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RESEARCH ARTICLE

ON THE RELATIONSHIP BETWEEN THE HEAT EQUATION, BLACK-SCHOLES MODEL AND CONTRIBUTORY PENSION PRICING

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ABSTRACT

This paper examines the relationship between heat equation and the Black-Scholes model. It shows from both intuitive reasoning and mathematical proof, a derivation of the Black-Scholes model from the combustion equation. It further places the relationship of underlying assets as a two-part diffusion equation having a constant drift and a random shock part. Amount change in the price is equal to the certainty of movement of the price plus the uncertainty caused by the volatility. From here a simulation of a pension pricing from Monte Carlo method of forecasting asset prices was generated and further modelled to have Retirement Savings Account(RSA) price as an underlying value to create an options product deemed suitable to resolve the need of including the Nigerian unorganised private sector into on-going contributory pension scheme. This recommendation doubles to increasing well desired financial inclusion through pension in Nigeria.

Keywords: Combustion equation, Black-Scholes model, Pension Pricing, Monte Carlo method of stock price forecasting, contributory pension scheme,Unorganised private sector, Financial inclusion.

INTRODUCTION

A pension portfolio contains a mix of both fixed and variable income securities including equity, money market securities, bonds treasury bills, open-end funds etc. The diversification to a large essence is to optimise portfolio performance of scarce resources called Asset Under Management (AUM). In Egbe (2013),Pension unit price is a function of the performance of the underlying securities that make up the portfolio. It can then be simulated as a derivative of the underlying stocks for a future expansion of the market for the trading of pension asset. This may be more useful for the informal market of Nigeria’s economy.

In this paper, before simulating pension pricing to Black-Scholes options pricing, which economy is also central on risky asset and riskless fixed income asset (see Black and Scholes, 1973),we first go through the derivation of the Black -Scholes formula from the intuition of the heat equation.

The heat equation

It is a well-known fact that the heat equation, also known as the diffusion equation, describes in typical application the evolution in time of density U of some quantity such as heat, chemical concentration, and etc. if $V \subset U$ is any smooth sub region, the rate of change of the total quantity within V equals the negative of the net flux through ∂V:

$$\frac{d}{dt}\int_V Udx = -\int_{\partial V} F.vds,$$

Fbeing the flux density. Thus,

$$U_t = -divF, \quad (1)$$

as$V$was arbitrary. In many situations F is proportional to the gradient of U, but points in the opposite direction since the flowis from regions of higher to lower concentration (see figure 1 below):

$$F = -aDU \ \ (a > 0). \quad (2)$$

Substituting (2) into (1), one obtains the partial differential equation

$$U_t = adiv(DU) = a\Delta U, \quad (3)$$

where for $a = 1$ is the heat equation.

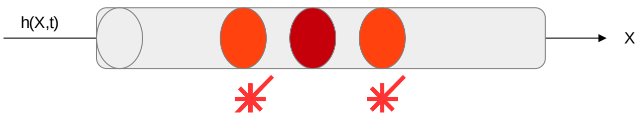

Figure 1: heat flow

The fundamental solution of the heat equation is the function;

$$\emptyset(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\pi/2}} e^{-\frac{|x|^2}{4t}} & (x \in R^n, t > 0) \\ 0 & (x \in R^n. \ t > 0) \end{cases}.$$

The heat equation appears as well in the study of Brownian motion. Relative to finance, a price K in the future is set up, what is the value of right which promise a holder of that right at the future the ability to buy or sell the asset at cost K when the asset is higher than K? This is Black-Scholes related. The famous Black-Scholes equation is the equation of the type;

$$\frac{\partial V}{\partial t} + (r-a)S\frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 = rV. \quad (4)$$

In the Black-Scholes equation, the symbols represent these variables: σ = volatility of returns of the underlying asset/commodity; S = its spot (current) price; a = rate of change; V = price of financial derivative; r = risk-free interest rate; t = time. The Black-Scholes equation was the mathematical justification for the trading that plunged the world's banks into catastrophe. It was the holy grail of investors. The Black-Scholes equation, brainchild of economists Fischer Black and Myron Scholes, provided a rational way to price a financial contract when it still had time to run. It was like buying or selling a bet on a horse, halfway through the race. It opened up a new world of ever more complex investments, blossoming into a gigantic global industry. But when the sub-prime mortgage market turned sour, the darling of the financial markets became the Black Hole equation, sucking money out of the universe in an unending stream.

Anyone who has followed the crisis will understand that the real economy of businesses and commodities is being upstaged by complicated financial instruments known as derivatives. These are not money or goods. They are investments in investments, bets about bets. Derivatives created a booming global economy, but they also led to turbulent markets, the credit crunch, the near collapse of the banking system and the economic slump. And it was the Black-Scholes equation that opened up the world of derivatives.A derivative is often based on a function. It coincides with the value of the option cost at time $t = 0$. It also coincides with the promised profit at the future time $t = T$. The profit is$x - K$ when x is greater than K. Figure 2 shows the position of in-the-money derivative, while figure 3 shows the common characteristics of heat flow and financial derivative.

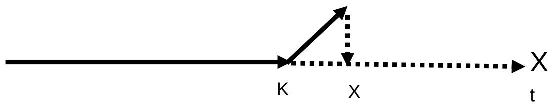

Figure 2: position of in-the-money derivative

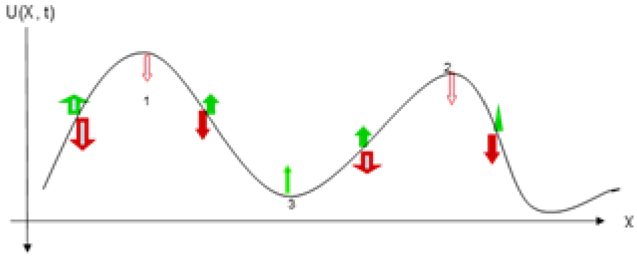

Figure 3: Common Characteristics (Heat and stock)

From figure 3 above, positions 1 & 2 are positions at which any side you move, temperature reduces. Position 3 guarantees temperature increase either way you move. Other positions may give higher or lower temperature depending on the direction of movement. We recall optimization local minimum and local maximum. In Egbe(2013) these are points on which define lowest points guaranteeing next upside and downside movements respectively. It follows that the number and arrow positions represent a kind of concavity of the temperature profile. Then we can say that, change in temperature is proportional to the concavity of temperature. The Black-Scholes equation was useful, it was precise, and its limitations were clearly stated. It provided an industry-standard method to assess the likely value of a financial derivative. So derivatives could be traded before they matured. The formula was fine if you used it sensibly and abandoned it when market conditions weren't appropriate. Black-Scholes underpinned massive economic growth. By 2007, the international financial system was trading derivatives valued at one quadrillion dollars per year. This is 10 times the total worth, adjusted for inflation, of all products made by the world's manufacturing industries over the last century. The downside was the invention of ever-more complex financial instruments whose value and risk were increasingly opaque. So companies hired mathematically talented analysts to develop similar formulas, telling them how much those new instruments were worth and how risky they were. Then, disastrously, they forgot to ask how reliable the answers would be if market conditions changed.

The relationship between the heat equation and the Black-Scholes Model

The idea behind many financial models goes back to Louis Bachelier in 1900, who suggested that fluctuations of the stock market can be modelled by a random process known as Brownian motion. At each instant, the price of a stock either increases or decreases, and the model assumes fixed probabilities for these events. They may be equally likely, or one may be more probable than the other. It's like someone standing on a street and repeatedly tossing a coin to decide whether to move a small step forwards or backwards, so they zigzag back and forth erratically. Just like the increase and decrease of temperature in figure 3, their position corresponds to the price of the stock, moving up or down at random. The most important statistical features of Brownian motion are its mean and its standard deviation. The mean is the short-term average price, which typically drifts in a specific direction, up or down depending on where the market thinks the stock is going. The standard deviation can be thought of as the average amount by which the price differs from the mean, calculated using a standard statistical formula. For stock prices this is called volatility, and it measures how erratically the price fluctuates. On a graph of price against time, volatility corresponds to how jagged the zigzag movements look.

Let $S(t)$ be the stock price of some assets at a specified time t, and μ, an expected rate of returns on the stock, and dt as the return or relative change in the price during the period of time. The dynamics of the stock price is:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \quad (5)$$

where$\mu S_t dt and \sigma S_t dW_t$ are the predictable and the unpredictable parts (respectively) of the stock return (Adeosun, et al, 2015). The solution to the StochasticDifferential Equation (SDE) (5) is given as;

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}, \quad (6)$$

with$W(t)$a standard Brownian motion on the probability space(Ω, β, μ). $\beta = \sigma$is the algebra generated by $W(t)$.

According to the properties of standard Brownian motionprocess for $n \geq 1$ any sequence of time$0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$, therefore by Euler's methodof discretization of the SDE, we have:

$$\ln S(t) - \ln S(t-1) = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma(W(t) - W(t-1)). \quad (7)$$

Lognormal distribution of stock-returns assumes that logarithmic stock-returns are normally distributed, that is;

$$\ln\left(\frac{s_{t+\Delta_t}}{s_t}\right) = \mu\Delta_t + \sigma\sqrt{\Delta_t}W_t, \quad (8)$$

with solution

$$S_{t+\Delta_t} = S_t e^{\mu\Delta_t + \sigma\sqrt{\Delta_t}W_t}. \quad (9)$$

In what follows, we state:

Theorem 1: Let $S(t)$ be a stochastic process which follows the SDE (4) with $\mu = 0$, then Black-Scholes equation related to the Heat Equation is given as;

$$\frac{\partial \bar{V}}{\partial t} + \frac{1}{2}\frac{\partial^2 \bar{V}}{\partial \bar{S}^2}\sigma^2 \bar{S}^2 = 0, \quad (10)$$

with solution

$$V(S,t) = V_0 e^{-rt}\left(e^{iyS\,e^{rt} - |y^2|\frac{e^{2rt}S^2\sigma^2}{2}t}\right). \quad (11)$$

Proof: In order to remove the effect of the discount rate (r) from (4), we let $\mu = 0$ and set

$$V = e^{-rt}\bar{V} \quad (12)$$

and

$$S = e^{-rt}\bar{S}. \quad (13)$$

A little calculation using (13) and (14) and substituting into (4) yields (10).

For the solution of (10), let us consider complex wave solutions of the form (Evans, 1997);

$$\bar{V}(\bar{S},t) = e^{i(y\bar{S} + \omega t)}. \quad (14)$$

Here,$\omega \in \mathbb{C}$ and $y = (y_1, y_2, \ldots, y_n) \in R^n$,$\omega$being the frequency and $\{y_i\}_i^n$ the wave numbers. It is not difficult to see that

$$\frac{\partial \bar{V}}{\partial t} + \frac{1}{2}\frac{\partial^2 \bar{V}}{\partial \bar{S}^2}\sigma^2 \bar{S}^2 = \left(i\omega + |y^2|\frac{\bar{S}^2\sigma^2}{2}\right)\bar{V} = 0,$$

provided $\omega = i|y^2|\frac{\bar{S}^2\sigma^2}{2}$. It follows that

$$\bar{V}(\bar{S},t) = e^{i(y\bar{S}-|y^2|\frac{\bar{S}^2\sigma^2}{2}t)}, \tag{15}$$

Solves the diffusion equation (4) for each $y \in R^n$. Taking the real and imaginary parts, we discover further that $e^{-|y^2|\frac{\bar{S}^2\sigma^2}{2}t}\cos(y\bar{S})$ and $e^{-|y^2|\frac{\bar{S}^2\sigma^2}{2}t}\sin(y\bar{S})$ are solutions as well.Plucking (15) into (12) and using (13) gives (11).

The Contributory Pension.

Generally, in Cochrane (1990),Pension is an amount of money paid regularly by a government or company to somebody, who is considered to be too old or too ill to work. Pension Scheme is a system in which someone and usually the employer pays money regularly into a pension fund while the employee is still under employment. This qualifies the employee to be paid pension after retirement. A pension scheme may be the system in which an employer pays money regularly into a pension fund while also committing the employee to pay some money into the same pension fund. The fund forms the aggregate of what the employee gets on retirement.

The system of populating the pension fund prior to employee's retirement is necessary as the basis for the scheme's survival. Reward systems are different sometimes based on the pension fund accumulation system. There are common factors defining reward systems in pension schemes which include the following: Annual income, Employment status (wage or salary), Age at employment Specified retirement age and Minimum years of service.

In Nigeria before 2004, we have practiced two regimes of pension scheme which are:

the traditional system where government and companies pay an amount of money regularly into a pension fund in trust for their employees until retirement. The retirees earn certain ratio of their last salary monthly up till their demise. E.g. the Federal Government Pension Bond.

The system where government and companies pay an amount of money regularly into a pension fund which also attract some regular payment of a fraction of the employee's monthly salary to be kept in trust by a government authority for the employees' retirement. E.g. Nigerian Social Insurance Trust Fund (NSITF)

The fully funded contributory pension scheme succeeds the first and second regimes of the practiced Pension Schemes in Nigeria to achieve the following objective below;

To ensure that every person who worked in either the public service of the federation, Federal Capital Territory or Private Sector service receives his retirement benefits as and when due.

Assist improvident individuals by ensuring that they save in order to cater for their livelihood during old age

To establish a uniform set of rules, regulations and standards for the administration and payments of retirement benefits for the public service of the Federation, Federal Capital Territory and the private sector

As Nigeria faces the challenges of meeting the Millennium Development Goals with all the poverty related great regional, sectarian and gender disparities. The realization that greater number of retired people suffer from late payment of pension and gratuity to absolute non-payment of pension, worse still, many private sector institutions do not have a pension scheme or may have a non-functional scheme. From corruption and poverty perception, most private sector employers program termination of employees some months to retirement age to avoid the payment of pension. Many private sector led companies and government owned companies alike liquidate and cause great frustration to employees retired and about to retire before the companies' liquidation.

The general short falls in the previous pension schemes led to the present fully funded contributory pension scheme which basic formula is that the employer pays into the pension account a minimum of the employees 10% of his/her personal salary which is made up of the following:
Basic salary, Housing allowance and Transport allowance.
The employee is also required to contribute 8% of his/her pensionable salary to the same pension fund account to make a total of 18% minimum of the pensionable salary.

The Contributory Pension Pricing.

A pension fund administrator's asset under management and invested in diversified nature is made out from the following contributions; Retirement Savings Account (RSA), Voluntary Contribution Account (VCA) and Accumulated Pension Account (Legacy Fund).
The aggregate of all funds from the sources of assets under management is viewed currently as follows:

1) Total Market Value of Securities based on the NSE daily official list as at the date of valuation,
2) Un-invested cash,
3) Un-distributed income to date less allowable expenses (asset based fees) and
4) Total value of money instruments.

Then the Net Asset Value (NAV) is calculated by summing items (1) to (4) and deducting the following:

I. Brokerage Commission
II. Stamp duties
III. SEC Fees (Securities & Exchange Commission)
IV. NSE (Nigerian Stock Exchange) fees
V. CSCS Fees
VI. VAT
VII. Accrued allowable expenses.

In what follows we have:

$$NAV = \sum_{m=1}^{4} X_m - \sum_{j=i}^{vii} Y_j . \quad (16)$$

Above Pension portfolio rate of return are calculated for 36, 24, or 12 months, and then be converted into an equivalent annual rate of returns (ARR), expressed as a percentage up to four decimal places Mehra and Prescott (1985).
The calculated rate of return over a 36–month period, converted, in equivalent annual rate of return is equal to the cube root of the value of the accounting unit at the end of the period for which it is calculated, divided by the value of the accounting unit (AU) as at the beginning of the 36 months, less one and expressed as percentage.

$$AU = \frac{NAV}{Totalunitofindividual RSA}. \quad (17)$$

To calculate for 24-months and 12months the powers are ½ and 1 respectively.
The Nigerian contributory pension scheme started in 2004 and since had been exclusively for the organised private sector and the employees of government ministries, departments and agencies. There has been this issue of involving the unorganised private sector which houses the majority of the Nigerian population. The non-attractiveness of this scheme to majority of the unorganised private sector includes the following;

1. Similarity to bank savings account
2. Inability of contributor to withdraw at will when in need of savings
3. No tangible evidence of their investment

4. Historical of Nigerian pension funds and legacies of misuse
5. Lack of clear understanding of the processes of engagement

On the **National Pension Commission** part, plans and draft regulations at some times have been reviewed in attempts to include the unorganised private sector into Nigeria's contributory pension scheme. These efforts have unresolved concerns including;

1. Method of collection of contribution
2. Safety of contributors' contribution on transit if cash should be acceptable
3. Avoidance of money laundry through contributions
4. How to trace income sources to avoid money laundry?
5. Narrowing financial exclusion etc.

It is in the light of these problems and attempts to provide a proposition through available mathematical solutions that the essence of this contribution takes its usefulness.

The corresponding portfolio value process of the NAV of equation (16) is the stochastic differential equation (SDE) written as

$$dV_t^i = V_t^i\left[\left(1-\sum_{i=1}^{n}\theta_t^i\right)\frac{dS_t^0}{S_t^0}+\sum_{i=1}^{n}\theta_t^i\frac{dS_t^i}{S_t^i}\right],\ V_0^0=v_0. \qquad (18)$$

Here

$$\frac{dS_t^0}{S_t^0}=rdt, S_0^0=s^0, r=r_t \text{ is the risk-free rate,}$$

$$\frac{dS_t^i}{S_t^i}=\mu^i dt+\sum_{i=1}^{k}\sigma^i dW_t^i,\ t\in[0,T],\ S_0^i=s^i. i=1,\dots,n.$$

The $\mathcal{F}_t$- predictable process $\theta=(\theta_t^0,\theta_t^1,\dots,\theta_t^n)$, is the investment strategy with $\theta_t^i(i=1,\dots,n)$ denotes the fraction of wealth invested in the risky asset i at time t (Akume et al, 2010),

Pension prices are the values of their underlying assets. The assets on themselves follow a decay process as theheat diffusion. We therefore view the net value asset prices as the Black-Scholes theory of the case of the multiple assets of the form;

$$S_t^{(i)}=S_0^{(i)}exp\left[\left(\mu^{(i)}-\frac{1}{2}\left[\sigma^{(i)}\right]^2\right)+\sigma W_t^{(i)}\right], t\epsilon[0,T] \qquad (19)$$

where$W_t^{(i)}, i=1,\dots,d$ are possibly correlated Brownian Motions. We may therefore concatenate to sell pension as a product of derivative to the unorganised private sector who holds and resells in a continuous process. Like this, the fears of both the commission and that of the contributors are dissolved in the control of Derivative Exchange processes.

However, we will use data (see tables 1-3below ;Source of original data: ARM Pension Managers (PFA) Limited) to forecast price movement of the prices of pension in Nigeria. We then use these prices to create options for sales and purchase to allow unorganised private sector participate through direct purchase. This enables them have corridor of selling and repurchasing/renewal.

Amount change in the price is equal to the certainty of movement of the price plus the uncertainty caused by the volatility. Therefore, we have;

$$\ln\left[\frac{S_t^{(i)}}{S_{t-1}^{(i)}}\right]=\bar{\mu}^{(i)}+\sigma W_t^{(i)}, \qquad (20)$$

where, $\boldsymbol{ln}\left[\frac{\boldsymbol{S_t^{(i)}}}{\boldsymbol{S_{t-1}^{(i)}}}\right]$are the multiple periodic return and continuously compounded,

$\bar{\mu}^{(i)}=\left(\mu^{(i)}-\frac{1}{2}\left[\sigma^{(i)}\right]^2\right)\tau$areconstant multiple drifts,$\sigma W_t^{(i)}$ multiple random shocks and the expected periodic rate of returns;

$$\mu^{(i)}=\frac{\bar{\mu}^{(i)}}{\tau}+\frac{1}{2}\left[\sigma^{(i)}\right]^2. \qquad (21)$$

Putting the emphasis on the risk quantification by the volatility can be both misleading because large risks are still looming and in addition damage profitability (Andersen and Sornette, 1999). Thus, in order to make our approach concrete, we assume that the net value asset price returns are $S_t^{(i)}$distributed according to the following probability distribution function (pdf).

$$P\left(S_t^{(i)}\right) = \frac{\lambda}{\sigma}\left(\frac{S_t^{(i)}-\mu}{\sigma}\right)^{\frac{\lambda}{2}-1} exp\left\{\left(\frac{S_t^{(i)}-\mu}{\sigma}\right)^{\lambda}\right\}, P\left(S_t^{(i)}\right) \geq 0, S_t^{(i)} \geq 0, \lambda > 0, \sigma > 0, -\infty < \mu < \infty. \quad (22)$$

λ is the shape parameter, σ is the scale parameter and μ is the location parameter of the distribution. Equation (22) is the well-known Weibull distribution with reliability function given by (Osu, 2010);

$$R\left(S_t^{(i)}\right) = exp\left\{\left(\frac{S_t^{(i)}-\mu}{\sigma}\right)^{\lambda}\right\},$$

and the Weibull negative rate function is given by;

$$H\left(S_t^{(i)}\right) = \frac{\lambda}{\sigma}\left(\frac{S_t^{(i)}-\mu}{\sigma}\right)^{\frac{\lambda}{2}-1}.$$

The mean time to failure (MTTF) directly proportional to the scale parameter is (see Osu, 2010)

$$\bar{S}_t^{(i)} = \mu + \sigma r\left(\frac{1}{\lambda}+1\right), \quad (23)$$

with standard deviation;

$$v = \sqrt{\sigma^2 r\left(1+\frac{2}{\lambda}\right) - \left[r\left(1+\frac{1}{\lambda}\right)\right]^2}, \quad (24)$$

where $r(n) = \int_0^\infty e^{-x}x^{n-1}dx$.

Notice that the parameters σ and λ can be estimated using (22) and (23), i.e.

$$\bar{S}_t^{(i)} - \mu = \sigma r\left(\frac{1}{\lambda}+1\right), v^2 + \left(\bar{S}_t^{(i)} - \mu\right)^2 = \sigma^2 r\left(1+\frac{2}{\lambda}\right).$$

Thus, λ is estimated from

$$\frac{\bar{S}_t^{(i)}-\mu}{v^2+\left(\bar{S}_t^{(i)}-\mu\right)^2} = \frac{r\left(\frac{1}{\lambda}+1\right)r\left(\frac{1}{\lambda}+1\right)}{r\left(\frac{2}{\lambda}+1\right)}. \quad (25)$$

Once λ is determined from (25), thenσ can be estimated from the formula

$$\sigma = \frac{\bar{S}_t^{(i)}-\mu}{r\left(\frac{1}{\lambda}+1\right)}. \quad (26)$$

μcan be seen as the smallest possible value of $S_t^{(i)}$, so it seems reasonable to estimate μ by the smallest observed value $S_{tmin}^{(i)}$ of $S_t^{(i)}$. That is

$$\bar{\mu} = Minimum of S_t^{(1)}, \ldots, S_t^{(d)} = S_{tmin}^{(i)} \quad (27)$$

The choice of the Weibull distribution is due to its flexibility to mimic other statistical distributions such as the exponential(when $\lambda = 1$) and the normal distributions (when λ=3.4) (Osu, 2009).

In fact, by using nonlinear change of variable, let us pose

$$y = \left(\bar{S}_t^{(i)} - \mu\right)^{\frac{\lambda}{2}}. \quad (28)$$

Inversely, we have

$$\bar{S}_t^{(i)} - \mu = y^{\frac{2}{\lambda}}. \quad (29)$$

The change of variable from$\bar{S}_t^{(i)}$ to y leads to a Gaussianpdf for the y-variable defined as

$$P(y) = \lambda\sigma^{-\frac{\lambda}{2}} exp\left\{-\frac{y^2}{\sigma^{2\lambda}}\right\}. \quad (30)$$

All the properties of the portfolio are contained in the probability distribution$P(y)$of y. We compute the moments of the distribution by

$$M_d\big(P(y)\big) = \int_0^{\infty} y^d \, \lambda\sigma^{-\frac{\lambda}{2}} exp\left\{-\frac{y^2}{\sigma^{2\lambda}}\right\} dy$$

$$= \lambda\sigma^{d+1-\frac{\lambda}{2}r\left(\frac{d}{2}+\frac{1}{2}\right)}, \qquad (31)$$

where

$$\bar{\mu} = M_1 = \lambda\sigma^{d+1-\frac{\lambda}{2}r\left(\frac{d}{2}+\frac{1}{2}\right)} = \lambda\sqrt{\pi}\sigma^{2-\frac{\lambda}{2}}, \qquad (32)$$

$$M_2 = \lambda\sigma^{3-\frac{\lambda}{2}r\left(1+\frac{1}{2}\right)} \qquad (33)$$

$$\bar{v} = M_2 - (M_1)^2 = \lambda\sigma^{3-\frac{\lambda}{2}r\left(1+\frac{1}{2}\right)} - \left(\lambda\sqrt{\pi}\sigma^{2-\frac{\lambda}{2}}\right)^2. \qquad (34)$$

Methodology of Data Collection and Analysis

1. Collate pension price data for 5 years on monthly basis
2. Use Monte Carlo method of forecasting asset-with-drift prices diffusion process (see table 1)
3. Use 10 runs to have average price as our pension options current price
4. Use 10 runs in the maturing month's prices to set our exercise price
5. Simulate 10 different options and compare with actual outcome for January 2016

Table 1: Data for Analysis& Results from Monte Carlo Asset price Forecasting method (S(T+1)

Date	Price	Price Log Return	N(0,1)	Log Return + Shock	S(T+)	Forecast Date
06-Oct-10	1.7287	S(T)	2.795			
09-Nov-10	1.7474	0.010759288	-0.31189	0.004398401	2.807320606	Dec-15
10-Dec-10	1.758	0.00604783	-0.44155	0.003126055	2.816110176	Jan-16
07-Jan-11	1.7926	0.019490281	-0.32177	0.004301487	2.828249727	Feb-16
14-Feb-11	1.8009	0.00461946	2.917018	0.036084288	2.93216875	Mar-16
08-Mar-11	1.7915	-0.005233282	1.22872	0.019516728	2.989957177	Apr-16
12-Apr-11	1.7891	-0.001340558	1.954264	0.026636611	3.070669685	May-16
18-May-11	1.8263	0.020579362	-0.24445	0.005060223	3.086247339	Jun-16
15-Jun-11	1.8326	0.003443661	0.555131	0.012906671	3.126338683	Jul-16
13-Jul-11	1.8197	-0.007064071	-0.73063	0.000289232	3.127243052	Aug-16
10-Aug-11	1.7971	-0.012497398	0.725878	0.014582244	3.173179387	Sep-16
14-Sep-11	1.7988	0.000945521	0.655464	0.013891262	3.217566437	Oct-16
17-Oct-11	1.7776	-0.011855636	0.596309	0.013310757	3.26068099	Nov-16
18-Nov-11	1.79	0.00695148	-1.45846	-0.006853061	3.238411736	Dec-16
09-Dec-11	1.7771	-0.007232798	-0.4108	0.003427816	3.249531462	Jan-17
20-Jan-12	1.8058	0.016020885	-0.05573	0.00691214	3.272070487	Feb-17
15-Feb-12	1.8151	0.005136856	-0.43219	0.003217887	3.2826166	Mar-17
06-Mar-12	1.835	0.010903919	0.896111	0.016252775	3.336404145	Apr-17
11-Apr-12	1.8439	0.004838412	-0.86964	-0.001074869	3.332819874	May-17
07-May-12	1.8911	0.025275777	-0.48903	0.002660112	3.341697351	Jun-17
13-Jun-12	1.8864	-0.00248842	0.827289	0.015577413	3.394159905	Jul-17
04-Jul-12	1.9035	0.009024046	-1.3948	-0.006228323	3.373085678	Aug-17
23-Jul-12	1.9276	0.012581409	1.972366	0.026814254	3.464755997	Sep-17
08-Aug-12	1.9358	0.004244972	-0.45677	0.002976699	3.4750849	Oct-17
06-Sep-12	1.9659	0.015429478	-1.25166	-0.004823731	3.45836239	Nov-17
11-Oct-12	2.0183	0.026305417	-0.35622	0.003963396	3.472096449	Dec-17
21-Nov-12	2.0356	0.008535043	-1.87814	-0.010971486	3.434210605	Jan-18

11-Dec-12	2.0552	0.009582551	0.364553	0.011036503	3.472322202	Feb-18
07-Jan-13	2.0942	0.018798452	-2.68916	-0.01893015	3.407208869	Mar-18
31-Jan-13	2.1543	0.028294226	0.311585	0.010516713	3.443230591	Apr-18
04-Mar-13	2.1824	0.012959344	-0.89579	-0.001331431	3.438649217	May-18
02-Apr-13	2.2043	0.00998481	1.203801	0.019272189	3.505562223	Jun-18
02-May-13	2.2148	0.004752108	0.230394	0.009719971	3.539802322	Jul-18
31-May-13	2.2827	0.030196847	0.576979	0.013121075	3.586554381	Aug-18
01-Jul-13	2.2706	-0.005314839	-0.31932	0.004325525	3.602101714	Sep-18
07-Aug-13	2.3136	0.018760639	-0.65507	0.001030805	3.605816692	Oct-18
18-Sep-13	2.3082	-0.002336753	-0.81745	-0.000562661	3.603788409	Nov-18
30-Sep-13	2.3228	0.006305355	-0.81999	-0.000587619	3.601671377	Dec-18
31-Oct-13	2.3525	0.012705238	0.971847	0.016995983	3.663408477	Jan-19
29-Nov-13	2.3829	0.012839641	1.436328	0.021554015	3.743226749	Feb-18
27-Dec-13	2.4161	0.013836437	-0.56752	0.001889925	3.750307858	Mar-19
29-Jan-14	2.4447	0.011767746	1.479917	0.021981766	3.833658993	Apr-19
27-Feb-14	2.4338	-0.004468594	-2.13468	-0.013488909	3.782294323	May-19
04-Mar-14	2.4401	0.0025852	-1.01775	-0.00252829	3.772743665	Jun-19
02-Apr-14	2.4467	0.002701156	-1.0327	-0.002674999	3.762665064	Jul-19
02-May-14	2.4751	0.011540621	-1.67764	-0.009003927	3.728938364	Aug-19
28-May-14	2.5115	0.014599385	-0.9531	-0.001893844	3.721883021	Sep-19
01-Jul-14	2.5651	0.02111728	1.304553	0.020260891	3.798060795	Oct-19
08-Aug-14	2.5939	0.01116507	2.242246	0.029462631	3.911626417	Nov-19
30-Sep-14	2.6188	0.009553663	-1.51065	-0.007365203	3.882922328	Dec-19
10-Oct-14	2.6217	0.001106765	0.668122	0.014015476	3.937726491	Jan-20
31-Nov-14	2.6087	-0.004970949	-0.18659	0.00562803	3.959950613	Feb-20
01-Dec-14	2.5922	-0.006345077	-0.52562	0.002301095	3.969073329	Mar-20
01-Dec-14	2.5922	0	-0.68157	0.000770738	3.972133624	Apr-20
31-Dec-14	2.6009	0.003350603	-1.04594	-0.0028049	3.961007798	May-20
06-Feb-15	2.5949	-0.002309559	-0.64564	0.001123261	3.965459544	Jun-20
13-Mar-15	2.6318	0.014120043	0.198813	0.009410068	4.002950909	Jul-20
25-Mar-15	2.6283	-0.001330773	-0.52331	0.002323745	4.012263561	Aug-20
05-May-15	2.7078	0.029799248	0.532715	0.012686698	4.0634902	Sep-20
04-Jun-15	2.7283	0.007542208	-1.44015	-0.006673344	4.036463413	Oct-20
02-Jul-15	2.7416	0.004862987	0.797045	0.015280625	4.098616757	Nov-20
28-Jul-15	2.7429	0.000474063	1.018431	0.017453117	4.170778284	Dec-20
03-Sep-15	2.7557	0.004655739	-0.37979	0.003732161	4.186373382	Jan-21
30-Sep-15	2.7854	0.010719996	-0.29724	0.004542208	4.205432013	Feb-21
02-Nov-15	2.795	0.003440617	-0.8067	-0.000457194	4.203509755	Mar-21

Procedure

1. Collate RSA Monthly closing prices
2. Compute the mean (using (27)), variance (using (26)) and standard deviation (using (24) of the periodic return if these values are discrete. If continuous, use equation (31) to (34).
3. Compute the periodic return (using (20)). This generates our log returns (see figure 4)
4. Generate set of random numbers
5. Forecast price using (19) (see figure 3).
6. Compute the value of price derivative using (10).

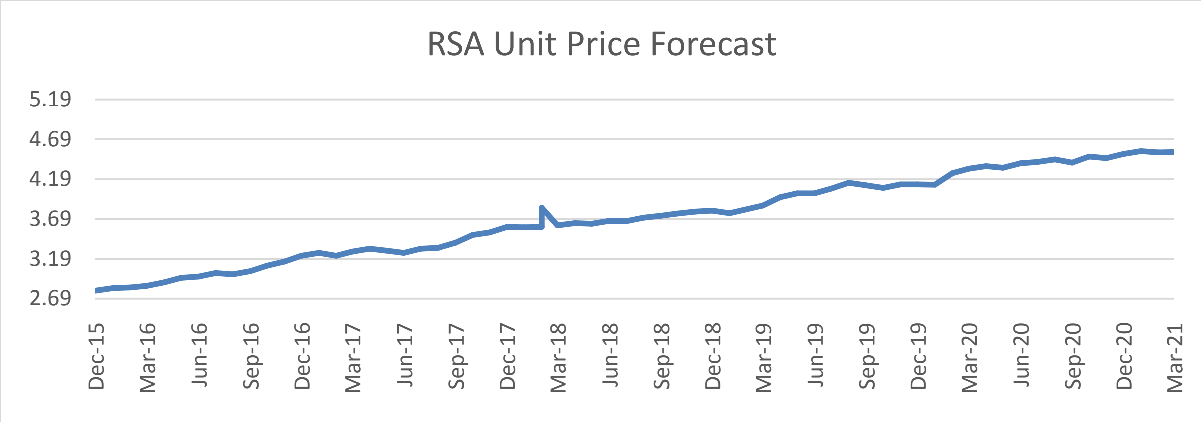

Figure 3: RSA price forecast between Dec 2015 to Mar 2021 on monthly basis

Graph of Data from table 1: Price forecast against date

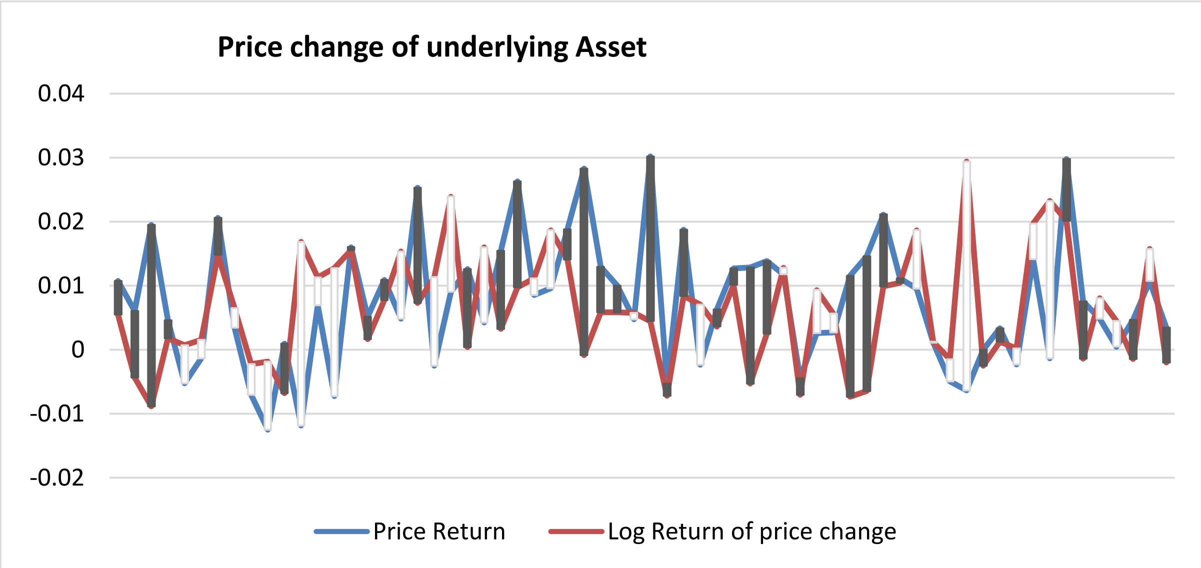

Figure4: Graph of randomness of price change and log of change in price returns

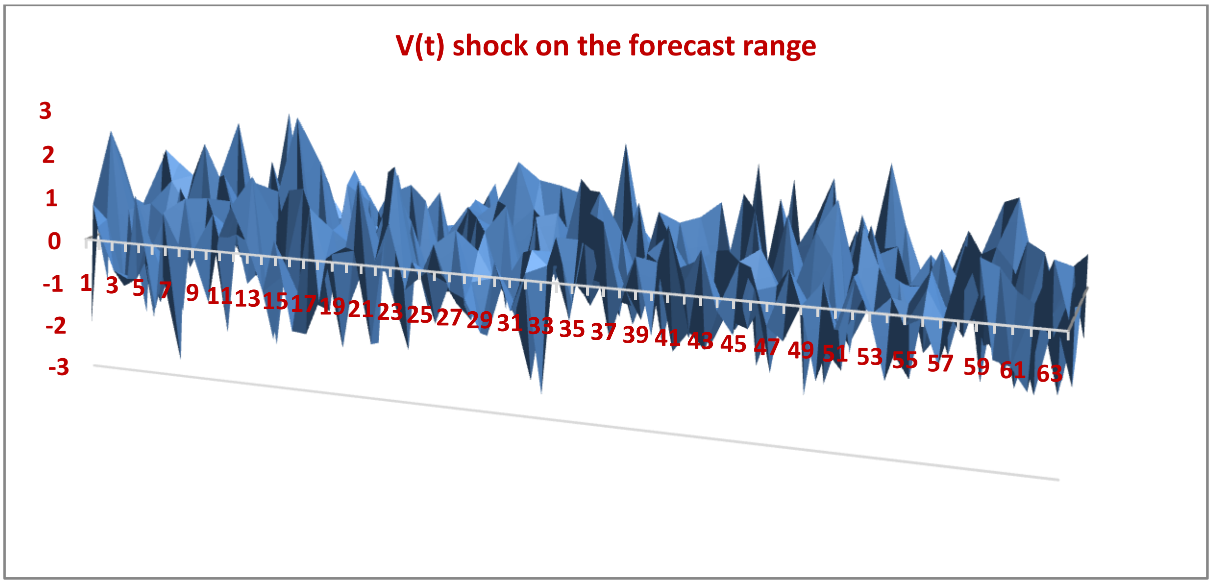

Figure 5: Simulation of Black –Scholes Model using price forecasts

Table 2: Forecast Prices from 10 runs and the averages with date

	1	2	3	4	5	6	7	8	9	10	Avg 10 Runs
Dec-15	2.785998695	2.818554748	2.803463781	2.797536164	2.791093227	2.792937022	2.844754626	2.777088284	2.809355613	2.775713108	2.799649527
Jan-16	2.826143801	2.831293306	2.785511769	2.807760505	2.82669442	2.779724535	2.866694894	2.815931354	2.806061778	2.82180054	2.81676169
Feb-16	2.812818798	2.848762736	2.831980565	2.830762977	2.863086698	2.812182077	2.930450164	2.840190979	2.836598156	2.837541055	2.84443742
Mar-16	2.843187794	2.82636069	2.830289233	2.839988256	2.871591025	2.823552959	2.958998398	2.868499246	2.877303765	2.842269782	2.858204115
Apr-16	2.862174577	2.824606451	2.842411587	2.91481928	2.885610782	2.797226077	2.983258252	2.933002838	2.858741318	2.862618224	2.876446938
May-16	2.876627775	2.809203618	2.909534548	2.957810834	2.888992855	2.794454524	2.956518513	2.961376465	2.868743014	2.892858824	2.891612097
Jun-16	2.89096957	2.819729407	2.943656058	3.035458724	2.875297882	2.791484284	2.998339924	2.92162223	2.923334859	2.942726262	2.91426192
Jul-16	2.850853998	2.861080081	2.989422938	3.020260402	2.919593881	2.775494635	3.045749854	2.890903855	2.914084027	2.994528595	2.926197227
Aug-16	2.89547866	2.920848383	3.011180079	3.052757519	2.888572146	2.776767537	3.044507324	2.921057525	2.964061268	3.019144475	2.949437491
Sep-16	2.951649165	2.927096084	2.967035089	3.106412488	2.906021215	2.787863289	3.081651924	2.903124703	3.005204917	2.995758796	2.963181767
Oct-16	2.97097934	3.019382846	3.009075786	3.130768112	2.892747551	2.838221604	3.054053596	2.951048369	3.056174438	3.075925185	2.999837683
Nov-16	2.970290624	3.052693478	3.049031354	3.143326533	2.900752759	2.829751674	3.012850934	2.955777143	3.053780506	3.103351616	3.007160662
Dec-16	2.998296366	3.188058	3.073542851	3.190567584	2.8957984	2.839416165	3.062145275	2.943841704	3.106570941	3.08076341	3.03790007
Jan-17	3.052311969	3.192840379	3.152014594	3.160411839	2.961965576	2.876429367	3.121928753	2.97576359	3.105937318	3.099654072	3.069925746
Feb-17	3.055744488	3.247273461	3.13030957	3.200912359	2.98506919	2.886590051	3.123470713	2.998255257	3.099955286	3.118620762	3.084620114
Mar-17	3.038139108	3.251529841	3.172131777	3.245416291	3.014835321	2.917983409	3.134996849	3.001771525	3.168352259	3.159262521	3.11044189
Apr-17	3.076185967	3.266201959	3.155669483	3.261048462	3.018006218	2.917199629	3.102623223	3.011433623	3.143752777	3.175813389	3.112793473
May-17	3.115636057	3.28906457	3.195047281	3.222906816	3.104598576	2.921302332	3.102957407	3.015740768	3.142988942	3.246430418	3.135667317
Jun-17	3.110680412	3.330464513	3.217475864	3.236869615	3.13254024	2.938944262	3.156235927	3.052497111	3.200174765	3.272805277	3.164868799
Jul-17	3.119240891	3.337907786	3.237691019	3.259988585	3.120678208	2.984245402	3.197300013	3.095561534	3.219895247	3.274993342	3.184750203
Aug-17	3.100454238	3.317083337	3.26893868	3.232062745	3.163002933	3.008798128	3.247318854	3.143919288	3.208070099	3.26928797	3.195893627
Sep-17	3.104374735	3.341713053	3.294131308	3.287237424	3.166941936	3.01096564	3.246959871	3.1144117	3.230262872	3.250597626	3.204759616
Oct-17	3.164157991	3.334983934	3.241486919	3.315767717	3.175510657	3.034571066	3.264131475	3.130479755	3.264395262	3.267154809	3.219263958
Nov-17	3.200688279	3.380579655	3.218045554	3.34871424	3.214697225	3.132334116	3.298031057	3.135968633	3.273395132	3.206713123	3.240916701
Dec-17	3.266389152	3.44985368	3.232193703	3.36809343	3.233767756	3.166038345	3.347044598	3.176071779	3.304111706	3.246527211	3.279009136
Jan-18	3.347024632	3.457893903	3.270346908	3.383133308	3.276882083	3.175300796	3.430101966	3.201936882	3.323778379	3.245078881	3.311147774
Feb-18	3.383934292	3.494953547	3.235761071	3.394160621	3.291046754	3.186593858	3.446685301	3.264539585	3.351758247	3.260462968	3.330989625
Mar-18	3.393554463	3.491961802	3.262773358	3.432488745	3.328378733	3.21359603	3.50316857	3.282244408	3.316240514	3.272191146	3.349659777
Apr-18	3.447919907	3.545543406	3.29876718	3.434018723	3.342746709	3.229231208	3.467457222	3.287176831	3.331990952	3.363715911	3.374856805
May-18	3.456099203	3.566623662	3.362056369	3.439280451	3.343617417	3.318301454	3.516855728	3.354266014	3.342107968	3.327893671	3.402710194
Jun-18	3.472277367	3.521256898	3.431421501	3.47250805	3.3738512	3.340025701	3.500185401	3.354752715	3.329400905	3.382218559	3.41778983
Jul-18	3.501080409	3.583479592	3.516022052	3.44022104	3.38957452	3.342328184	3.59174832	3.385315497	3.315504113	3.386307583	3.445158131
Aug-18	3.434848715	3.588207124	3.561924457	3.422910877	3.407363856	3.304072535	3.636651681	3.401496279	3.390522144	3.418275392	3.456627306
Sep-18	3.479321178	3.566732426	3.608327004	3.459662231	3.422820105	3.3331657	3.717992264	3.44926602	3.42625423	3.467033873	3.493057503
Oct-18	3.50625238	3.547048262	3.713813023	3.478390504	3.442625505	3.395087013	3.70998488	3.495597698	3.466606477	3.512530668	3.526793641
Nov-18	3.579419837	3.583379976	3.724800116	3.512509675	3.468834996	3.433741938	3.771758699	3.500912093	3.513904857	3.441576958	3.553083914
Dec-18	3.568077701	3.589469901	3.79664737	3.495192739	3.53314394	3.447964845	3.745250887	3.502759988	3.580993489	3.518296424	3.577779728
Jan-19	3.539788862	3.539380439	3.842655367	3.568276885	3.552445351	3.523171641	3.758103168	3.47120537	3.622382817	3.607705711	3.602511561
Feb-18	3.648788757	3.57476072	3.824709146	3.590646822	3.571328039	3.536601679	3.781988899	3.551855115	3.700789649	3.667646722	3.644911555
Mar-19	3.630009034	3.618665061	3.862140721	3.616154	3.548072195	3.563278081	3.72835509	3.580307417	3.760595199	3.709292237	3.661686904
Apr-19	3.685503548	3.67349685	3.886325979	3.680947811	3.567603473	3.576529876	3.7344003	3.629845954	3.729275755	3.770073954	3.69340035
May-19	3.685306833	3.742705397	3.904249298	3.676441461	3.648152141	3.581942626	3.724518225	3.670925892	3.704655775	3.840603414	3.717950106
Jun-19	3.666446871	3.84656026	3.85421392	3.679717544	3.677543971	3.565947653	3.746890524	3.721325682	3.781950634	3.797924201	3.733852126
Jul-19	3.740466859	3.860594892	3.806309077	3.723078195	3.663399597	3.563373271	3.746413875	3.745510756	3.757595471	3.844048459	3.745079045
Aug-19	3.751466597	3.945725734	3.837681845	3.751909282	3.667508239	3.593053808	3.758652078	3.776668781	3.80891329	3.896436872	3.778801653
Sep-19	3.781370412	3.98802113	3.995870756	3.834148106	3.714217299	3.640356657	3.772372199	3.737407398	3.888016414	3.99752432	3.834930469
Oct-19	3.741805166	4.050804268	4.020574869	3.8492865	3.679632176	3.668644124	3.765795636	3.798715587	3.892598703	4.001705154	3.846956218
Nov-19	3.801842697	4.025850466	4.093553353	3.860878724	3.712283111	3.754560475	3.823765411	3.92539947	3.951901948	4.015013531	3.896504918
Dec-19	3.835037979	4.017646209	4.101097854	3.888334774	3.706434655	3.784312275	3.808741215	3.984033415	4.010016462	4.0600951	3.919574994
Jan-20	3.979757898	4.090361968	4.074968024	3.891595184	3.781370046	3.861933361	3.832614832	4.021653015	4.027247823	4.107262259	3.966876441
Feb-20	4.052752262	4.086015856	4.098418033	3.986585561	3.808336142	3.859208982	3.78457934	4.100532716	4.042829887	4.132834579	3.995209336
Mar-20	4.073738742	4.139890249	4.080256163	4.043776869	3.834110616	3.844035847	3.817896001	4.193895868	4.109745527	4.229646015	4.03669919
Apr-20	4.123464672	4.212150126	4.108639037	4.122254965	3.870054054	3.870923968	3.826191475	4.246073026	4.126506669	4.210188189	4.071644618
May-20	4.129912206	4.305893523	4.157695669	4.16712648	3.846331773	3.860087275	3.798052475	4.277073154	4.145811108	4.237446815	4.092543048
Jun-20	4.22715508	4.269390202	4.221023117	4.244199381	3.902299552	3.936794697	3.8271376	4.211908821	4.123880778	4.274383297	4.123817252
Jul-20	4.305680666	4.242039946	4.304119673	4.252941451	3.990387237	3.979779416	3.83800869	4.281746834	4.241554655	4.329750349	4.176600892
Aug-20	4.297908073	4.262539284	4.301407375	4.28729456	3.960058888	4.037997762	3.874930971	4.284367223	4.250203778	4.348454654	4.190516257
Sep-20	4.293533857	4.324726612	4.326314926	4.362562127	3.988476544	4.05061256	3.8625442	4.320164856	4.31969492	4.419158991	4.226778959
Oct-20	4.297598371	4.385382171	4.348898464	4.411117345	4.079846989	4.087480543	3.897458482	4.341541141	4.401263178	4.548745664	4.279933235
Nov-20	4.3434528	4.409172857	4.384944589	4.451663928	4.106777515	4.113927439	3.904022106	4.365300446	4.432307499	4.568284892	4.307985407
Dec-20	4.406822029	4.381815019	4.431585468	4.435943735	4.173379931	4.155913491	3.973149465	4.379156688	4.397715829	4.599949191	4.333543085
Jan-21	4.44372699	4.408201956	4.449699613	4.531271676	4.197475682	4.213836269	4.018823046	4.398387948	4.467875173	4.645038788	4.377433714
Feb-21	4.42269543	4.449073054	4.572468859	4.599606203	4.224657287	4.274419198	4.049843207	4.428212623	4.487421367	4.683880521	4.419227775
Mar-21	4.474299691	4.446418948	4.59358882	4.639057632	4.305570694	4.269067578	4.110550198	4.481455914	4.490682986	4.664687138	4.44753796

Table 3: Treasury bill

Auction Date	Security Type	Maturity Date	Rate
1/20/2016	NTB	4/21/2016	3.2500 - 4.2900
1/20/2016	NTB	7/21/2016	4.0000 - 7.5900

Source: Central Bank of Nigeria Website

RSA is to be soldby (01May 2016) at the price if NGN2.891on expiration in 6 months from (02 November 2015) at exercise price of NGN2.795. From table 3 we take average rate at 3.6. From Monte Carlo simulation we know that Volatility is 1%. We are using Excel assisted modelled solution to solve for the options pricing to enhance decision making by the unorganised private sectors and the pension fund administrators arrive at selling and purchase prices. The value of the Call option is NGN0.15 with intrinsic value of NGN0.10. This gives a speculative premium of NGN0.05. It therefore follows that this option was sold at NGN0.15. Comparing to forecast price we can see that for a call option of this RSA2795 offers a good NGN0.18 between 02 November 2015 and 1 May 2016. By this analysis the PFAs (Pension Fund administrators gains from the put option and only pays N0.03 as actual gain to the unorganised private sectors who repurchases for another six months at a prevailing current price of NGN3.07. (See Table 1 May 2016).

Conclusion

We have shown in this work how the Black-Scholes equation relates the recommended price of the option to four other quantities. Three can be measured directly: time, the price of the asset (A pension fund administrator's asset under management and invested in diversified nature which is made out from the following contributions; Retirement Savings Account (RSA), Voluntary Contribution Account (VCA) and Accumulated Pension Account (Legacy Fund)upon which the option is secured and the risk-free interest rate. This is the theoretical interest that could be earned by an investment with zero risk, such as government pension policies. The fourth quantity is the volatility of the asset. This is a measure of how erratically its market value changes. We have also shown how the Black-Scholes equation is related to the heat equation. In fact, the Black-Scholes equation has its roots in mathematical physics, where quantities are infinitely divisible, time flows continuously and variables change smoothly.

Table 4: Result of Simulation 1

Black-Scholes Option		
Time to Expiration		0.49315068
Exercise Price	NGN	2.80
Current Stock Price	NGN	2.89
Volatility		1.00%
Risk-Free Rate		3.60%
d1		7.34050088
d2		7.33347841
N(d1)		1
N(d2)		1
Call Option Value	NGN	0.15
Intrinsic Value	NGN	0.10
Speculative Prem.	NGN	0.05
Put Option Value	NGN	0.00
Intrinsic Value	NGN	-
Speculative Prem.	NGN	0.00

	Call	
	Delta	1.0000
	Gamma	0.0000
	Theta	-0.0988
	Vega	0.0000
	Rho	1.35410156
	Put	
	Delta	0.0000
	Gamma	0.0000
	Theta	0.0000
	Vega	0.0000
	Rho	-1.5183E-13

Recommendation

We recommend that the Nigerian National Pension Commission evaluates this model to start the involvement of unorganised private sector into its contributory pension scheme. Nigeria having its peculiarities outlined in the statement of problems needs a unique model to solve this financial exclusion of a large population from participating in the ongoing contributory pension scheme. With this model, it could be seen that there is a balance between the contributors through six monthly graded options with pension stock as its underlying asset. We however encourage more research to improve any deeper peculiarity required for implementation. For example, our model mimics the homogeneous heat equation. In an on-going research work, we found out if the nonhomogeneous case will make better model.

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