Vol.4.Issue.2.2016 (April-June)

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com

Email:editorbomsr@gmail.com

RESEARCH ARTICLE

ISSN:2348-0580

ON THE GEOMETRY OF RIEMANNIAN SUBMERSIONS OVER ORBIT OF KILLING VECTOR FIELDS

TURSUNOV BAYRAMALI AKBAROVICH

National University of Uzbekistan, 100174, st. Talabalar 2, National University of Uzbekistan, Tashkent, Uzbekistan

TURSUNOV BAYRAMALI AKBAROVICH

ABSTRACT

In the paper it is studied the geometry of submersions which is mapping of Euclidean space to a orbit Killing vector fields. It is proved that there exists a Riemannian metric on the orbit with respect to which the submersion will be Riemannian, and foliation generated by submersion will be isoparametric.

Keywords: Killing vector field, an orbit, submersion, Riemannian submersion, Riemannian foliations, isoparametric foliation, sectional curvature, basic vector field.

1. INTRODUCTION

In this paper we study geometry of some submersions, which arise at study of geometry of Killing vector fields. Geometry of vector fields is subject of numerous studies in connection its importance in geometry and other areas of mathematics (Zoyidov A.N., Tursunov B.A., 2015; Narmanov A.Ya., Saitova S.S., 2014; Gromoll D., Walschap G. 2008).

Let M be a smooth Riemannian manifold of dimension n with the Riemannian metric g, ∇ - the Levi-Civita connection, $\langle\cdot,\cdot\rangle$ - inner product defined by the Riemannian metric g.

We denote by $V(M)$ the set of all smooth vector fields defined on M, through a $[X,Y]$ Lie bracket of vector fields $X, Y \in V(M)$. The set $V(M)$ is a Lie algebra with Lie bracket.

Throughout the paper, the smoothness means smoothness of a class C^{∞}.

Definition 1. Differentiable mapping $\pi: M \to B$ of a maximal rank, where B is smooth manifold of dimension m, $n > m$, is called submersion.

By the theorem on the rank of a differentiable function for each point $p \in B$ the full inverse image $\pi^{-1}(p)$ is a submanifold of dimension $k = n - m$. Thus submersion $\pi: M \to B$ generates a foliation F of dimension $k = n - m$, whose leaves are submanifolds $L_p = \pi^{-1}(p), p \in B$.

To the study of the geometry of submersions were devoted numerous papers (Zoyidov A.N., Tursunov B.A., 2015 – Reinhart B. L.,1959), in particular in paper (O'Neil B., 1966) derived the fundamental equations of submersion.

Let F be a foliation of dimension k, where $0 < k < n$ (Gromoll D., Walschap G., 2008). We denote by $L_p(q)$ leaf of foliation F, passing through a point $q \in M$, where $\pi(q) = p$, by T_qF tangent space of leaf L_p at the point $q \in L_p$, by H_qF orthogonal complement of subspace T_qF.

As result arise subbundle's $TF = \{T_qF\}$, $HF = \{H_qF\}$ of the tangent bundle TM and we have an orthogonal decomposition $TM = TF \oplus HF$.

Thus every vector field X is decomposable as: $X = X^v + X^h$, where $X^v \in TF$, $X^h \in HF$. If $X^h = 0$ (respectively $X^v = 0$), then the field X is called as vertical (respectively horizontal) vector field.

The submersion $\pi: M \to B$ is said to be Riemannian if differential $d\pi$ preserves lengths of horizontal vectors. As it is known those Riemannian submersions generate Riemannian foliation (Reinhart B.L., 1959).

We remark that foliation F is called Riemannian if every geodesic, orthogonal in some point to leaves, remains orthogonal to leaves in all points.

The curve is called as horizontal if it's tangential vector is horizontal.

Let $\gamma: [a, b] \to B$ is smooth curve in B, and $\gamma(a) = p$. Horizontal curve $\tilde{\gamma}: [a, b] \to M$, $\tilde{\gamma}(a) \in \pi^{-1}(\,p)$ is called as horizontal lift of a curve $\gamma: [a, b] \to B$, if $\pi(\tilde{\gamma}(t)) = \gamma(t)$ for all $t \in [a, b]$.

The map $: V(F) \times HF \to V(F)$, defined by the formula $S(U, X) = \nabla_U^v X$, is called second basic tensor, where $V(F)$, HF set of vertical and horizontal vector fields respectively, where $\nabla_U^v X$ is vertical component of vector field $\nabla_U X$.

At the fixed field of normal $X \in HF$, map $S(U, X)$ generates tensor field S_X of type (1,1):

$$S(U, X) = S_X U = \nabla_U^v X.$$

The tensor field S_X is linear map and consequently it is defined by the matrix A: $S(U, X) = AU$.

Horizontal vector field X is called basic if vector field $[U, X]$ is also vertical for each vector field $U \in V(F)$. Eigenvalues of matrix A is called the principal curvature of foliation F, when vector field X is basic. If the principal curvatures are locally constant along leaf, then foliation F is called isoparametric.

2. MAIN RESULT

Let's consider some set $D \subset V(M)$, which contains finite or infinite number of smooth vector fields. For a point $x \in M$ through $t \to X^t(x)$ we will denote the integral curve of a vector field X passing through a point x at $t = 0$. Map $t \to X^t(x)$ is defined in some domain $I(x) \subset R$, which generally depends on field X and point x.

Definition 2. The orbit $L(x)$ of set D, passing through the point x, is defined as set of such points $y \in M$, such that there exists $t_i \in R$, and vector fields $X_i \in D$

$$y = X_k^{t_k}\left(X_{k-1}^{t_{k-1}}\left(\dots\left(X_1^{t_1}(x)\right)\dots\right)\right).$$

In (Sussmann. H., 1973) it is proved that each orbit of a set of smooth vector fields has a differential structure of the smooth immersed submanifold of M.

Recall that the vector field X on M is called the Killing vector field, if the group of local transformations $x \to X^t(x)$ consists of isometries (Narmanov A.Ya., Saitova S.S. 2014).

Note that the Lie bracket of two fields of the field of Killing gives a field of Killing and a linear combination of Killing fields over the field of real numbers is also a field of Killing

Therefore, the set of all Killing vector field on the manifold M, denoted $K(M)$, generates a Lie algebra over the field of real numbers. It is known that the Lie algebra $K(M)$ is finite-dimensional. We will denote through $A(D)$ the smallest Lie subalgebra of algebra $K(M)$, containing set D.

Since the algebra $K(M)$ finite, there exist vector fields X_1, X_2, …, X_m that vectors $X_1(x)$, $X_2(x)$, …, $X_m(x)$ forms bases for the subspace $A_x(D)$ for each $x \in M$.
In (Narmanov A.Ya., Saitova S.S. 2014) proved the following theorem, which shows that each point in the orbit $L(x_0)$ can be reached from x_0 by finitely many "switches" with the use of the vector fields $X_1, X_2, \dots, X_m$ in a certain order.

Theorem 1. Set of points of form

$$y = X_m^{t_m}\left(X_{m-1}^{t_{m-1}}\left(\dots\left(X_1^{t_1}(x_0)\right)\dots\right)\right),$$

where $(t_1, t_2, \dots, t_m) \in R^m$, coincides with the orbit $L(x_0)$.
This theorem allows constructing various submersions $\pi: R^m \to L(x_0)$ using the vector fields X_1, X_2, …, X_m by the formula

$$\pi(t_1, t_2, \dots, t_m) = X_m^{t_m}\left(X_{m-1}^{t_{m-1}}\left(\dots\left(X_1^{t_1}(x_0)\right)\dots\right)\right).$$

Now, let us to construct the submersion $\pi: R^{n+k} \to R^n$, where $n \geq 2k$, $k \in N$.
Let's consider the Killing vector fields

$$Y_1 = \frac{\partial}{\partial x_1},\quad Y_2 = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2},\quad Y_3 = \frac{\partial}{\partial x_3},\quad Y_4 = -x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4},\quad \dots,\quad Y_{2k-1} = \frac{\partial}{\partial x_{2k-1}},$$

$$Y_{2k} = -x_{2k}\frac{\partial}{\partial x_{2k-1}} + x_{2k-1}\frac{\partial}{\partial x_{2k}},\quad Y_{2k+1} = \frac{\partial}{\partial x_{2k+1}},\quad Y_{2k+2} = \frac{\partial}{\partial x_{2k+2}},\quad \dots,\quad Y_n = \frac{\partial}{\partial x_n}$$

on R^n. It is easy to check that the orbit of vector fields Y_i, $i = \overline{1,n}$ is coincides with space R^n and the basis of subalgebra $A(D)$ consists of following vector fields

$$X_1 = \frac{\partial}{\partial x_1},\quad X_2 = \frac{\partial}{\partial x_2},\quad X_{n+1} = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2},\quad X_3 = \frac{\partial}{\partial x_3},\quad X_4 = \frac{\partial}{\partial x_4},\quad X_{n+2} = -x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4},$$

$$\dots,\quad X_{2k-1} = \frac{\partial}{\partial x_{2k-1}},\quad X_{2k} = \frac{\partial}{\partial x_{2k}},\quad X_{n+k} = -x_{2k}\frac{\partial}{\partial x_{2k-1}} + x_{2k-1}\frac{\partial}{\partial x_{2k}},$$

$$X_{2k+1} = \frac{\partial}{\partial x_{2k+1}},\quad X_{2k+2} = \frac{\partial}{\partial x_{2k+2}},\quad \dots,\quad X_{2k+l} = \frac{\partial}{\partial x_{2k+l}}$$

where $n = 2k + l$, and consequently the orbit $L(p)$ for each point $p \in R^n$ coincides with space R^n.
We will define following submersion $\pi: R^{n+k} \to R^n$ with formula

$$\pi(t_1, t_2, \dots, t_{n+k}) = X_{n+k}^{t_{n+k}}\left(X_{n+k-1}^{t_{n+k-1}}\left(\dots\left(X_{2k+l}^{t_{2k+l}}\left(X_{n+k}^{t_{n+k}}\left(X_{2k}^{t_{2k}}\left(X_{2k-1}^{t_{2k-1}}\left(\dots\left(\left(X_2^{t_2}\left(X_1^{t_1}(O)\right)\right)\dots\right)\right)\right)\right)\right)\dots\right)\right).$$

where O - origin of coordinates in R^n.

Theorem 2. There exists a Riemannian metric $\tilde{g}$ on R^n that:

1) Submersion $\pi: R^{n+k} \to R^n$ is Riemannian submersion and it generates riemannian foliation;
2) Submersion $\pi: R^{n+k} \to R^n$ generates on R^{n+k} isoparametric foliation;
3) $(R^n, \tilde{g})$ is manifold of nonnegative curvature;
   If $k \geq 2$, then
4) Submersion $\pi: R^{n+k} \to R^n$ generates on R^{n+k} a foliation of zero curvature.

Proof of theorem 2.

1) Mapping π has the form

$$\pi(t_1, t_2, \dots, t_{n+k}) = (x_1, x_2, \dots, x_n),$$

where $t_1, t_2, \dots, t_{n+k} \in R^{n+k}$, $x_1, x_2, \dots, x_n \in R^n$ and

$$x_1 = t_1 \cos t_{n+1} - t_2 \sin t_{n+1}, \qquad x_2 = t_1 \sin t_{n+1} + t_2 \cos t_{n+1},$$
$$x_3 = t_3 \cos t_{n+2} - t_4 \sin t_{n+2}, \qquad x_4 = t_3 \sin t_{n+2} + t_4 \cos t_{n+2},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$x_{2k-1} = t_{2k-1} \cos t_{n+k} - t_{2k} \sin t_{n+k}, \qquad x_{2k} = t_{2k-1} \sin t_{n+k} + t_{2k} \cos t_{n+k},$$
$$x_{2k+1} = t_{2k+1} \qquad x_{2k+l} = t_{2k+l}$$

The easy calculation shows, the rank of the Jacobi matrix of mapping π at each point $q \in R^{n+k}$ is equal n.

Therefore for each point $p = (x_1, x_2, \dots, x_n) \in R^n$ the full inverse image $\pi^{-1}(p)$ is a k-dimensional submanifold L_p in R^{n+k} and has the form

$$\pi^{-1}(p) = L_p = \{(t_1, t_2, \dots, t_{n+k})\}$$

where

$$t_1 = x_1 \cos u_1 + x_2 \sin u_1, \qquad t_2 = -x_1 \sin u_1 + x_2 \cos u_1, \qquad t_{n+1} = u_1,$$
$$t_3 = x_3 \cos u_2 + x_4 \sin u_2, \qquad t_4 = -x_3 \sin u_2 + x_4 \cos u_2, \qquad t_{n+2} = u_2,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$t_{2k-1} = x_{2k-1} \cos u_k + x_{2k} \sin u_k, \qquad t_{2k} = -x_{2k-1} \sin u_k + x_{2k} \cos u_k, \qquad t_{n+k} = u_k,$$
$$t_{2k+1} = x_{2k+1} \qquad t_{2k+l} = x_{2k+l}$$

where $u_i \in R$, $i = \overline{1,k}$, $2k + 1 = n$.

It is easy to check that the foliation F, generated by the submersion $\pi: R^{n+k} \to R^n$, consists of k-dimensional surface in the R^{n+k}, and the vector-speeds of curves u_i (a vertical fields) on this surface has the form,

$$V_i = t_{2i} \frac{\partial}{\partial x_{2i-1}} - t_{2i} \frac{\partial}{\partial x_{2i}} + \frac{\partial}{\partial x_{n+i}}.$$

This vector fields is a Killing field. Really, it is known that, the vector field $X = \xi_i \frac{\partial}{\partial x_i}$ in R^n is the Killing vector field if and only if the following conditions are satisfied (Narmanov A.Ya., Saitova S.S. 2014):

$$\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} = 0, \quad i \neq j; \qquad \frac{\partial \xi_i}{\partial x_i} = 0, \quad i, j = 1,2, \dots, n.$$

The vertical fields V_i satisfies these conditions and consequently is a Killing field.

Thus foliation F is a Riemannian.

Let $\gamma: [a, b] \to R^n$, $\gamma(a) = p$ a smooth curve. Then for each point $q \in \pi^{-1}(p)$ there is it's horizontal lift $\tilde{\gamma}: [a, b] \to R^{n+k}$ such that $\tilde{\gamma}(a) = q$ (Zoyidov A.N., Tursunov B.A. 2015).

Let X, Y vector fields on R^n, and X^*, Y^* - horizontal lifting of the vector fields, i.e. X^*, Y^* are horizontal vector fields on R^{n+k} and $d\pi(X^*) = X$, $d\pi(Y^*) = Y$. Since the vector field V_i are Killing fields, a inner product $\langle X^*, Y^* \rangle$ is constant along $L_p = \pi^{-1}(p)$ (O'Neil B., 1966). Hence, if we will put $\langle X, Y \rangle(p) = \langle X^*, Y^* \rangle(q)$, where $q \in L_p$, $\langle X, Y \rangle$ is correctly defined inner product, and we get Riemannian metric $\tilde{g}$ on R^n. Concerning this Riemannian metric submersion $\pi: R^{n+k} \to R^n$ will be Riemannian.

2) Vector fields

$$H_i^1 = t_{2i-1} \frac{\partial}{\partial x_{2i-1}} + t_{2i} \frac{\partial}{\partial x_{2i}}, \quad H_i^2 = -t_{2i} \frac{\partial}{\partial x_{2i-1}} + t_{2i-1} \frac{\partial}{\partial x_{2i}} + \left(t_{2i}^2 + t_{2i-1}^2\right) \frac{\partial}{\partial x_{n+i}}, \quad H_j^3 = \left(t_{2i}^2 + t_{2i-1}^2\right) \frac{\partial}{\partial x_j},$$

where $i = 1, \dots, k$, $j = 2k + 1, \dots, n$ are basic fields, as:

$$[V_i, H_i^1] = 0, \quad [V_i, H_i^2] = 0, \quad [V_i, H_j^3] = 0.$$

We calculate the second fundamental tensor $S_{H_i^j}$ with respect to fields H_i^j :

$$\nabla_{V_i} H_i^1 = t_{2i} \frac{\partial}{\partial x_{2i-1}} - t_{2i-1} \frac{\partial}{\partial x_{2i}}, \quad \nabla_{V_i} H_i^2 = t_{2i-1} \frac{\partial}{\partial x_{2i-1}} + t_{2i} \frac{\partial}{\partial x_{2i}}, \quad \nabla_{V_i} H_j^3 = 0,$$

respectively,

$$S(V_i, H_i^1) = \frac{\langle V_i, \nabla_{V_i} H_i^1 \rangle}{V_i^2} V_i = \frac{t_{2i-1}^2 + t_{2i}^2}{t_{2i-1}^2 + t_{2i}^2 + 1} V_i,$$

and others tensors $S(V_i, H_i^2)$, $S(V_i, H_j^3)$ are equal zero.

In this case eigenvalues λ_i^1 corresponding matrixes A_i^1 are equal:

$$\lambda_i^1 = \frac{t_{2i-1}^2 + t_{2i}^2}{t_{2i-1}^2 + t_{2i}^2 + 1},$$

respectively others eigenvalues are equal zero.

It is easy to check that $V_i(\lambda_i) = 0$. Thus foliation F is isoparametric.

3) We will calculate sectional curvature of manifold $(R^n, \tilde{g})$ in the two-dimensional direction, defined by vectors $V_i^*(p), V_j^*(p)$ at the point $p \in R^n$.

By the formula O'Neill (O'Neil B., 1966), for the Riemannian submersion $\pi: M \to B$, sectional curvatures K, K_* of manifolds M and B are connected by the relation

$$K(X, Y) = K_*(X^*, Y^*) - \frac{3}{4} \frac{|[X, Y]^v|^2}{\|X \wedge Y\|^2},$$

where – X, Y are horizontal vector fields of M, $X \wedge Y$-bivector constructed on vectors X, Y.

As an Euclidean space R^{n+k} is space of zero sectional curvature $K(X, Y) = 0$ for any arbitrary two-dimensional direction. Therefore

$$K_*(X^*, Y^*) = \frac{3}{4} \frac{|[X, Y]^v|^2}{\|X \wedge Y\|^2} \geq 0.$$

Thus manifold $(R^n, \tilde{g})$ is n-dimensional manifold of nonnegative curvature.

Really, consider the vector fields $H_i^* = d\pi(H_i)$ on $(R^n, \tilde{g})$. By the fact that the mapping $\pi: R^{n+k} \to R^n$ has maximum rank, vector fields H_i^* are linearly independent in each point of manifold $(R^n, \tilde{g})$. In this case we can calculate sectional curvature. As $K(H_i, H_j) = 0$, $i, j = 1, \ldots, n$, $i \neq j$ we will receive following expression for the curvature

$$K_*(H_i^*, H_j^*)(p) \geq 0.$$

4) By the formula O'Neill (O'Neil B., 1966), for the Riemannian submersion $\pi: M \to B$, sectional curvatures K, $\widehat{K}$ of manifold M and the foliation F are connected by the relation

$$K(U, V) = \widehat{K}(U, V) - \frac{\langle \nabla_U^h U, \nabla_V^h V \rangle - \left\| \nabla_U^h V \right\|^2}{\|U \wedge V\|^2},$$

where - U and V are vertical vector fields, $U \wedge V$-bivector constructed on vectors U, V, U^v and U^h vertical and horizontal complement of vector U.

As an Euclidean space R^{n+k} is space of zero sectional curvature $K(U, V) = 0$ for any arbitrary two-dimensional direction. Therefore

$$\widehat{K}(U, V) = \frac{\langle \nabla_U^h U, \nabla_V^h V \rangle - \left\| \nabla_U^h V \right\|^2}{\|U \wedge V\|^2}.$$

Thus, L_p is manifold of zero curvature.

The simple calculation shows that $\nabla_U^h V = 0$, $\langle \nabla_U^h U, \nabla_V^h V \rangle = 0$. Thus, $\widehat{K}(U,V) = 0$ and the foliation F is k-dimensional manifold of zero curvature. The theorem 2 is proved.

ACKNOWLEDGMENT

This paper is partially supported by Ministry of High Education of Republic of Uzbekistan, Grant F4-04.

REFERENCES

[1]. Zoyidov A.N., Tursunov B.A. Geometry of submersions on manifolds of nonnegative curvature. Uzbek mathematical journal. 2015, № 2, pp.27-34.

[2]. Narmanov A.Ya., Saitova S.S. On the Geometry of Orbits of Killing Vector Fields. Differential Equations, 2014, Vol. 50, No. 12, pp. 1–8. Original Russian Text published in Differentsial'nye Uravneniya, 2014, Vol. 50, No. 12, pp. 1582-1589.

[3]. O'Neil B. The Fundamental equations of a submersions. Michigan Mathematical Journal, v.13, 1966, p. 459-469

[4]. Gromoll D., Walschap G. Metric Foliations and Curvature. Basel-Boston-Berlin, 2008, 185 p.

[5]. Hermann R. A sufficient condition that a mapping of Riemannian manifolds to be a fiber bundle. Proc. Amer. Math. Soc.11 (1960), 236-242.

[6]. Reinhart B. L. Foliated manifolds with bundle-like metrics. Annals of Mathematics, Second Series, Vol. 69, No. 1,1959, pp. 119-132

[7]. Sussmann. H. Orbits of family of vector fields and integrability of systems with singularities. Bull. Amer. Math. Soc., 1973, 79, p. 197-199.