them on other reference books, such as [9], [10] and [11], etc..

2. Proof of Proposition 1

In this section, we derive an explicit expression for the solution to problem (1). First we give a Lemma regarding the radial form of the Hessian operator.

Lemma 1. If $u: B_R(o)\to R$ is radially symmetric, then the Hessian operator σ_k takes the following form

$$\sigma_k(D^2u)=\frac{1}{k}C_{n-1}^{k-1}r^{-n+1}[r^{n-k}(u')^k]',$$

where C_{n-1}^{k-1} is combinatorial coefficient.

Proof of the Lemma. The Lemma is well-known, but for the convenience of the reader, we still give a short proof here.

Since u is radially symmetric, we may assume

$$u(x)=u(|x|)=u(r),\qquad \text{for}\quad |x|=r,$$

For $1\le i,j\le n$, we take the first and second derivatives of r with respect to the variables x_i and x_j to get

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r},$$

$$\frac{\partial^2 r}{\partial x_i \partial x_j} = -r^{-3} x_i x_j + r^{-1}\delta_{ij}.$$

Then from the above, it follows that

$$u_{ij} = [u''(r) r^{-2} - u'(r) r^{-3}] x_i x_j + u'(r) r^{-1} \delta_{ij}.$$

Therefore, at the point $x = (r, 0, \cdots, 0)$, the Hessian matrix $D^2 u$ is diagonal with $u_{11} = u''(r)$ and $u_{ii}(r) = \frac{u'(r)}{r}$, for $2 \le i \le n$. Since the Hessian operator σ_k is rotationally invariant, it follows that

$$\sigma_k(D^2 u) = C_{n-1}^{k-1} u'' (\frac{u'}{r})^{k-1} + C_{n-1}^{k} (\frac{u'}{r})^{k} = \frac{1}{k} C_{n-1}^{k-1} r^{-n+1} [r^{n-k} (u')^k]'. \square$$

Now we begin to solve the equation and derive an explicit expression for the solution to problem (1).

Proof of Proposition 1.

Since the domain of the problem is the ball $B_R(o)$ with radius R and center o, then using the results of Chou-Wang [12] or Tso [13], we know that the equation (1) has a unique negative admissible solution $u \in C^\infty(B_R) \bigcap C^{1,1}(\overline{B_R})$ which is radially symmetric.

Using the radial form of the Hessian operator in the above Lemma, we rewrite the equation (1) as

$$\begin{cases} \frac{1}{k} C_{n-1}^{k-1} r^{-n+1} [r^{n-k} (u')^k]' = C_n^k \quad in \quad r \in [0, R), \\ u(r) < 0 \qquad in \qquad r \in [0, R), \\ u(r) = 0 \qquad at \quad r = R, \end{cases} \tag{2}$$

where $1 \le k \le n$.

The first equation of problem (2) is equivalent to

$$[r^{n-k} (u')^k]' = \frac{k C_n^k}{C_{n-1}^{k-1}} r^{n-1} = n r^{n-1}.$$

Integrating the variable r from 0 to s, we obtain

$$s^{n-k} [u'(s)]^k = s^n + C_1, \tag{3}$$

where C_1 is a constant to be determined. Since u is rotationally symmetric, then we know that $u'(0) = 0$. Substituting it into equation (3), we get $C_1 = 0$ and therefore we get

$$u'(s) = s. \tag{4}$$

Then integrating the variable s from 0 to r, we finally obtain $u(r) = \frac{r^2}{2} + C_2$, where C_2 is also a constant to be determined. Noting the condition that $u(R) = 0$ in problem (2), we get $C_2 = -\frac{R^2}{2}$. Therefore the admissible solution to problem (1) is

$$u(x) = \frac{1}{2}(x_1^2 + x_2^2 + \cdots + x_n^2 - R^2) = \frac{1}{2}(|x|^2 - R^2),$$

where we denote $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. Hence we have completed the proof of **Proposition 1**. $\square$

3. Proof of Theorem 1

In this section, we prove Theorem 1 regarding the $\frac{1}{2}$-power convexity of the smooth admissible solution to problem (1) by computing the principal curvatures of the graph with respect to the function $v=-(-u)^{\frac{1}{2}}$.

We know from Proposition 1 that $u=\frac{|x|^2-R^2}{2}$ is the unique admissible solution to problem (1). We now prove that the function $v=-\sqrt{-u}=-\sqrt{\frac{R^2-|x|^2}{2}}$ is strictly convex by computing the principal curvatures of the surface corresponding to the function v. To calculate the principal curvatures, we take the first and second derivatives of v with respect to x_i and x_j to get

$$v_i=\frac{x_i}{\sqrt{2(R^2-|x|^2)}},$$

$$v_{ij}=\frac{1}{\sqrt{2(R^2-|x|^2)}}(\delta_{ij}+\frac{x_i x_j}{R^2-|x|^2}).$$

If we write the position vector of the graph v as $X=(x,v(x))$, $x\in R^n$, then the tangent vector and the unit normal vector of the graph are given by

$$X_i=(0,\cdots,0,1,0,\cdots,0,\frac{x_i}{\sqrt{2(R^2-|x|^2)}}), \text{the } i-\text{th component being } 1, \text{ for } \quad 1\le i\le n$$

and

$$\vec{n}=(-\frac{x_1}{\sqrt{2R^2-|x|^2}},\cdots,-\frac{x_n}{\sqrt{2R^2-|x|^2}},\frac{\sqrt{2(R^2-|x|^2)}}{\sqrt{2R^2-|x|^2}})$$

respectively. We also obtain

$$X_{ij}=(0,\cdots,0,\frac{1}{\sqrt{2(R^2-|x|^2)}}(\delta_{ij}+\frac{x_i x_j}{R^2-|x|^2})).$$

Therefore the first fundamental form and the second fundamental form of the graph are given by

$$g_{ij}=<X_i,X_j>=\delta_{ij}+\frac{1}{2(R^2-|x|^2)}x_i x_j$$

and

$$h_{ij}=<X_{ij},\vec{n}>=\frac{2}{\sqrt{2R^2-|x|^2}}(\delta_{ij}+\frac{x_i x_j}{R^2-|x|^2})$$

respectively.

We denote the inverse matrix of $\{g_{ij}\}$ by $\{g^{ij}\}$. By direct computation, we know that

$$g^{ij}=\delta_{ij}-\frac{x_i x_j}{2R^2-|x|^2}$$

and hence the shape operator of the graph v is given by

$$h_j^i=\sum_{k=1}^n g^{ik}h_{kj}=\frac{2}{\sqrt{2R^2-|x|^2}}(\delta_{ij}+\frac{x_i x_j}{2R^2-|x|^2}).$$

By calculation, we obatain that the principal curvatures of the graph v which are the eigenvalues of the shape operator are $\frac{1}{\sqrt{2R^2-|x|^2}}$ with n-1 multiplicities and $\frac{2R^2}{(2R^2-|x|^2)^{\frac{3}{2}}}$.

In the following calculations, we give upper and lower bound estimates for the principal curvatures and the Gaussian curvatures. Since we have the following comparison of the principal curvatures:

$$\frac{2R^2}{(2R^2-|x|^2)^{\frac{3}{2}}}=\frac{2R^2}{(2R^2-|x|^2)}\frac{1}{\sqrt{2R^2-|x|^2}}\geq\frac{1}{\sqrt{2R^2-|x|^2}},$$

then we obtain the upper and lower bound estimates for the principal curvatures of the graph:

$$\frac{1}{\sqrt{2}R}\leq\kappa(x)\leq\frac{2}{R},$$

where $\kappa(x)$ denotes arbitrary principal curvature of the graph at x.

Since all the principal curvatures are positive, we conclude that the graph $v=-(-u)^{\frac{1}{2}}$ is strictly convex and the proof of **Corollary 1** follows.

We further compute the Gaussian curvature of the graph v by multiplying all the principal curvatures and obtain

$$K_G(x)=(\frac{1}{\sqrt{2R^2-|x|^2}})^{n-1}\cdot\frac{2R^2}{(2R^2-|x|^2)^{\frac{3}{2}}}=2R^2(2R^2-|x|^2)^{-\frac{n+2}{2}}.$$

Therefore, we also obtain upper and lower bound estimates for the Gaussian curvature K_G, i.e.,

$$2^{-\frac{n}{2}}R^{-n}\leq K_G(x)\leq 2R^{-n}.$$

Therefore we have completed the proof of Theorem 1 and Corollary 1.

4. Some final remarks

In this paper, we consider convexity estimates of a Hessian equation in the n-dimensional ball and obtain upper and lower bound estimates for the principal curvatures and the Gaussian curvature. However, for the corresponding results of general Hessian operator and general convex domains, we meet some technical difficulties. However, we believe that the same kind results still hold true. For example, if we assume u be the admissible solution to the following problem

$$\begin{cases}\sigma_k(D^2u)=\lambda(-u)^p & in \quad B_R(o)\subset R^n,\\ u<0 & on \quad B_R(o),\\ u=0 & on \quad \partial B_R(o),\end{cases}$$

where $1\leq k<n$ and $1\leq p\leq k$, then can prove the function $v=-(-u)^{\frac{k-p}{2k}}$ is strictly convex in $B_R(o)$. For general 3-dimensional convex domain, we can also prove the same result. In our further studies, we will focus on the study of convexity and curvature estimates for general Hessian equations and general convex domain. These kind of convexities and convexity estimates are of great interest in the study of fully-nonlinear elliptic equations. Finding their various applications of the convexities will help us understanding of the geometry of the solution surfaces.

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