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WIENER INDEX OF SOME GRAPH TRANSFORMATIONS

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ABSTRACT

The Wiener index of a graph G, denoted by W(G) is the sum of the distance between all (undirected) pairs of vertices of G. In this paper, we investigate wiener index of some graph transformations or operations like plick graph and lict graph.

Keywords: Wiener index, Plick graph, Lict graph.

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1. INTRODUCTION

Let G = (V,E) be a simple, connected, undirected graph. The number of vertices and edges of G, we denote by n and m, respectively. Thus, $|V(G)| = n$ and $|E(G)| = m$. As usual, n is said to be the order and m the size of G. For the undefined terminologies and notations we refer reader to Harary[8]. The distance between two vertices v_i and v_j, denoted by $d(v_i, v_j)$ is the length of shortest path between the vertices v_i and v_j in G. The diameter diam(G) of a connected graph G is the length of any longest geodesic. The degree of a vertex v_i in G is the number of edges incident to v_i and is denoted by $d_i = \deg(v_j)$ [2,10].

The Wiener index (or Wiener number) [18] of a graph G, denoted by W(G) is the sum of the distance between all (unordered) pair of vertices of G. That is

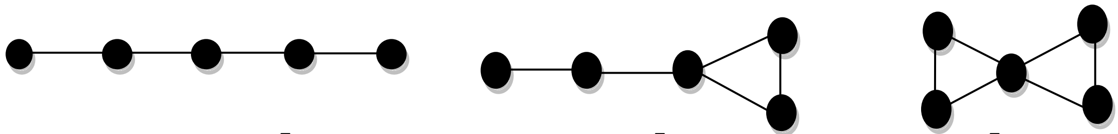
$$W(G)= \sum_{i \leq j} d(v_i, v_j).$$

The wiener index is a graph invariant that belongs to the molecules structure descriptors called topological indices, which are used for the design of molecules with desired properties [16]. For details on wiener index see [4, 10,15].

The line graph L(G) of a graph G such that the vertices of L(G) are the edges of G and two vertices of L(G) are adjacent if and only if their corresponding edges in G share a common vertex[10]. The concept of line graph has various applications in physical chemistry [7, 8].

If G is a connected graph, then the distance between two edges $d(e_i , e_j)$ is, the distance between the corresponding vertices of the line graph of G [4].

Let F_1 be the 5-vertex path, F_2 be the graph obtained by identifying a vertex of a triangle with an end vertex of the 3-vertex path and F_3 the graph obtained by identifying a vertex of a triangle with a vertex of another triangle (see Figure. 1).



F_1 F_2 F_3

(Figure.1)

2. PLICK GRAPH AND LICT GRAPH

Let G = (V, E) be a graph with n vertices and m edges. In the mathematical literature several "Transformation Graphs" have been considered, constructed so that their vertex set is equal to $E(G)\bigcup B(G)$, where B(G) is the number of blocks in graph G and $E(G)\bigcup C(G)$, where C(G), is the number of cut-vertices in graph. In what follows we define the best known representatives of such graphs.

The Plick graph P(G)[11] of a graph G, is obtained from the line graph by adding a new vertex corresponding to each block of the original graph and joining this vertex to the vertex of the line graph which correspond to the edges of the blocks of the original graph.

The cut-vertices and edges of a graph G are called its members. The Lict graph $L_c(G)$[12] of a graph G, is defined as the graph whose vertex set is the union of edges and the set of cut-vertices of G in which two vertices are adjacent if and only if the corresponding edges of G are adjacent or the corresponding members of G are incident.

3 RESULTS

We begin with the some of following previously known, auxiliary results.

Remark 3.1. [11,12] Line graph is subgraph of plick graph P(G) and lict graph $L_c(G)$.

Theorem 3.2. [15] If diam(G) ≤ 2 and none of the graphs F_1 , F_2 , F_3 of figure.1 is an induced subgraph of G then diam(L(G)) ≤ 2.

Theorem 3.3. [1] For every tree T on n vertices $W(L(G))=W(G)-\binom{n}{2}$.

Theorem 3.4. [6] If G is connected graph with n vertices and m edges then,

$$W(L(G)) \geq W(G) - n(n-1) + \frac{m(m+1)}{2}.$$

Theorem 3.5. [14] If G be a connected graph with n vertices, m edges and $d_i = \deg(v_i)$; i = 1,2, 3,…,n. If diam(G)≤ 2 and G does not contain F_k ; k = 1, 2, 3 (of figure.1) as an induced subgraph then,

$$W(G) = m^2 - \frac{1}{2}\sum_{i=1}^{n} d_i^2 .$$

Corollary 3.6. [14] If G is a connected r-regular graph on n vertices with diam(G) ≤ 2 and none of F_k; k = 1,2, 3 (of figure.1) as an induced subgraph of G then,

$$W(L(G)) = \frac{nr^2(n-2)}{4}.$$

Theorem 3.7. [14] Let G be a connected graph with vertex set V (G) = v_1 , v_2, v_3 ,…, v_n and edge set E(G) = e_1 ,e_2 , e_3 ,…, e_m . Let $d_i = \deg(v_i)$ then,

$$W(L(G)) \geq \sum_{i=1}^{n} \frac{d_i(d_i-1)}{2} + m(m-1) - \sum_{i=1}^{m} \deg(e_i).$$

Theorem 3.8.[14] If T is a tree with vertices V (G) = v_1 ,v_2 ,v_3 ,…,v_n and d_i = deg(v_i); i = 1, 2, 3,…,n. Then,

$$W(L(G)) = \sum_{i=1}^{n} \frac{d_i(d_i-1)}{2} + \sum_{i<j;i,j=1}^{n} [1 + d(v_i, v_j)](d_i - 1)(d_j - 1).$$

Theorem 3.9. [14] If T is a tree having k-vertices with degree s and remaining with degree 1. Then,

$$W(L(T)) = \frac{ks(s-1)}{2} + (s-1)^2 [\binom{k}{2} + W(T')].$$

Where T ' is the tree obtained from T by removing all its end vertices.

Theorem 3.10. [17] Let G be a graph of order n and size m. Then W(G)= n(n-1)-m if and only if diam(G) ≤ 2.

Theorem 3.11. Let G be a graph with n vertices m edges, d_i = deg(v_i); i =1,2,3,…,n and $B(G) = \sum_{i=1}^{m} b_i$. If diam(G) ≤ 2 and G does not contain F_k; k=1,2,3 as an induced subgraph then,

$$W(P(G)) = (m + \sum_{i=1}^{m} b_i)^2 - ((-m + \frac{1}{2}\sum_{i=1}^{n} d_i^2) + \sum_{h,i=1}^{m} (m_h \times b_i))$$

Here $(m_h \times b_i)$ where h, i=1,2,3,…,m is the number of edges incident with blocks b_i ; i=1,2,3,...

Proof. Let P(G) be the plick graph. By the definition, vertex set of P(G) is,

$V(P(G)) = E(G) \cup B(G)$.

And the number of edges in P(G) is

$$E(P(G)) = -m + \frac{1}{2}\sum_{i=1}^{n} d_i^2 + \sum_{h,i=1}^{m} (m_h \times b_i)$$

If diam(G) ≤ 2, then by the Theorem 3.10,

$$W(G) = n(n-1) - m$$

$$W(P(G)) = (m + \sum_{i=1}^{m} b_i)(m + \sum_{i=1}^{m} b_i - 1) - (-m + \frac{1}{2}\sum_{i=1}^{n} d_i^2 + \sum_{h,i=1}^{m} (m_h \times b_i))$$

$$= m^2 + \sum_{i=1}^{m} b_i^2 + 2m\sum_{i=1}^{m} b_i - \sum_{i=1}^{m} b_i - m + m - \frac{1}{2}\sum_{i=1}^{n} d_i^2 - \sum_{h,i-1}^{m} (m_h \times b_i)$$

$$W(P(G)) = (m + \sum_{i=1}^{m} b_i)^2 - \sum_{i=1}^{m} b_i - \frac{1}{2}\sum_{i=1}^{n} d_i^2 - \sum_{h,i=1}^{m} (m_h \times b_i).$$

Corollary 3.12. If G is a connected r-regular graph on n-vertices with diam(G) ≤ 2, $B(G) = \sum_{i=1}^{m} b_i$ and none of F_k; k = 1, 2, 3(of figure.1) as an induced subgraph of G then,

$$W(P(G)) = \frac{nr^2}{2}(\frac{n}{2} - 1) + \sum_{i=1}^{m} b_i(b_i + nr + 1) - \sum_{h,i=1}^{n} (\frac{n_h r}{2} \times b_i).$$

Proof. Let G is an r-regular graph on n-vertices the number of edges of G is $m = \frac{nr}{2}$ and d_i=deg(v_i) ; i=1,2,3,...,n=r. From Theorem 3.11,

$$W(P(G)) = (m + \sum_{i=1}^{m} b_i)^2 - \sum_{i=1}^{m} b_i - \frac{1}{2}\sum_{i=1}^{n} d_i^2 - \sum_{h,i=1}^{m}(m_h \times b_i)$$

$$= (\frac{nr}{2} + \sum_{i=1}^{m} b_i)^2 - \sum_{i=1}^{m} b_i - \frac{1}{2}\sum r^2 - \sum_{h,i=1}^{n}(\frac{n_h r}{2} \times b_i)$$

$$= (\frac{nr}{2})^2 + \sum_{i=1}^{m} b_i^{\,2} + 2(\frac{nr}{2})\sum_{i=1}^{m} b_i - \sum_{i=1}^{m} b_i - \frac{1}{2}nr^2 - \sum_{h,i=1}^{n}(\frac{n_h r}{2} \times b_i)$$

$$= (\frac{nr}{2})^2 + \sum_{i=1}^{m} b_i^{\,2} + nr\sum_{i=1}^{m} b_i - \sum_{i=1}^{m} b_i - \frac{1}{2}nr^2 - \sum_{h,i=1}^{n}(\frac{n_h r}{2} \times b_i)$$

$$W(P(G)) = \frac{nr^2}{2}(\frac{n}{2} - 1) + \sum_{i=1}^{m} b_i(b_i + nr + 1) - \sum_{h,i=1}^{n}(\frac{n_h r}{2} \times b_i).$$

Theorem 3.13. If T is a tree with vertices v_1 , v_2 , v_3 ,…, v_n, edge set e_1 , e_2 , e_3,…, $e_{n-1}$and d_i=deg(v_i); i=1,2,3,…,n then,

$$W(P(T)) = \sum_{i=1}^{n}\frac{d_i(d_i - 1)}{2} + \sum_{i<j}[1 + d(v_i, v_j)](d_i - 1)(d_j - 1) + \sum_{i<j}[2 + d(e_i, e_j)] + \sum_{i,j=1}^{n-1}[1 + d(e_i, e_j)]$$

(if i=j then d(e_i , e_j)=0).

Proof. By the Remark 3.1 and Theorem 3.8,

$$W(L(T)) = \sum_{i=1}^{n}\frac{d_i(d_i - 1)}{2} + \sum_{i<j;i,j=1}^{n}[1 + d(v_i, v_j)](d_i - 1)(d_j - 1) \quad (1)$$

In plick graph, the vertices are the union of blocks and edges in G. Therefore the distance between pair of vertices in P(T) corresponding to the blocks in T is,

$$\sum_{i<j}[2 + d(e_i, e_j)] \quad (2)$$

and the distance between the vertices in P(T) corresponding to the pair of blocks and edges in T is,

$$\sum_{i,j=1}^{n-1}[1 + d(e_i, e_j)] , \text{ if i=j then } d(e_i , e_j)=0 \quad (3)$$

Therefore from (1),(2) and (3)

$$W(P(T)) = \sum_{i=1}^{n}\frac{d_i(d_i - 1)}{2} + \sum_{i<j}[1 + d(v_i, v_j)](d_i - 1)(d_j - 1) + \sum_{i<j}[2 + d(e_i, e_j)] + \sum_{i,j=1}^{n-1}[1 + d(e_i, e_j)] ,$$

(if i=j then d(e_i , e_j)=0).

Theorem 3.14. If T is a tree having k-vertices with degree S and remaining with degree 1 then,

$$W(P(T)) = \frac{ks(s-1)}{2} + (s-1)^2[\binom{k}{2} + W(T')] + \sum_{i<j}[2 + d(e_i, e_j)] + \sum_{i,j=1}^{n-1}[1 + d(e_i, e_j)] ,$$

(if i=j then d(e_i,e_j)=0).

Proof. By the Remark 3.1 and Theorem 3.9,

$$W(L(T)) = \frac{ks(s-1)}{2} + (s-1)^2[\binom{k}{2} + W(T \quad)]. \quad (4)$$

By the Theorem 3.12, distance between vertices in P(T) corresponding to pair of blocks in T is,

$$\sum_{i<j}[2+d(e_i,e_j)] \tag{5}$$

and the distance between the vertices in P(T) corresponding to the pair of blocks and edges in T is,

$\sum_{i,j=1}^{n-1}[1+d(e_i,e_j)]$,(if i = j then $d(e_i , e_j)=0$). (6)

Therefore, from (4),(5) and (6)

$$W(P(T))=\frac{ks(s-1)}{2}+(s-1)^2[\binom{k}{2}+W(T^{'})]+\sum_{i<j}[2+d(e_i,e_j)]+\sum_{i,j=1}^{n-1}[1+d(e_i,e_j)] \text{ ,}$$

(if i=j then $d(e_i\ e_j)=0$).

Theorem 3.15. Let G be a connected graph with n vertices m edges, d_i =degv_i ; i = 1, 2, 3, …, n and $\sum_{i=1}^{n-2}c_i$ be the cut-vertices. If diamG ≤ 2 and G does not contain F_k; k = 1,2, 3 as an induced sub-graph then, $W(L_c(G))=(m+\sum_{i=1}^{n-2}c_i)^2-\sum_{i=1}^{n-2}c_i-\frac{1}{2}\sum_{i=1}^{n}d_i^2-\sum_{i=1}^{n-2}(\deg c_i).$

Proof. Let $L_c(G)$ be the lict graph, By the definition, the vertex set is $V(L_c(G))=E(G)\bigcup C(G)$, where C(G) = $\sum_{i=1}^{n-2}c_i$ is the number of cut-vertices in G and the edge set of $L_c(G)$ is

$$E(L_c(G))=-m+\frac{1}{2}\sum_{i=1}^{n}d_i^2-\sum_{i=1}^{n-2}(\deg c_i)$$

If diamG ≤ 2 then, by Theorem 3.10

$$W(G)=n(n-1)-m$$

$$\begin{aligned}W(L_c(G)) &= (m+\sum_{i=1}^{n-2}c_i)(m+\sum_{i=1}^{n-2}c_i-1)-(-m+\frac{1}{2}\sum_{i=1}^{n}d_i^2+\sum_{i=1}^{n-2}(\deg c_i)\\ &= m^2+\sum_{i=1}^{n-2}c_i^2+2m\sum_{i=1}^{n-2}c_i-\sum_{i=1}^{n-2}c_i-\frac{1}{2}\sum_{i=1}^{n}d_i^2-\sum_{i=1}^{n-2}(\deg c_i)\\ &= (m+\sum_{i=1}^{n-2}c_i)^2-\sum_{i=1}^{n-2}c_i-\frac{1}{2}\sum_{i=1}^{n}d_i^2-\sum_{i=1}^{n-2}(\deg c_i).\end{aligned}$$

Therefore,

$$W(L_c(G))=(m+\sum_{i=1}^{n-1}c_i)^2-\sum_{i=1}^{n-2}c_i-\frac{1}{2}\sum_{i=1}^{n}d_i^2-\sum_{i=1}^{n-2}(\deg c_i).$$

Theorem 3.16. If T is a tree with vertices v_1 , v_2 , v_3 ,…, v_n, edge set e_1 , e_2 , e_3,…, $e_{n-1}$and cut-vertices C_i ; i=1,2,3,…,n-1 and d_i=deg(v_i); i=1,2,3,…,n then,

$$W(L_c(T))=\sum_{i=1}^{n}\frac{d_i(d_i-1)}{2}+\sum_{i<j}[1+d(v_i,\ v_j)](d_i-1)(d_j-1)+\sum_{i,j=1}^{n-2}[1+d(c_i,c_j)]+$$

$$\sum_{i\neq j,i,j=1}^{n}d(c_i,v_j)\text{ .}$$

Proof. By Theorem 3.8, we know that

$$W(L(T))=\sum_{i=1}^{n}\frac{d_i(d_i-1)}{2}+\sum_{i<j;i,j=1}^{n}[1+d(v_i,\ v_j)](d_i-1)(d_j-1)\text{ .} \tag{7}$$

And the sum of the distance between the pair of vertices of $L_c(G)$ corresponding to the cut-vertices of G is

$$\sum_{i,j=1}^{n-2}[1+d(c_i,c_j)] \ . \quad (8)$$

And distance between the vertices of $L_c(G)$ corresponding to cut-vertices and edges in G is

$$\sum_{i\neq j,i,j=1}^{n} d(c_i,v_j)\ . \quad (9)$$

Therefore, from (7),(8) and (9)

$$W(L_c(T)) = \sum_{i=1}^{n}\frac{d_i(d_i-1)}{2} + \sum_{i<j}[1+d(v_i,\, v_j)](d_i-1)(d_j-1) + \sum_{i,j=1}^{n-2}[1+d(c_i,c_j)] + \sum_{i\neq j,i,j=1}^{n} d(c_i,v_j).$$

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4. REFERENCES

[1]. Buckley, F., 1981, *Mean distance of line graphs*, Congr. Numer., 32 153-162.

[2]. Buckley, F., Harary, F., 1990, *Distance in Graphs*, Addison-Wesley, Red-wood,.

[3]. Cohen, N., Dimitrove, D., Krakovski, R., Vilkasinovic, V., 2010, *On Wiener index of graphs and their line graphs*, Inst. Math. Phys. Mech., 48 , 11-13.

[4]. Dankelmann, P., Gutman, I., Mukwembi, S., Swart, H.C., 1981, *The edge-Wiener index of a graph*, Discrete Math., 309 3452-3457.

[5]. Dodrynin, A. A., Melnikov, L. S., 2005, *Some results on wiener index of trees; theory and applications*, Electron. Notes Discrete Math., 22, 469-475.

[6]. Gutman, I., 1996, *Distance of line graphs*, Graph Theory Notes New York , 31, 49-52.

[7]. Gutman,I., 1996, E. Estrada, *Topological indices based on the line graph of the molecular graph*, J.Chem. Inf. and Compute. sci., 36(3) , 541-543.

[8]. Gutman, I., Popovic, L., Mishra, B.K., Kuanar, M., Estrada.E., Guevara, N., 1997, *Applications of line graphs in physical chemistry: Predicting the surface tensions of alkanes,* J. Serb. Chem. Soc., 62(3) , 1025-1029.

[9]. Gutman, I., Yeh, Y., Lee, S., Luo, Y., 1993, *Some recent results in the theory of Wiener number*, Indian J. Chem., 32A , 651-661.

[10]. Harary,F., 1969,*Graph Theory,* Addison-Wesley, Reading.

[11]. Kulli. V.R., Basavanagoud, B., 2004, *Characterizations of planar plick graphs,* Discussion Mathematics,, Vol.24 pp.41-45 .

[12]. Kulli V.R., Muddebihal, M.H., 2006, *Lict and litact graph of graph*, Journal of Analysis and computation,, Vol.2 No 1, 33-43.

[13]. Nikolic, S., Trinajstic,N., Mihalic, Z., 1996, *The Wiener index; development and applications*, Croat. Chem. Acta 68, 105-129.

[14]. Ramane, H.S., Revankar, D.S., Ganagi, A.B., 2012, *On the Wiener index of a graph*, J.Indones. Math.SocVol.18, No1, 57-66.

[15]. Ramane, H.S., Revankar D.S., Gutman I., Walikar, H.B., 2009, *Distance spectra and distance energies of line graphs of regular graph*, Publ.de L'Inst. Math., 85(99), 39-46.

[16]. Randic,M., 1993*, In search for graph invariants of chemical interest*, J.Mol.struct 300, 551-571.

[17]. Walikar, H.B., Shigihalli, V.S., Ramane,*H.S., 2004, Bounds on the Wiener number of a graph,* MATH Comm. Math. Comp. Chem. 50,117-132.

[18]. Wiener,H., 1947, *Structural determination of parafin boiling points,* J. Amer. Chem. Soc., 69 , 17-20.