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RESEARCH ARTICLE

SOME PROBLEMS RELATE TO CURVATURE ON SMOOTH MANIFOLDS

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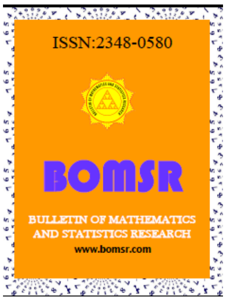

ABSTRACT

The classification themes allows us to work on low-dimensional manifolds for certain curvature related problems. In this paper such an attempt is maid by surveying some problems with new insights.

Keywords: Positive scalar curvature, topological space, non-negative curvature.

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1. INTRODUCTION

In this note we give some highlights of curvature which is an important geometric attribute. The modern theory of differential geometry emphasizes the importance of curvature on many ways. We come across the problem of determining the global features of the manifold which unless one thrice to know the topology of manifolds with specified curvature values, it will not be to the complete understanding of the manifolds. For instances, the complete three dimensional manifolds with positive scalar curvature. The topology over determine.

In case of complete manifold of dimensional greater than three are Schoen and S.T.Yau of give the first result. On structure of manifold with positive scalar curvature. Later on Gromov-Lawson provided some of this results using the arguments of harmonic spinners. This generalized the workshop Lichnerowicz [4] and Hitchin[3]. These works enabled you and Schoen to come we the complete understanding of such manifold with positive scalar curvature.

To give an account of these exploitation. We cities the following results due to them.

Theorem 1: Let M be a complete orientable three dimensional manifold with scalar curvature ≥ 1 .Then we can write M as an increasing union of compact subdomains Ω_1, each of which is diffeomorphic to the complement of 'S' finite number of disjoint balls of a three-dimensional manifold of the from $M_1 \# M_2 \#, \ldots\ldots\ldots, \# M_k$, $N_1, \ldots\ldots, N_k$, where " # " means connected sum, $M_1 = S^2 \times S^1$ and N_i is a compact three-dimensional manifold with finite fundamental group.

We say that it is a conjecture in topology that a compact three-dimensional manifold with finite fundamental group is a spherical space from. If this conjecture is true, theorem 1 gives a complete classification of complete three-dimensional manifold with scalar curvature ≥ 1 . The last assumption can be replaced by requiring only scalar curvature≥ 0. Under this weaker assumption, we have to allow handle bodies as possible connected summands in the above theorem. In this way, we can even allow our three-dimensional manifold to have compact boundaries whose mean curvature is nonnegative with respect to the outward normal.

As a consequence of the theorem in [9], we can also prove the following proposition which will be used in understanding manifolds with positive scalar curvature in higher dimension.

Proposition: Let M be a complete three-dimensional manifold with scalar curvature≥ 1. Then we can write M as an increasing union of compact sub-domains Ω_1 so that each component of $\partial\Omega_1$ has an area less than some constant C which is independent of i.

Theorem 2: Let M be a complete three-dimensional manifold which is connected at infinity. If the scalar curvature of M is greater than one, then there exists no distance-increasing map from the real line into M.

All of the above theorems have a finite version, i.e., we do not have to assume that M is complete and the conclusion is focused on the part of M which is not close to ∂M .

When M > 3, we proved the following theorem quite a while ago.

Theorem 3: Let M be a compact manifold with positive scalar curvature. Then M cannot be represented as a homology class in a compact manifold with non-positive sectional curvature.

We proved this theorem by following our arguments in [8] using minimal hyper surfaces. We note that the technical assumption of dimensions ≤ 7 in [8] was dropped by us a few years ago. Independently, Gao, in his Stony Brook thesis, was able to prove Theorem 3 following an argument of Gromov-Lawson. Some special cases of the following two theorems were also obtained by Gromov-Lawson independently.

2. Curvature Manifestation

In this section we will give a simple exploitation to presence of curvature from the differential geometry point of view. We know that the real n-dimensional Euclidean space R^n are the familiar example of smooth manifolds. The Euclidean metric in R^n will enable us to develop geometry by introducing co-ordinates. This Euclidean geometry is a model with curvature being zero. The situation dramatically changed with presence of curvature. We are going to examine this fact. We recall few definitions with regard to basic topological space and function define on them.

Definition: Let X be a topological space a family $U_\alpha : \alpha \in \Lambda$ of open sets (each U_α open) is said to be an open cover of X if $X = \bigcup U_\alpha \quad \forall \alpha \in \Lambda.$

Definition: The topological space X is said to be compact if every open cover of X has a finite subcover.

If $U_\alpha : \alpha \in \Lambda$ is a family of open cover for X then a subcover is a family $V_\alpha : \alpha \in \Lambda$ of $U_\alpha : \alpha \in \Lambda$ whenever $U_\alpha \subset V_\alpha$ for each α and for $1 \leq i_1 < i_2 < \cdots\cdots < i_k = n$ if $X = \bigcup_{i=1....n} V\alpha_i$ then for each such a finite subcover we say that X is compact.

Example of compact topological space :

1. The [a, b] in R is compact.
2. The unit circle $S^1 = \{(x, y) \in R^2 / x^2 + y^2 = 1\}$.
3. The sphere $S^2 = \{(x, y, z) \in R^3 / x^2 + y^2 + z^2 = 1\}$ is compact.

However, the real line R, the plane R^2, the space R^3 are not compact.

Interestly, the compact sets in the example give above are the subsets of R the real line, the plane R^2 and the space R^3.

Topologically the [a, b] the unit circle S^1 and the unit sphere S^2 are different from R ,R^2 ,R^3.

This topological distances lead to the classification problems of the spaces up to homomorphism. We will precisely explain this fact in our following description.

For a case of 1- dimensional spaces we have seen that the R and the unit circle S^1 or not homomorphic by this we mean there is no continuous map from S^1 into the real line R whose inverse is also continuous map. Thus if X is a 1- dimensional spaces then X is either homomorphic to the real line R or homomorphic to the circle S^1.i.e.,

$$X\ compact \Rightarrow X \cong [a\ ,b]$$

$$X\ compact \Rightarrow X \cong \mathrm{R}$$

What about the case of two dimensional spaces?

If X is a 2-dimensional space.

We have seen this in one of our example S^2, the unit sphere which is two dimensional and compact.

Now, the plane R^2 is also 2- dimensional, but it is not compact.

Therefore, S^2 and R^2 are different as topological spaces. So, if X is 2- dimensional and compact then X $\cong S^2$ this is one sonorous, and if X is not compact then X $\cong R^2$.

X is a not compact.

If X is a 2-dimensional space of compact the X $\cong S^2$ or S^2 with n-handles if X is as compact the X $\cong$ R^2, $S^2 \subset R^3$, $t^2 \subset R^3$ or R^4 observe that

As subset they are all compact for by R, R^2,………, R^n are non-compact.

As quotient spaces, topology on S^1,S^2 and t^2 are induced topology of R and R^2 respectively.

We also provide a brief description of differentiable structure on S^1 and S^2, with that differentiable structure they become differentiable manifolds.

a) differentiable structure a S^1

Recall the definition of a smooth (differentiable) manifold and its atlas description for S^1, we will give these details.

If n = 1, and since S^1 is connected from the very definition, as a manifold it is compact and connected. The atlas for S^1 contains only two charts. We shall write them as (U, f) and (V, g) where U, V, f and g are determined as follows: We define f^{-1}, as a map from the open interval (0 , 2π) into S^1, by $\theta \to (\cos\theta, \sin\theta)$ that is, $f^{-1} : (0,2\pi) \to S^1$. is a local diffeomorphism.

i.e., f^{-1} is continuous, invertible and is therefore homomorphism.

We take the open set $U = S^1 - \{(1,0)\}$

Thus, we have defined U and f

Next, we shall define V and g for this, we take g^{-1}, here it differs from f^{-1} in the sense, instead of (0 , 2π). We take $(-\pi, \pi)$. The definition for g^{-1} is same as that of f^{-1} given earlier. Thus we here , $g^{-1}: (-\pi\ , \pi) \to S^1$ as $\theta \to (\cos\theta, \sin\theta)$.

The image of $g^{-1}(-\pi\ , \pi)$ in S^1 is the set $S^1 - \{(-1, 0)\}$. This gives as V and g. Further, with that $U \cup V = S^1$. Thus, the structure $\{\ (\ U\ , f)\ and\ (\ V, g\ )\}$ from a smooth atlas on S^1 and S^1 becomes a smooth manifold of dimension one.

For higher dimension examples, one can as well take the cartesian product of the lower dimensional manifolds. Since R and S^1 are one-dimensional manifolds.

R ×R, $S^1 \times S^1$, $\underbrace{\mathrm{R} \times \mathrm{R} \times \cdots\cdots \times \mathrm{R}}_{n\ times}$, $\underbrace{S^1 \times S^1 \times \cdots\cdots \times S^1}_{n\ times}$

give high dimensional manifolds as examples.

Thus, $\underbrace{R \times R \times \cdots\cdots \times R}_{n\ times}$ = R^n is an n-dimensional manifold and $\underbrace{S^1 \times S^1 \times \cdots\cdots \times S^1}_{n\ times}$ = t^n is an n-dimensional manifold.

b) We shall respect to $S^1 \times S^1$ which is a torus t^2, as a manifold, it is smooth two-dimensional compact manifold. We will not go into the definite of providing the smooth atlas description of the manifold.

c) the sphere S^n, as an n-dimensional manifold is not a product of a one - dimensional manifolds.

We have noticed that these manifolds arise quite naturally and are seen every where. In fact whenever one deals with differentiable function of one or more real or complex variables there is usually a manifold in which these function are defined.

We conclude this section in the notion of orientability for manifolds.

Orientability has its genesis in the properties of surface, two-dimensional manifolds in R^3 . For instance, a disc give by $x^2 + y^2 \leq a^2, z = 0$ which is a surface has a top-side and a bottom-side. Where as in case of a sphere S^2 (example 1,2,3) it has an inside and outside. Same is the case with the torus t^2 such two indeed surfaces are called orientable. Since we can use their two sidedness to define orientations or directions in the R^3.

For the disc, the unit normal n to the surface has two possible directions from top to bottom as far bottom with top corresponding to $n = \pm K$. Similarly, the sphere S^2 has two normal direction at each part inward or outward pointing normal $n = \pm e_\gamma$ where e_γ is a unit vector in the normal direction.

However, if the surface has not got two sides, then we are into difficulty the properties of the surface will not help us to define orientation on such one indeed surface. We avoid this orientation.

Having got clarity about orientability of the two dimensional manifolds one would generalise this concept for manifolds of any dimension. This fact will be described now non-orientability change the direction for a normal at a point with a fixed orientation. This is the case with the Mobins strip.

To this end consider a vector space V with two bases $\{e_1, \ldots\ldots, e_n\}$ and $\{e_1', \ldots\ldots, e_n'\}$ of V. Let M be the matrices which performs the change of basis.

$e_i' = Me_i, \qquad i = 1, \ldots\ldots\ldots, n \qquad (**)$

the determinant connected with M it is also called the Jacobins determinant.

Then from (**), the determinant of M is either 0 or 1, side M is invertible (M most be invertible).

If the determinant M > 0 then we say this $\{e_i\}$ and $\{e_i'\}$. If determine n < 0, then the bases $\{e_i\}$ and $\{e_i'\}$ are said to be opposite orientation.

For example, $\{e_1', \ldots\ldots, e_n'\} = \{e_1, \ldots, e_{n-1}, -e_n\}$

So that the corresponding metric M given by

$$M = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & -1 \end{pmatrix}$$

and its determinant is -1

or $\{e_1', \ldots\ldots, e_n'\} = \{e_1, e_2, \ldots, e_{n-2}, -e_{n-1}, -e_n\}$

then determinant open matrix given by

$$M = \begin{pmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & -1 & \\ 0 & & & & & -1 \end{pmatrix}$$

has det = 1.

Thus this set of all bases for V is divided into two exponential classes: each class is transformed into itself by matrices is transformed into the other by matrices of negative determinant.

Suppose we have curve $\gamma: I \to R^n$, and we wish to know the history of the system of K-particles in R^n. Is there a relation that ensures us to know this fact such problems are if they are physically in relative system of differential equations, with initial condition, determined by the position and movements of there particles in space. The geometry of the space and attributes like curvature (associated with the curves / surfaces or integral an hyper surface) curve to fore. In our study we first by to understand this geometric driven problem. The following is a particular description for a 3- dimensional need Euclidean space.

Let K particular move in R^3 under the known forces. Suppose that the state of the system of there K-particle at a given time is determined by knowing the position and momentum of each particle.

Thus, at a given time the system is determined by a point in R^{6k}

The different equation giving the system is the Hamilton equation

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = \frac{\partial H}{\partial q_i}()$$

if ( ) is solved uniquely for a given initial condition, then we get the entire history of the system determined by a curve in R^{6k} to this end let us recall the group action on manifolds. similarly, such group action on spaces are of interest if they presence the structure of the manifold (or space). Topological structure would export the qualitative property of action of groups on space, measure theoretic and smooth structure are structure conditions of the group action on spaces.

Each element of the group acts as a transformation on the space and presence the given structure. In above context we have cited the motion of K-particles in R^3. Suppose the group R action on R^3, thus for each real t ∈ R, (a time parameter) we have a map t_t: $R^{6k} \to R^{6k}$. Thus, a family of transformation { t_t : t ∈ R} of the state space.

If x is a point in the state space representing the system of a time to, then $t_t(x)$ we shall denote the point of the state space representing the system at time $t + t_0$.

Then, we see that t_t is a transformation of the state space and $T_o = id$ and $T_{t+s} = T_t \ o \ T_s$.

Thus, $t \to T_t$ determine the action of the group R on the state space we know that, the Hamilton H is constant along solution curves, each energy surface $H^{-1}(e)$ is invariant for the transformation t_t.

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