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NEW APPROXIMATION OPERATORS USING MIXED DEGREE SYSTEMS

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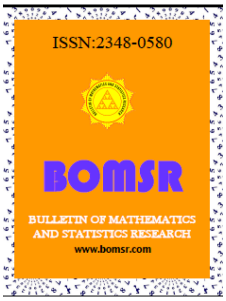

ABSTRACT

This paper is concerned with introducing and studying the first new approximation operators using mixed degree system and second new approximation operators using mixed degree system which are the core concept in this paper. In addition, the approximations of graphs using the operators first lower and first upper are accurate then the approximations obtained by using the operators second lower and second upper sincefirst accuracy less then second accuracy. For this reason, we study in detail the properties of second lower and second upper in this paper. Furthermore, we summarize the results for the properties of approximation operators second lower and second upper when the graph *G* is arbitrary, serial 1, serial 2, reflexive, symmetric, transitive, tolerance, dominance and equivalence in table.

Key words: Digraph, Out-degree set, In-degree set, Mixed degree set, first approximation operators, second approximation operators, first accuracy and second accuracy.

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1. INTRODUCTION AND PRELIMINARIES

The theory of rough sets. Proposed by Pawlak [8], is a good mathematical tool for data representation. Its methodology is concerned with the classification and analysis of missing attribute values, we introduce a new definitions of the lower and upper approximation operators using mixed degree systems, for example in structural analysis [21].We built on some of the results in [1], [3], [6], [7], [10], [11], [12], [13],[14], [15], [17], [18], [19] [20].

A directed graph or digraph [16] is pair $G = (V(G), E(G))$ where $V(G)$ is a non-empty set (called vertex set) and $E(G)$ of ordered pairs of elements of $V(G)$ (called edge set). An edge of the from(v, v) is called a loop. If $v \in V(G)$, the out-degree of v is $|\{u \in V(G) : (v, u) \in E(G)\}|$ and in-degree of v

is |{u∈$V(G)$: (u, v)∈$E(G)$}|. A digraph is reflexive if (v, v)∈$E(G)$ for each v∈$V(G)$, symmetric if (v, u)∈$E(G)$ implies (u, v) ∈$E(G)$, transitive if (v, u)∈$E(G)$ and (u, w)∈$E(G)$ implies (v, w)∈ $E(G)$, tolerance if it is reflexive and symmetric, dominance if it is reflexive and transitive, equivalence if it is reflexive and symmetric and transitive, serial if for all v∈$V(G)$ there exists u∈$V(G)$ such that (v, u)∈$E(G)$. A subgraph of a graph G is a graph each of whose vertices belong to $V(G)$ and each of whose edges belong to $E(G)$. An empty graph [2] if the vertices set and edge set is empty. The out-degree set of v is denoted by OD and defined by: OD = {u∈$V(G)$: (v, u) ∈ $E(G)$} and in-degree set of v is denoted by ID and defined by: ID = {u ∈ $V(G)$: (u, v) ∈ $E(G)$}.Let G = ($V(G)$,$E(G)$) be a digraph, the digraph inverse G^{-1} [5] is specified by the same set of vertices $V(G)$ and a set of edge $E(G)^{-1}$ = {(u, v) : (v, u)∈ $E(G)$}.

2. New Approximation Operators Using Mixed Degree Systems

In the rough set theory, one starts with an equivalence relation. A universe is divided into a family of disjoint subsets. The granulation structure adopted is a partition of the universe. By weakening the equivalence relations, we can have more general granulation structures such as coverings of the universe. Out-degree (resp. In-degree) systems provide an even more general granulation structures. For each vertex v of graph G, one associate it with a nonempty family of out-degree (resp. in-degree) granules, which is called an out-degree (resp. in-degree) system of v and is denoted by $OD(v)$ (resp. $ID(v)$). From this point of view, rough set theory is a special form of out-degree (resp. in-degree) system space theorem sees [4, 9 and 17].

2.1. First New Approximation Operators Using Mixed Degree Systems.

The main objective of this section is to propose a set-theoretic framework for granular computing using mixed degree systems. Some types of degree system based on arbitrary are used. Consider the generalized approximation space G = ($V(G)$, $E(G)$), we introduce a new definitions of the lower and upper approximation operators using mixed degree systems. The properties of the suggested operators are obtained. Also, we give new definitions of the accuracy of the introduced approximations. Furthermore, an interesting theorem is proved. The approximation are constructed using out, in and mixed degree systems. A comparison between these three approaches is superimposed.

Definition 2.1.1. Let G = ($V(G)$, $E(G)$) be a generalization approximation space and H⊆G. Then

(a) the firstlower and upper approximations of H using out degree systems are denoted by $L_o^1(V(H))$ and $U_o^1(V(H))$ and defined by

$$L_o^1(V(H)) = \{v \in V(H) ; OD(v) \subseteq V(H)\},$$
$$U_o^1(V(H)) = V(H) \cup \{v \in V(G) - V(H) ; OD(v) \cap V(H) \neq \phi\},$$

(b) the first lower and upper approximations of H using in degree systems are denoted by $L_i^1(V(H))$ and $U_i^1(V(H))$ and defined by

$$L_i^1(V(H)) = \{v \in V(H) ; ID(v) \subseteq V(H)\},$$
$$U_i^1(V(H)) = V(H) \cup \{v \in V(G) - V(H) ; ID(v) \cap V(H) \neq \phi\},$$

(c) the first lower and upper approximations of H using mixed degree systems are denoted by $L_m^1(V(H))$ and $U_m^1(V(H))$ and defined by

$$L_m^1(V(H)) = \{v \in V(H) ; \text{for some } MD(v) \subseteq V(H)\},$$
$$U_m^1(V(H)) = V(H) \cup \{v \in V(G) - V(H) ; \text{for all } MD(v) \cap V(H) \neq \phi\}.$$

Definition 2.1.2. Let G = ($V(G)$, $E(G)$) be a generalization approximation space and H⊆G. Then

(a) the first boundary, positive and negative regions of H using out degree systems are denoted by $Bd_o^1(V(H))$, $POS_o^1(V(H))$ and $NEG_o^1(V(H))$ and defined by

$$Bd_o^1(V(H)) = U_o^1(V(H)) - L_o^1(V(H)),$$
$$POS_o^1(V(H)) = L_o^1(V(H)),$$
$$NEG_o^1(V(H)) = V(G) - U_o^1(V(H)),$$

(b) the first boundary, positive and negative regions of H using in degree systems are denoted by $Bd_i^1(V(H))$, $POS_i^1(V(H))$ and $NEG_i^1(V(H))$ and defined by

$$Bd_i^1(V(H)) = U_i^1(V(H)) - L_i^1(V(H)),$$
$$POS_i^1(V(H)) = L_i^1(V(H)),$$
$$NEG_i^1(V(H)) = V(G) - U_i^1(V(H)),$$

(c) the first boundary, positive and negative regions of H using mixed degree systems are denoted by $Bd_m^1(V(H))$, $POS_m^1(V(H))$ and $NEG_m^1(V(H))$ and defined by

$$Bd_m^1(V(H)) = U_m^1(V(H)) - L_m^1(V(H)),$$
$$POS_m^1(V(H)) = L_m^1(V(H)),$$
$$NEG_m^1(V(H)) = V(G) - U_m^1(V(H)).$$

Definition 2.1.3. Let $G = (V(G), E(G))$ be a generalization approximation space. The first accuracy of the approximations of a subgraph $H \subseteq G$ using (out, in and mixed) degree systems are denoted by ($\eta_o^1(V(H))$, $\eta_i^1(V(H))$ and $\eta_m^1(V(H))$) and defined by

$$\eta_o^1(V(H)) = 1 - \frac{|Bd_o^1(V(H))|}{|V(G)|},$$
$$\eta_i^1(V(H)) = 1 - \frac{|Bd_i^1(V(H))|}{|V(G)|},$$
$$\eta_m^1(V(H)) = 1 - \frac{|Bd_m^1(V(H))|}{|V(G)|}.$$

It is obvious that $0 \le \eta_o^1(V(H)) \le 1$, $0 \le \eta_i^1(V(H)) \le 1$ and $0 \le \eta_m^1(V(H)) \le 1$. Moreover, if $\eta_o^1(V(H)) = 1$ or $\eta_i^1(V(H)) = 1$or $\eta_m^2(V(H)) = 1$, then H is called H-definable (H-exact) graph. Otherwise, it is called H-rough.

Example 2.1.4. Let $G = (V(G), E(G))$: $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{(v_1, v_2), (v_1, v_4), (v_2, v_2), (v_2, v_3), (v_2, v_4), (v_4, v_5), (v_4, v_3), (v_5, v_2), (v_5, v_5)\}$.

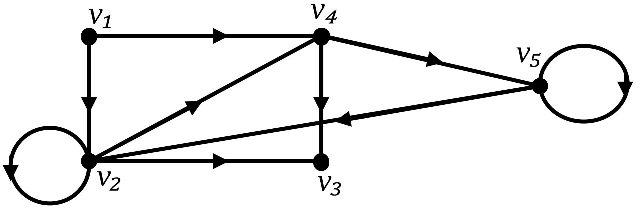

Figure 2.1.1 : Graph G given in Example 2.1.4.

We get

$OD(v_1) = \{v_2, v_4\}$, $OD(v_2) = \{v_2, v_3, v_4\}$, $OD(v_3) = \phi$, $OD(v_4) = \{v_3, v_5\}$, $OD(v_5) = \{v_2, v_5\}\}$

Also we have

$ID(v_1) = \{\phi\}$, $ID(v_2) = \{v_1, v_2, v_5\}$, $ID(v_3) = \phi$, $ID(v_4) = \{v_1, v_2\}$, $ID(v_5) = \{v_4, v_5\}$

Then we obtain

$MDS(v_1) = \{\{v_2, v_4\}, \phi\}$, $MDS(v_2) = \{\{v_2, v_3, v_4\}, \{v_1, v_2, v_5\}\}$, $MDS(v_3) = \{\phi, \{v_2, v_4\}\}$, $MDS(v_4) = \{\{v_3, v_5\}, \{v_1, v_2\}\}$, $MDS(v_5) = \{\{v_2, v_5\}, \{v_4, v_5\}\}$.

Accordingly, we can obtain the following table

Table 2.1.1: η_o^1*(V(H))*, η_i^1*(V(H))* and η_m^1*(V(H))* for all $H \subseteq G$.

V(H)	η_o^1(V(H))	η_i^1(V(H))	η_m^1(V(H))
$\{v_1\}$	4/5	3/5	1
$\{v_2\}$	2/5	2/5	4/5
$\{v_3\}$	3/5	4/5	1
$\{v_4\}$	2/5	2/5	4/5
$\{v_5\}$	3/5	3/5	4/5
$\{v_1,v_2\}$	2/5	2/5	4/5
$\{v_1,v_3\}$	2/5	2/5	3/5
$\{v_1,v_4\}$	2/5	1/5	3/5
$\{v_1,v_5\}$	2/5	2/5	3/5
$\{v_2,v_3\}$	1/5	2/5	3/5
$\{v_2,v_4\}$	1/5	1/5	2/5
$\{v_2,v_5\}$	2/5	1/5	3/5
$\{v_3, v_4\}$	2/5	2/5	4/5
$\{v_3, v_5\}$	2/5	2/5	3/5
$\{v_4, v_5\}$	1/5	2/5	3/5
$\{v_1, v_2, v_3\}$	1/5	2/5	3/5
$\{v_1,v_2, v_4\}$	2/5	2/5	3/5
$\{v_1,v_2, v_5\}$	2/5	2/5	4/5
$\{v_1,v_3, v_4\}$	2/5	1/5	3/5
$\{v_1,v_3, v_5\}$	1/5	1/5	2/5
$\{v_1,v_4, v_5\}$	1/5	2/5	3/5
$\{v_2,v_3, v_4\}$	2/5	2/5	3/5
$\{v_2,v_3, v_5\}$	2/5	1/5	3/5
$\{v_2,v_4, v_5\}$	2/5	2/5	3/5
$\{v_3, v_4, v_5\}$	2/5	2/5	4/5
$\{v_1,v_2, v_3, v_4\}$	3/5	3/5	4/5
$\{v_1,v_2, v_3, v_5\}$	2/5	2/5	4/5
$\{v_1,v_2, v_4, v_5\}$	3/5	4/5	1
$\{v_1,v_3, v_4, v_5\}$	2/5	2/5	4/5
$\{v_2,v_3, v_4, v_5\}$	4/5	3/5	1
V(G)	1	1	1
ϕ	1	1	1

Theorem 2.1.5. Let *G* = (*V(G), E(G)*) be a generalization approximation space and $H \subseteq G$. Then

(a) $L_m^1(V(H)) = L_o^1(V(H)) \cup L_i^1(V(H))$,

(b) $U_m^1(V(H)) = U_o^1(V(H)) \cap U_i^1(V(H))$,

(c) $Bd_m^1(V(H)) = Bd_o^1(V(H)) \cap Bd_i^1(V(H))$ and

(d) $\eta_m^1(V(H)) \geq \max\{\eta_o^1(V(H)), \eta_i^1(V(H))\}$.

Proof.

(a) Let $v \in (L_o^1(V(H)) \cup L_i^1(V(H)))$

$\Leftrightarrow v \in L_o^1(V(H)) \vee v \in L_i^1(V(H))$

$\Leftrightarrow OD(v) \subseteq V(H) \vee ID(v) \subseteq V(H)$

$\Leftrightarrow \exists MD(v)$ such that $MD(v) \subseteq V(H)$

$\Leftrightarrow v \in L_m^1(V(H))$

Hence, $L_m^1(V(H)) = L_o^1(V(H)) \cup L_i^1(V(H))$.

(b) Let $v \in U_m^1(V(H))$, then there are two cases:

(1) $v \in V(H) \Longrightarrow v \in U_o^1(V(H)) \wedge v \in U_i^1(V(H))$

$\Longrightarrow v \in U_o^1(V(H)) \cap U_i^1(V(H))$.

(2) $v \in V(G) - V(H)$. Then $v \in U_m^1(V(H))$

$\Longrightarrow$for all $MD(v)$, $MD(v) \cap V(H) \neq \phi$

$\Longrightarrow (OD(v) \cap V(H) \neq \phi) \wedge (ID(v) \cap V(H) \neq \phi)$

$\Longrightarrow v \in (U_o^1(V(H))) \wedge v \in (U_i^1(V(H)))$

$\Longrightarrow v \in (U_o^1(V(H)) \cap U_i^1(V(H)))$

Conversely, let $v \in (U_o^1(V(H)) \cap U_i^1(V(H)))$,

Then there are two cases:

(1) $v \in V(H) \Longrightarrow v \in U_m^1(V(H))$

(2) $v \in V(G) - V(H)$. Then $v \in (U_o^1(V(H)) \cap U_i^1(V(H)))$

$\Longrightarrow (OD(v) \cap V(H) \neq \phi) \wedge (ID(v) \cap V(H) \neq \phi)$

$\Longrightarrow$for all $MD(v)$, $MD(v) \cap V(H) \neq \phi$

$\Longrightarrow v \in U_m^1(V(H))$

Consequently, $U_m^1(V(H)) = U_o^1(V(H)) \cap U_i^1(V(H))$.

The proof of (c) and (d) is similar to proof (c) and (d) in Theorem (2.17) in [21].

Some properties of the first approximation operators $L_m^1(V(H))$ and $U_m^1(V(H))$ are imposed in the following properties.

Proposition 2.1.6. Let $G = (V(G), E(G))$ be a generalization approximation space and $H, K \subseteq G$.

(L_1) $L_m^1(V(H)) \subseteq V(H)$,

(L_2) $L_m^1(V(G)) = V(G)$,

(L_3) $L_m^1(\phi) = \phi$,

(L_4) If $V(H) \subseteq V(K)$, then $L_m^1(V(H)) \subseteq L_m^1(V(K))$,

(L_5) $L_m^1(V(H) \cap V(K)) \subseteq L_m^1(V(H)) \cap L_m^1(V(K))$,

(L_6) $L_m^1(V(H) \cup V(K)) \supseteq L_m^1(V(H)) \cup L_m^1(V(K))$,

(L_7) $L_m^1(V(H)) = V(G) - [U_m^1(V(G) - V(H))]$,

(U_1) $V(H) \subseteq U_m^1(V(H))$,

(U_2) $U_m^1(V(G)) = V(G)$,

(U_3) $U_m^1(\phi) = \phi$,

(U_4) If $V(H) \subseteq V(K)$, then $U_m^1(V(H)) \subseteq U_m^1(V(K))$,

(U_5) $U_m^1(V(H) \cap V(K)) \subseteq U_m^1(V(H)) \cap U_m^1(V(K))$,

(U_6) $U_m^1(V(H) \cup V(K)) \supseteq U_m^1(V(H)) \cup U_m^1(V(K))$,

(U_7) $U_m^1(V(H)) = V(G) - [L_m^1(V(G) - V(H))]$ and

(LU) $L_m^1(V(H)) \subseteq U_m^1(V(H))$.

Proof.

The proof (L_1), (L_2) and (L_3) by Definition(2.1.1).

(L_4) let $V(H) \subseteq V(K)$ and $v \in L_m^1(V(H))$, then $\exists MD(v)$ such that $MD(v) \subseteq V(H)$so$v \in L_m^1(V(H)) \subseteq V(H) \subseteq V(K)$. Thus we have $v \in V(K)$ and there exist $MD(v)$ such that $MD(v) \subseteq V(H) \subseteq V(K)$. Hence, $v \in L_m^1(V(H))$ and so $L_m^1(V(H)) \subseteq L_m^1(V(K))$.

(L_5) let $v \in U_m^1(V(H) \cap V(K))$, then $\exists MD(v)$ such that $MD(v) \subseteq (V(H)) \cap V(K))$ so $MD(v) \subseteq V(H)$ and $MD(v) \subseteq V(K)$. Thus we have $v \in U_m^1(V(H)) \wedge v \in U_m^1(V(K))$. Hence, $v \in U_m^1(V(H)) \cap U_m^1(V(K))$ and so $U_m^1(V(H) \cap V(K)) \subseteq U_m^1(V(H)) \cap U_m^1(V(K))$.

(L_6)let $V(H) \subseteq V(H) \cup V(K)$ or $V(K) \subseteq V(H) \cup V(K)$

then,$L_m^1(V(H)) \subseteq L_m^1(V(H) \cup V(K)) \vee L_m^1(V(K)) \subseteq L_m^1(V(H) \cup V(K))$.

Hence, $L_m^1(V(H) \cup V(K)) \supseteq L_m^1(V(H)) \cup L_m^1(V(H))$.

(L_7) let $v \in L_m^1(V(H)) \Leftrightarrow v \in V(H), \exists MD(v) \subseteq V(H)$

$\Leftrightarrow v \in V(G) - [V(G) - V(H)], \exists MD(v)$: $MD(v) \cap [V(G) - V(H)] = \phi$

$\Leftrightarrow v \notin L_m^1[V(G) - V(H)]$

$\Leftrightarrow v \in V(G) - [U_m^1(V(G) - V(H))]$

$\Leftrightarrow L_m^1(V(H)) = V(G) - [U_m^1(V(G) - V(H))]$.

The proof (U_1), (U_2) and (U_3) by Definition(2.1.1).

(U_4) let $V(H) \subseteq V(K)$ and $v \in U_m^1(V(H))$, we have:

(1) $v \in V(H) \Longrightarrow v \in V(H) \subseteq V(K) \Longrightarrow v \in V(K) \subseteq U_m^1(V(K)) \Longrightarrow v \in U_m^1(V(K))$.

(2) $v \in V(G) - V(H)$. Then $v \in U_m^1(V(H)) \Longrightarrow \forall MD(v)$: $MD(v) \cap V(H) \neq \phi$ and since $V(H) \subseteq V(K)$ thus we have $\forall$ $MD(v)$: $MD(v) \cap V(H) \neq \phi$ and hence we have

(1) $v \in V(K) - V(H) \Longrightarrow v \in V(K) \Longrightarrow v \in U_m^1(V(K))$.

(2) $v \in V(G) - V(K)$. So $\forall$ $MD(v)$, $MD(v) \cap V(K) \neq \phi \Longrightarrow v \in U_m^1(V(K))$. Hence, by (1) and (2) we have $U_m^1(V(H)) \subseteq U_m^1(V(K))$.

(U_5) let $V(H) \cap V(K) \subseteq V(H)$ and $V(H) \cap V(K) \subseteq V(K)$

then,$L_m^1(V(H) \cap V(K)) \subseteq L_m^1(V(H)) \wedge L_m^1(V(H) \cap V(K)) \subseteq L_m^1(V(K))$

Hence, $L_m^1(V(H) \cap V(K)) \subseteq L_m^1(V(H)) \cap L_m^1(V(H))$.

(U_6) let $v \notin U_m^1(V(H) \cup V(K))$, then $v \notin (V(H) \cup V(K))$ and $v \in V(G) - (V(H) \cup V(K))$, $\exists MD(v)$, $MD(v) \cap [V(H) \cup V(K)] = \phi$, so $v \in [V(G) - V(H)]$,$\exists MD(v)$, $(MD(v) \cap V(H)) \cup (MD(v) \cap V(K)) \neq \phi$. Thus, $v \in V(G) - V(H)$, $\exists MD(v)$, $MD(v) \cap V(H) = \phi \wedge v \in V(G) - V(H)$, $\exists MD(v)$, $MD(v) \cap V(K) = \phi \Longrightarrow v \notin U_m^1(V(H)) \wedge v \notin U_m^1(V(K)) \Longrightarrow v \notin (U_m^1(V(H)) \cup U_m^1(V(K)))$. Hence, we have $U_m^1(V(H) \cup V(K)) \supseteq U_m^1(V(H)) \cup U_m^1(V(K))$.

(U_7) By substituting $V(G) - V(H)$ for $V(H)$ in (L_7) we have $U_m^1(V(H)) = V(G) - [L_m^1(V(G) - V(H))]$.

(LU) Obviously, by (L_1) and (U_1) we get $L_m^1(V(H)) \subseteq U_m^1(V(H))$.

Remark 2.1.7. Let $G = (V(G), E(G))$ be a generalization approximation space and $H, K \subseteq G$. Then the following are not necessarily true.

(L_8) $L_m^1(V(H)) = L_m^1(L_m^1(V(H))$,

(L_9) $L_m^1(V(H)) = U_m^1(L_m^1(V(H))$,

(L_{10}) $V(H) \subseteq L_m^1(U_m^1(V(H))$,

(L_{11}) $L_m^1(V(H)) \subseteq L_m^1(L_m^1(V(H))$,

(L_{12}) $L_m^1(V(H) \cup V(K)) = L_m^1(V(H)) \cup L_m^1(V(K))$,

(U_8)$U_m^1(V(H)) = U_m^1(U_m^1(V(H))$,

(U_9) $U_m^1(V(H)) = L_m^1(U_m^1(V(H))$,

(U_{10}) $V(H) \supseteq U_m^1(L_m^1(V(H))$,

(U_{11}) $U_m^1(V(H)) \supseteq U_m^1(U_m^1(V(H))$ and

(U_{12}) $U_m^1(V(H) \cup V(K)) = U_m^1(V(H)) \cup U_m^1(V(K))$.

The following example is employed as a conter example to show this remark

Example 2.1.8. According to Example (2.1.4), we have

(L_8) if H = ($V(H)$, $E(H)$): $V(H)$ = {v_1, v_2, v_4}, $E(H)$ = {(v_1, v_2), (v_1, v_4), (v_2, v_2), (v_2, v_4)}, then $L^1_m(V(H))$ = {v_1, v_4}, $L^1_m(L^1_m(V(H))$ = {v_1}. Therefore, $L^1_m(V(H)) \neq L^1_m(L^1_m(V(H))$.

(L_9) if H = ($V(H)$, $E(H)$): $V(H)$ = {v_1, v_3}, $E(H) = \phi$, then $L^1_m(V(H))$ = {v_1, v_3}, $U^1_m(L^1_m(V(H))$ = {v_1, v_2, v_3, v_4}. Therefore, $L^1_m(V(H)) \neq U^1_m(L^1_m(V(H))$.

(L_{10}) if H = ($V(H)$, $E(H)$): $V(H)$ = {v_2, v_4, v_5}, $E(H)$ = {(v_2, v_2), (v_2, v_4), (v_4, v_5), (v_5, v_2), (v_5, v_5)}, then $L^1_m(U^1_m(V(H))$ = {v_5}. Therefore, $V(H) \not\subseteq L^1_m(U^1_m(V(H))$.

(L_{11}) if H = ($V(H)$, $E(H)$): $V(H)$ = {v_2, v_5}, $E(H)$ = {(v_2, v_2), (v_5, v_2), (v_5, v_5)}, then $L^1_m(V(H))$ = {v_5}, $L^1_m(L^1_m(V(H)) = \phi$. Therefore, $L^1_m(V(H)) \not\subseteq L^1_m(L^1_m(V(H))$.

(L_{12}) let$H$ = ($V(H)$, $E(H)$): $V(H)$ = {v_1, v_2, v_3, v_4}, $E(H)$ = {(v_1, v_2), (v_1 v_4), (v_2, v_2), (v_2, v_3), (v_2, v_4), (v_4, v_3)}and K = ($V(K)$, $E(K)$): $V(K)$ = {v_1, v_2, v_3, v_5}, $E(K)$ = {(v_1, v_2), (v_2, v_2), (v_2, v_3), (v_5, v_2), (v_5, v_5)}then $L^1_m(V(H))$ ={v_1, v_2, v_3, v_4}and $L^1_m(V(K))$ = {v_1, v_2, v_3, v_5} But, $H \cap K$ = ($V(H) \cap V(K)$,$E(H) \cap E(K)$): $V(H) \cap V(K)$ = {v_1, v_2, v_3}, $E(H) \cap E(K)$ = {(v_1, v_2), (v_2, v_2), (v_2, v_3)} such that $L^1_m(V(H) \cap V(K))$ = {v_1, v_3} and so $L^1_m(V(H) \cap V(K)) \neq L^1_m(V(H)) \cap L^1_m(V(H))$.

(U_8) if H = ($V(H)$, $E(H)$): $V(H)$ = {v_1, v_5}, $E(H) = \phi$, then $U^1_m(V(H))$ = {v_1, v_4, v_5}, $U^1_m(U^1_m(V(H)))$ = {v_1, v_2, v_4, v_5}. Therefore, $U^1_m(V(H)) \neq U^1_m(U^1_m(V(H)))$.

(U_9) if H = ($V(H)$, $E(H)$): $V(H)$ = {v_3, v_4}, $E(H)$ = {(v_4, v_3)}, then $U^1_m(V(H))$ = {v_3, v_4}, $L^1_m(U^1_m(V(H)))$ = {v_3}. Therefore, $U^1_m(V(H)) \neq L^1_m(U^1_m(V(H)))$.

(U_{10}) if H = ($V(H)$, $E(H)$): $V(H)$ = {v_1, v_2, v_5}, $E(H)$ = {(v_1, v_2), (v_2, v_2), (v_5, v_2), (v_5, v_5)}, then $L^1_m(V(H)$ = {v_1, v_2, v_5}, $U^1_m(L^1_m(V(H))$ = {v_1, v_2, v_4, v_5}. Therefore, $V(H) \not\supseteq U^1_m(L^1_m(V(H))$.

(U_{11}) if H = ($V(H)$, $E(H)$): $V(H)$ = {v_2, v_3}, $E(H)$ = {(v_2, v_2), (v_2, v_3)}, then $U^1_m(V(H))$ = {v_2, v_3, v_4}, $U^1_m(U^1_m(V(H)))$ = {v_2, v_3, v_4, v_5}. Therefore, $U^1_m(V(H)) \not\supseteq U^1_m(U^1_m(V(H)))$.

(U_{12}) let$H$ = ($V(H)$, $E(H)$): $V(H)$ = {v_4}, $E(H) = \phi$and K = ($V(K)$, $E(K)$): $V(K)$ = {v_5}, $E(K) = \phi$,then $U^1_m(V(H))$ = {v_4} and $U^1_m(V(K))$ = {v_5} But, $H \cup K$ = ($V(H) \cup V(K)$,$E(H) \cup E(K)$): $V(H) \cup V(K)$ = {v_4, v_5}, $E(H) \cup E(K)$ = {(v_4, v_5), (v_5, v_5)} such that $U^1_m(V(H) \cup V(K))$ = {v_2, v_4, v_5} and so $U^1_m(V(H) \cup V(K)) \neq U^1_m(V(H)) \cup U^1_m(V(K))$.

2.2. Second New Approximation Operators Using Mixed Degree Systems.

This section is devoted to propose a set-theoretic framework for granular computing using mixed degree systems. Considering the generalized approximation space G = ($V(G)$, $E(G)$), we introduce a new definition of the lower and upper approximation operators using mixed degree systems. The approximations are constructed using (out, in and mixed) degree systems. A comparison between these three approaches is given.

Definition 2.2.1. Let G = ($V(G)$, $E(G)$) be a generalization approximation space and $H \subseteq G$. Then

(a) the second lower and upper approximations of H using out degree systems are denoted by $L^2_O(V(H))$ and $U^2_O(V(H))$ and defined by

$$L^2_O(V(H)) = \{v \in V(G) \ ; OD(v) \subseteq V(H)\},$$
$$U^2_O(V(H)) = \{v \in V(G) \ ; OD(v) \cap V(H) \neq \phi\},$$

(b) the second lower and upper approximations of H using in degree systems are denoted by $L^2_i(V(H))$ and $U^2_i(V(H))$ and defined by

$$L^2_i(V(H)) = \{v \in V(G) \ ; ID(v) \subseteq V(H)\},$$
$$U^2_i(V(H)) = \{v \in V(G) \ ; ID(v) \cap V(H) \neq \phi\},$$

(c) the second lower and upper approximations of H using mixed degree systems are denoted by $L^2_m(V(H))$ and $U^2_m(V(H))$ and defined by

$$L^2_m(V(H)) = \{v \in V(G) \ ; \text{for some } MD(v) \subseteq V(H)\},$$
$$U^2_m(V(H)) = \{v \in V(G) \ ; \text{for all } MD(v) \cap V(H) \neq \phi\}.$$

Definition 2.2.2. Let G = ($V(G)$, $E(G)$) be a generalization approximation space and $H \subseteq G$. Then

(a) the second boundary, positive and negative regions of H using out degree systems are denoted by $Bd_o^2(V(H))$, $POS_o^2(V(H))$ and $NEG_o^2(V(H))$ and defined by

$$Bd_o^2(V(H)) = U_o^2(V(H)) - L_o^2(V(H)),$$
$$POS_o^2(V(H)) = L_o^2(V(H)),$$
$$NEG_o^2(V(H)) = V(G) - U_o^2(V(H)),$$

(b) the second boundary, positive and negative regions of H using in degree systems are denoted by $Bd_i^2(V(H))$, $POS_i^2((V(H))$ and $NEG_i^2(V(H))$ and defined by

$$Bd_i^2(V(H)) = U_i^2(V(H)) - L_i^2(V(H)),$$
$$POS_i^2((V(H)) = L_i^2(V(H)),$$
$$NEG_i^2(V(H)) = V(G) - U_i^2(V(H)),$$

(c) the second boundary, positive and negative regions of H using mixed degree systems are denoted by $Bd_m^2(V(H))$, $POS_m^2(V(H))$ and $NEG_m^2(V(H))$ and defined by

$$Bd_m^2(V(H)) = U_m^2(V(H)) - L_m^2(V(H)),$$
$$POS_m^2(V(H)) = L_m^2(V(H)),$$
$$NEG_m^2(V(H)) = V(G) - U_m^2(V(H)).$$

Definition 2.2.3. Let G = ($V(G)$, $E(G)$) be a generalization approximation space. The second accuracy of the approximations of a subgraph $H \subseteq G$ using (out, in and mixed) degree systems are denoted by ($\eta_o^2(V(H))$, $\eta_i^2(V(H))$ and $\eta_m^2(V(H))$) and defined by

$$\eta_o^2(V(H)) = 1 - \frac{|Bd_o^2(V(H))|}{|V(G)|},$$
$$\eta_i^2(V(H)) = 1 - \frac{|Bd_i^2(V(H))|}{|V(G)|},$$
$$\eta_m^2(V(H)) = 1 - \frac{|Bd_m^2(V(H))|}{|V(G)|}.$$

It is obvious that $0 \le \eta_o^2(V(H)) \le 1$, $0 \le \eta_i^2(V(H)) \le 1$ and $0 \le \eta_m^2(V(H)) \le 1$. Moreover, if $\eta_o^2(V(H)) = 1$ or $\eta_i^2(V(H)) = 1$ or $\eta_m^2(V(H)) = 1$, then H is called H-definable (H-exact) graph. Otherwise, it is called H-rough.

Example 2.2.4. Accordingly, to Example (2.1.4) we have the following table

Table 2.2.1: $\eta_o^2(V(H))$, $\eta_i^2(V(H))$ and $\eta_m^2(V(H))$ for all $H \subseteq G$.

$V(H)$	$\eta_o^2(V(H))$	$\eta_i^2(V(H))$	$\eta_m^2(V(H))$
$\{v_1\}$	1	3/5	1
$\{v_2\}$	2/5	2/5	4/5
$\{v_3\}$	3/5	1	1
$\{v_4\}$	3/5	3/5	1
$\{v_5\}$	3/5	3/5	4/5
$\{v_1,v_2\}$	2/5	3/5	4/5
$\{v_1,v_3\}$	3/5	3/5	3/5
$\{v_1,v_4\}$	3/5	1/5	4/5
$\{v_1,v_5\}$	3/5	2/5	3/5
$\{v_2,v_3\}$	1/5	2/5	3/5
$\{v_2,v_4\}$	3/5	2/5	3/5
$\{v_2,v_5\}$	2/5	1/5	3/5
$\{v_3, v_4\}$	2/5	3/5	1

$\{v_3, v_5\}$	3/5	3/5	3/5
$\{v_4, v_5\}$	1/5	3/5	4/5
$\{v_1, v_2, v_3\}$	1/5	3/5	4/5
$\{v_1, v_2, v_4\}$	3/5	3/5	3/5
$\{v_1, v_2, v_5\}$	2/5	3/5	1
$\{v_1, v_3, v_4\}$	2/5	1/5	3/5
$\{v_1, v_3, v_5\}$	3/5	2/5	3/5
$\{v_1, v_4, v_5\}$	1/5	2/5	3/5
$\{v_2, v_3, v_4\}$	3/5	2/5	3/5
$\{v_2, v_3, v_5\}$	3/5	1/5	4/5
$\{v_2, v_4, v_5\}$	3/5	3/5	3/5
$\{v_3, v_4, v_5\}$	2/5	3/5	4/5
$\{v_1, v_2, v_3, v_4\}$	3/5	3/5	4/5
$\{v_1, v_2, v_3, v_5\}$	3/5	3/5	1
$\{v_1, v_2, v_4, v_5\}$	3/5	1	1
$\{v_1, v_3, v_4, v_5\}$	2/5	2/5	4/5
$\{v_2, v_3, v_4, v_5\}$	1	3/5	1
$V(G)$	1	1	1
ϕ	1	1	1

Theorem 2.2.5. Let $G = (V(G), E(G))$ be a generalization approximation space and $H \subseteq G$. Then

(a) $L_m^2(V(H)) = L_o^2(V(H)) \cup L_i^2(V(H))$,

(b) $U_m^2(V(H)) = U_o^2(V(H)) \cap U_i^2(V(H))$,

(c) $Bd_m^2(V(H)) = Bd_o^2(V(H)) \cap Bd_i^2(V(H))$ and

(d) $\eta_m^2(V(H)) \geq \max\{\eta_o^2(V(H)), \eta_i^2(V(H))\}$.

Proof.

(a) Let $v \in (L_o^2(V(H)) \cup L_i^2(V(H)))$

$\Leftrightarrow v \in L_o^2(V(H)) \vee v \in L_i^2(V(H))$

$\Leftrightarrow OD(v) \subseteq V(H) \vee ID(v) \subseteq V(H)$

$\Leftrightarrow \exists MD(v)$ such that $MD(v) \subseteq V(H)$

$\Leftrightarrow v \in L_m^2(V(H))$

So, $L_m^2(V(H)) = L_o^2(V(H)) \cup L_i^2(V(H))$.

(b) Let $v \in (U_o^2(V(H)) \cap U_i^2(V(H)))$

$\Leftrightarrow v \in (U_o^2(V(H)) \wedge U_i^2(V(H)))$

$\Leftrightarrow (OD(v) \cap V(H) \neq \phi) \wedge (ID(v) \cap V(H) \neq \phi)$

$\Leftrightarrow$ for each $MD(v)$, $MD(v) \cap V(H) \neq \phi$

$\Leftrightarrow v \in U_m^2(V(H))$

So, $U_m^2(V(H)) = U_o^2(V(H)) \cap U_i^2(V(H))$.

(c) and (d) is similar to the proof (c) and (d) in Theorem (2.17) in [21].

In the following proposition, we investigate the relation between the approximation operatorsL_m^1, U_m^1 which introduced in the previous section and the approximation operators L_m^2,U_m^2.

Proposition 2.2.6. Let $G = (V(G), E(G))$ be a generalization approximation space and $H \subseteq G$. Then

(a) $L_m^1(V(H)) \subseteq L_m^2(V(H))$,

(b) U_m^2*(V(H))*⊆U_m^1*(V(H)),*
(c) Bd_m^2*(V(H))*⊆Bd_m^1*(V(H))* and
(d) η_m^1*(V(H))* ≤η_m^2*(V(H)).*

Proof.

(a) Let *v*∈L_m^1*(V(H))*, then by Definition (2.1.1), we have *v*∈*V(H)* and *MD(v)* ; *MD(v)*⊆*V(H)*, so by Definition (2.2.1), we get *v*∈L_m^2*(V(H))*. Hence, L_m^1*(V(H))*⊆ L_m^2*(V(H)).*

(b) Suppose that *v*∉U_m^1*(V(H))*, then by Definition (2.1.1), we have *v*∉*V(H)* and *v*∈*V(G)*–*V(H)* such that ∃*MD(v)* ; *MD(v)*⋂*V(H)* = ϕ. Thus, by Definition (2.2.1), *v*∉U_m^2*(V(H))*. Therefore, U_m^2*(V(H))*⊆U_m^1*(V(H)).*

(c) By using (a) and (b), we have B_m^2*(V(H))*⊆B_m^1*(V(H)).*
Since B_m^2*(V(H))* = U_m^2*(V(H))*–L_m^2*(V(H))*
⊆U_m^1*(V(H))*–L_m^1*(V(H)).*

(d) By using (c), we have Bd_m^2*(V(H))*⊆Bd_m^1*(V(H))*
$\Rightarrow |Bd_m^2(V(H))| \leq |Bd_m^1(V(H))|$
$\Rightarrow \frac{|Bd_m^2(V(H))|}{|V(G)|} \leq \frac{|Bd_m^1(V(H))|}{|V(G)|}$
$\Rightarrow 1 - \frac{|Bd_m^2(V(H))|}{|V(G)|} \geq 1 - \frac{|Bd_m^1(V(H))|}{|V(G)|}$
$\Rightarrow \eta_m^2(V(H)) \geq \eta_m^1(V(H)).$

Remark 2.2.7. Let *G* = (*V(G), E(G)*) be a generalization approximation space and *H*⊆*G*, then the following are not necessarily true.

(a) L_m^1*(V(H))*=L_m^2*(V(H)),*
(b) U_m^2*(V(H))*=U_m^1*(V(H)),*
(c) Bd_m^2*(V(H))*= Bd_m^1*(V(H))* and
(d) η_m^1*(V(H))* =η_m^2*(V(H)).*
(e)

The following example illustrates this remark

Example 2.2.8. According to Examples (2.1.4) and (2.2.4), if *H* = (*V(H), E(H)*): *V(H)* = {v_4, v_5}, *E(H)* = {(v_4, v_5), (v_5, v_5)}

(a) L_m^1*(V(H))* = {v_5}, L_m^2*(V(H))* = {v_1, v_3, v_5}. Hence, L_m^1*(V(H))*≠L_m^2*(V(H)),*
(b) U_m^1*(V(H))* = {v_2, v_4, v_5}, U_m^2*(V(H))* = {v_2, v_5}. Hence, U_m^1*(V(H))*≠U_m^2*(V(H)),*
(c) Bd_m^1*(V(H))* = {v_2, v_5}, Bd_m^2*(V(H))* = {v_2}. Hence, Bd_m^1*(V(H))*≠Bd_m^2*(V(H)),*
(d) η_m^1*(V(H))* = *4/5*, η_m^2*(V(H))* = *3/5*. Hence, η_m^1*(V(H))*≠η_m^2*(V(H)).*

Consider the generalized approximation space *G* = (*V(G), E(G)*), Proposition (2.2.6) proves that the approximations of graphs using the operators L_m^1 and U_m^1 are accurate then the approximations obtained by using the operators L_m^2 and U_m^2 since η_m^1*(V(H))* ≤ η_m^2*(V(H))*. For this reason, we study in detail the properties of L_m^2 and U_m^2 in the next section.

2.3 Properties ofthe Second New Approximation Operators $L_m^2$and U_m^2.

The core concepts of classical rough set theory are lower and upper approximation operators based on equivalence relation. This section studies in detail the properties of the approximation operators $L_m^2$and U_m^2. In this setting, some of common properties of classical lower and upper approximation operators are no longer satisfied. So, we investigate conditions for a relation under which these properties hold for the approximation operators $L_m^2$and U_m^2.

Some properties of the approximation operators L_m^2 and U_m^2 are given in the following proposition.

Proposition 2.3.1. Let $G = (V(G), E(G))$ be a generalization approximation space and $H, K \subseteq G$. Then

(L_2) $L_m^2(V(G)) = V(G)$,

(L_4) If $V(H) \subseteq V(K)$, then $L_m^2(V(H)) \subseteq L_m^2(V(K))$,

(L_5) $L_m^2(V(H) \cap V(K)) \subseteq L_m^2(V(H)) \cap L_m^2(V(K))$,

(L_6) $L_m^2(V(H) \cup V(K)) \supseteq L_m^2(V(H)) \cup L_m^2(V(K))$,

(L_7) $L_m^2(V(H)) = V(G) - [U_m^2(V(G) - V(H))]$,

(U_3) $U_m^2(\phi) = \phi$,

(U_4) If $V(H) \subseteq V(K)$, then $U_m^2(V(H)) \subseteq U_m^2(V(K))$,

(U_5) $U_m^2(V(H) \cap V(K)) \subseteq U_m^2(V(H)) \cap U_m^2(V(K))$,

(U_6) $U_m^2(V(H) \cup V(K)) \supseteq U_m^2(V(H)) \cup U_m^2(V(K))$ and

(U_7) $U_m^2(V(H)) = V(G) - [L_m^2(V(G) - V(H))]$.

Proof.

The proof (L_2) by Definition(2.2.1).

(L_4) let $V(H) \subseteq V(K)$ and $v \in L_m^2(V(H))$, then $\exists MD(v)$ such that $MD(v) \subseteq V(H)$ so $v \in L_m^2(V(H)) \subseteq V(H) \subseteq V(K)$. Thus we have $v \in V(K)$ and there exist $MD(v)$ such that $MD(v) \subseteq V(H) \subseteq V(K)$. Hence, $v \in L_m^2(V(H))$ and so $L_m^2(V(H)) \subseteq L_m^2(V(K))$.

(L_5) let $V(H) \cap V(K) \subseteq V(H)$ and $V(H) \cap V(K) \subseteq V(K)$

then, $L_m^2(V(H) \cap V(K)) \subseteq L_m^2(V(H)) \wedge L_m^2(V(H) \cap V(K)) \subseteq L_m^2(V(K))$

Hence, $L_m^2(V(H) \cap V(K)) \subseteq L_m^2(V(H)) \cap L_m^2(V(H))$.

(L_6) let $V(H) \subseteq V(H) \cup V(K)$ or $V(K) \subseteq V(H) \cup V(K)$

then, $L_m^2(V(H)) \subseteq L_m^2(V(H) \cup V(K)) \vee L_m^2(V(K)) \subseteq L_m^2(V(H) \cup V(K))$.

Hence, $L_m^2(V(H) \cup V(K)) \supseteq L_m^2(V(H)) \cup L_m^2(V(H))$.

(L_7) let $v \in L_m^2(V(H)) \Leftrightarrow v \in V(H), \exists MD(v) \subseteq V(H)$

$\Leftrightarrow v \in V(G) - [V(G) - V(H)], \exists MD(v): MD(v) \cap [V(G) - V(H)] = \phi$

$\Leftrightarrow v \notin L_m^2[V(G) - V(H)]$

$\Leftrightarrow v \in V(G) - [U_m^2(V(G) - V(H))]$

$\Leftrightarrow L_m^2(V(H)) = V(G) - [U_m^2(V(G) - V(H))]$.

The proof (U_3) by Definition(2.2.1).

(U_4) let $V(H) \subseteq V(K)$ and $v \in U_m^2(V(H))$, we have:

(1) $v \in V(H) \Longrightarrow v \in V(H) \subseteq V(K) \Longrightarrow v \in V(K) \subseteq U_m^2(V(K)) \Longrightarrow v \in U_m^2(V(K))$.

(2) $v \in V(G) - V(H)$. Then $v \in U_m^1(V(H)) \Longrightarrow \forall\ MD(v)$: $MD(v) \cap V(H) \neq \phi$ and since $V(H) \subseteq V(K)$ thus we have $\forall\ MD(v)$: $MD(v) \cap V(H) \neq \phi$ and hence we have

(1) $v \in V(K) - V(H) \Longrightarrow v \in V(K) \Longrightarrow v \in U_m^1(V(K))$.

(2) $v \in V(G) - V(K)$. So $\forall\ MD(v)$, $MD(v) \cap V(K) \neq \phi \Longrightarrow v \in U_m^1(V(K))$. Hence, by (1) and (2) we have $U_m^1(V(H)) \subseteq U_m^1(V(K))$.

(U_5) let $V(H) \cap V(K) \subseteq V(H)$ and $V(H) \cap V(K) \subseteq V(K)$

then, $L_m^2(V(H) \cap V(K)) \subseteq L_m^2(V(H)) \wedge L_m^2(V(H) \cap V(K)) \subseteq L_m^2(V(K))$

Hence, $L_m^2(V(H) \cap V(K)) \subseteq L_m^2(V(H)) \cap L_m^2(V(H))$.

(U_6) let $V(H) \subseteq (V(H) \cup V(K))$ or $V(K) \subseteq (V(H) \subseteq V(K))$

then, $U_m^2(V(H)) \subseteq U_m^2(V(H) \cup V(K)) \vee U_m^2(V(K)) \subseteq U_m^2(V(H) \cup V(K))$

Hence, $U_m^2(V(H) \cup V(K)) \supseteq U_m^2(V(H)) \cup L_m^2(V(H))$.

(U_7) By substituting $V(G) - V(H)$ for $V(H)$ in (L_7) we have $U_m^2(V(H)) = V(G) - [L_m^2(V(G) - V(H))]$.

(LU) Obviously, by (L_1) and (U_1) we get $L_m^2(V(H)) \subseteq U_m^2(V(H))$.

Remark 2.3.2. Let G = ($V(G)$, $E(G)$) be a generalization approximation space and $H, K\subseteq G$. Then the following are not necessarily true.

(L_1) $L_m^2(V(H)) \subseteq V(H)$,
(L_3) $L_m^2(\phi) = \phi$,
(L_8) $L_m^2(V(H)) = L_m^2(L_m^2(V(H))$,
(L_9) $L_m^2(V(H)) = U_m^2(L_m^2(V(H))$,
(L_{10}) $V(H) \subseteq L_m^2(U_m^2(V(H))$,
(L_{11}) $L_m^2(V(H)) \subseteq L_m^2(L_m^2(V(H))$,
(L_{12}) $L_m^2(V(H) \cup V(K)) = L_m^2(V(H)) \cup L_m^2(V(K))$,
(U_1) $V(H)\subseteq U_m^2(V(H))$,
(U_2) $U_m^2(V(G)) = V(G)$,
(U_8)$U_m^2(V(H)) = U_m^2(U_m^2(V(H))$,
(U_9) $U_m^2(V(H)) = L_m^2(U_m^2(V(H))$,
(U_{10}) $V(H) \supseteq U_m^2(L_m^2(V(H))$,
(U_{11})$U_m^2(V(H)) \supseteq U_m^2(U_m^2(V(H))$,
(U_{12}) $U_m^2(V(H) \cup V(K)) = U_m^2(V(H)) \cup U_m^2(V(K))$ and
(LU) $L_m^2(V(H)) \subseteq U_m^2(V(H))$.

The following example illustrates this remark

Example 2.3.3. According to Example (2.2.4).

(L_1) if H = ($V(H)$, $E(H)$): $V(H) = \{v_2\}$, $E(H) = \{(v_2, v_2)\}$, then $L_m^2(V(H)) = \{v_1, v_3\}$. Therefore, $L_m^2(V(H)) \not\subseteq V(H)$.

(L_3)if H = ($V(H)$, $E(H)$): $V(H) = \phi$, $E(H) = \phi$, then $L_m^2(V(H)) = \{v_1, v_3\}$. Therefore, $L_m^2(\phi) \neq \phi$.

(L_8) if H = ($V(H)$, $E(H)$): $V(H) = \{v_1, v_2\}$, $E(H) = \{(v_1, v_2), (v_2, v_2)\}$, then $L_m^2(V(H)) = \{v_1, v_3, v_4\}$, $L_m^2(L_m^2(V(H)) = \{v_1, v_3\}$. Therefore, $L_m^2(V(H)) \neq L_m^2(L_m^2(V(H))$.

(L_9) if H = ($V(H)$, $E(H)$): $V(H) = \{v_1, v_3, v_5\}$, $E(H) = \{(v_5, v_5)\}$, then $L_m^2(V(H)) = \{v_1, v_3, v_4\}$, $U_m^2(L_m^2(V(H)) = \{v_2, v_4\}$. Therefore, $L_m^2(V(H)) \neq U_m^2(L_m^2(V(H))$.

(L_{10}) if H = ($V(H)$, $E(H)$): $V(H) = \{v_1, v_4\}$, $E(H) = \{(v_1, v_4)\}$, then $U_m^2(V(H) = \{v_2\}$, $L_m^2(U_m^2(V(H)) = \{v_1, v_3\}$. Therefore, $V(H) \not\subseteq L_m^2(U_m^2(V(H))$.

(L_{11}) if H = ($V(H)$, $E(H)$): $V(H) = \{v_2, v_5\}$, $E(H) = \{(v_2, v_2), (v_5, v_2), (v_5, v_5)\}$, then $L_m^2(V(H)) = \{v_1, v_3, v_5\}$, $L_m^2(L_m^2(V(H)) = \{v_1, v_3, v_4\}$. Therefore, $L_m^2(V(H)) \not\subseteq L_m^2(L_m^2(V(H))$.

(L_{12}) let$H$ = ($V(H)$, $E(H)$): $V(H) = \{v_3, v_4, v_5\}$, $E(H) = \{(v_4, v_3), (v_4, v_5), (v_5, v_5)\}$and K = ($V(K)$, $E(K)$): $V(K) = \{v_1, v_2, v_3, v_4\}$, $E(K) = \{(v_1, v_2), (v_1, v_4), (v_2, v_2), (v_2, v_3), (v_2, v_4), (v_4, v_3)\}$then $L_m^2(V(H)) = \{v_1, v_3, v_4, v_5\}$and $L_m^2(V(K)) = \{v_1, v_2, v_3, v_4\}$ But, $H\cap K$ = ($V(H)\cap V(K)$,$E(H)\cap E(K)$): $V(H)\cap V(K) = \{v_3, v_4\}$, $E(H)\cap E(K) = \{(v_4, v_3)\}$ such that $L_m^2(V(H) \cap V(K)) = \{v_1, v_3\}$ and so $L_m^2(V(H) \cap V(K)) \neq L_m^2(V(H)) \cap L_m^2(V(H))$.

(U_1)if H = ($V(H)$, $E(H)$): $V(H) = \{v_1, v_3\}$, $E(H) = \phi$, then $U_m^2(V(H)) = \{v_2, v_4\}$. Therefore, $V(H)\not\subseteq U_m^2(V(H))$.

(U_2)if H = ($V(H)$, $E(H)$): $V(H) = V(G)$, $E(H) = \{(v_1, v_2), (v_1, v_4), (v_2, v_2), (v_2, v_3), (v_2, v_4), (v_4, v_3), (v_4, v_5), (v_5, v_2), (v_5, v_5)\}$, then $U_m^2(V(H)) = \{v_2, v_4, v_5\}$. Therefore, $U_m^2(V(G))\neq V(G)$.

(U_8) if H = ($V(H)$, $E(H)$): $V(H) = \{v_1, v_3\}$, $E(H) = \phi$, then $U_m^2(V(H)) = \{v_2, v_4\}$, $U_m^2(U_m^2(V(H))) = \{v_2, v_5\}$. Therefore, $U_m^2(V(H))\neq U_m^2(U_m^2(V(H)))$.

(U_9) if H = ($V(H)$, $E(H)$): $V(H) = \{v_2, v_3\}$, $E(H) = \{(v_2, v_2), (v_2, v_3)\}$, then $U_m^2(V(H)) = \{v_2, v_4\}$, $L_m^2(U_m^2(V(H))) = \{v_1, v_3\}$. Therefore, $U_m^2(V(H)) \neq L_m^2(U_m^2(V(H)))$.

(U_{10}) if H = ($V(H)$, $E(H)$): $V(H) = \{v_1, v_2, v_3\}$, $E(H) = \{(v_1, v_2), (v_2, v_2), (v_2, v_3)\}$, then $L_m^2(V(H) = \{v_1, v_3, v_4\}$, $U_m^2(L_m^2(V(H)) = \{v_2, v_4\}$. Therefore, $V(H) \not\supseteq U_m^2(L_m^2(V(H))$.

(U_{11}) if $H = (V(H), E(H))$: $V(H) = \{v_3, v_4, v_5\}$, $E(H) = \{(v_4, v_3), (v_4, v_5), (v_5, v_5)\}$, then $U^2_m(V(H)) = \{v_2, v_5\}$, $U^2_m(U^2_m(V(H))) = \{v_2, v_4, v_5\}$. Therefore, $U^2_m(V(H)) \not\supseteq U^2_m(U^2_m(V(H)))$.

(U_{12}) let $H = (V(H), E(H))$: $V(H) = \{v_4, v_5\}$, $E(H) = \{(v_4, v_5), (v_5, v_5)\}$ and $K = (V(K), E(K))$: $V(K) = \{v_1, v_2, v_4\}$, $E(K) = \{(v_1, v_2), (v_1, v_4), (v_2, v_2), (v_2, v_4)\}$, then $U^2_m(V(H)) = \{v_2, v_5\}$ and $U^2_m(V(K)) = \{v_2, v_5\}$ But, $H \cup K = (V(H)\cup V(K), E(H)\cup E(K))$: $V(H)\cup V(K) = \{v_1, v_2, v_4, v_5\}$, $E(H)\cup E(K) = \{(v_1, v_2), (v_1, v_4), (v_2, v_2), (v_2, v_4), (v_4, v_5), (v_5, v_2), (v_5, v_5)\}$ such that $U^2_m(V(H) \cup V(K)) = \{v_2, v_4, v_5\}$ and so $U^2_m(V(H) \cup V(K)) \neq U^2_m(V(H)) \cup U^2_m(V(K))$.

(LU) if $H = (V(H), E(H))$: $V(H) = \{v_3\}$, $E(H) = \phi$, then $L^2_m(V(H)) = \{v_1, v_3\}$, $U^2_m(V(H)) = \phi$. Therefore, $L^2_m(V(H)) \neq U^2_m(V(H))$.

Proposition 2.3.4. Let $G = (V(G), E(G))$ be a non-empty finite serial graph. If $V(G) = \bigcup_{v \in V(G)} OD(v)$ then the following hold:

(L_3) $L^2_m(\phi) = \phi$,

(U_2) $U^2_m(V(G)) = V(G)$.

Proof.

(L_3) Since G is a serial graph and $V(G) = \bigcup_{v \in V(G)} OD(v)$ then $MD(v) \neq \phi$ for all $v \in v(G)$ and hence $L^2_m(\phi) = \phi$.

(U_2) Since $L^2_m(\phi) = \phi \Longrightarrow [L^2_m(\phi)]' = \phi'$

$\Longrightarrow V(G) - L^2_m(\phi) = V(G) - \phi$

$\Longrightarrow V(G) - L^2_m(V(G) - V(G)) = V(G)$. But $U^2_m(V(H)) = V(G) - L^2_m(V(G) - V(H))$ for all $H \subseteq G$. So $U^2_m(V(G)) = V(G) - L^2_m(V(G) - V(G))$. Accordingly, $U^2_m(V(G)) = V(G)$.

Remark 2.3.5. Let $G = (V(G), E(G))$ be a non-empty finite serial graph. If $V(G) = \bigcup_{v \in V(G)} OD(v)$, then the properties ($L_1$), ($L_8$), ($L_9$), ($L_{10}$), ($L_{11}$), ($L_{12}$), ($U_1$), ($U_8$), ($U_9$), ($U_{10}$), ($U_{11}$), ($U_{12}$) and ($LU$) are not true in general for every $H, K \subseteq G$.

The next example illustrates this remark.

Example 2.3.6. Let $G = (V(G), E(G))$: $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{(v_1, v_1), (v_2, v_3), (v_2, v_4), (v_3, v_4), (v_3, v_5), (v_4, v_1), (v_4, v_4), (v_5, v_2), (v_5, v_5)\}$.

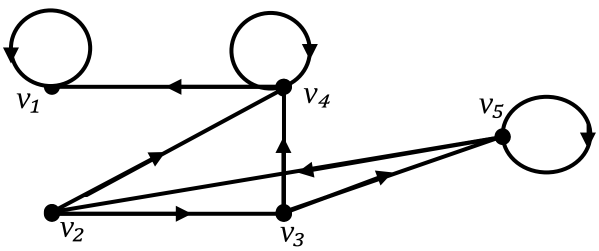

Figure 2.3.1 : Graph G given in Example 2.3.6.

We get

$OD(v_1) = \{v_1\}$, $OD(v_2) = \{v_3, v_4\}$, $OD(v_3) = \{v_4, v_5\}$, $OD(v_4) = \{v_1, v_4\}$, $OD(v_5) = \{v_2, v_5\}$

Also we have

$ID(v_1) = \{v_1, v_4\}$, $ID(v_2) = \{v_5\}$, $ID(v_3) = \{v_2\}$, $ID(v_4) = \{v_2, v_3, v_4\}$, $ID(v_5) = \{v_3, v_5\}$

Then we obtain

$MDS(v_1) = \{\{v_1\},\{v_1, v_4\}\}$, $MDS(v_2) = \{\{v_3, v_4\}, \{v_5\}\}$, $MDS(v_3) = \{\{v_4, v_5\}, \{v_2\}\}$, $MDS(v_4) = \{\{v_1, v_4\}, \{v_2, v_3, v_4\}\}$, $MDS(v_5) = \{\{v_2, v_5\}, \{v_3, v_5\}\}$.

Therefore, we have

(L_1) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_2, v_5\}$, $E(H) = \{(v_1, v_1), (v_5, v_2), (v_5, v_5)\}$, then $L_m^2(V(H)) = \{v_1, v_2, v_3, v_5\}$. Therefore, $L_m^2(V(H)) \nsubseteq V(H)$.

(L_8) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_2, v_3\}$, $E(H) = \{(v_1, v_1), (v_2, v_3)\}$, then $L_m^2(V(H)) = \{v_1, v_3\}$, $L_m^2(L_m^2(V(H)) = \{v_1\}$. Therefore, $L_m^2(V(H)) \neq L_m^2(L_m^2(V(H))$.

(L_9) if $H = (V(H), E(H))$: $V(H) = \{v_4, v_5\}$, $E(H) = \{(v_4, v_4), (v_5, v_5)\}$, then $L_m^2(V(H)) = \{v_2, v_3\}$, $U_m^2(L_m^2(V(H)) = \{v_5\}$. Therefore, $L_m^2(V(H)) \neq U_m^2(L_m^2(V(H))$.

(L_{10}) if $H = (V(H), E(H))$: $V(H) = \{v_2, v_4\}$, $E(H) = \{(v_2, v_4), (v_4, v_4)\}$, then $U_m^2(V(H) = \{v_3, v_4\}$, $L_m^2(U_m^2(V(H)) = \{v_2\}$. Therefore, $V(H) \nsubseteq L_m^2(U_m^2(V(H))$.

(L_{11}) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_3, v_4\}$, $E(H) = \{(v_1, v_1), (v_3, v_4), (v_4, v_1), (v_4, v_4)\}$, then $L_m^2(V(H)) = \{v_1, v_2, v_4\}$, $L_m^2(L_m^2(V(H)) = \{v_1, v_3\}$. Therefore, $L_m^2(V(H)) \nsubseteq L_m^2(L_m^2(V(H))$.

(L_{12}) let $H = (V(H), E(H))$: $V(H) = \{v_2, v_3, v_4\}$, $E(H) = \{(v_2, v_3), (v_2, v_4), (v_3, v_4), (v_4, v_4)\}$ and $K = (V(K), E(K))$: $V(K) = \{v_2, v_3, v_5\}$, $E(K) = \{(v_2, v_3), (v_3, v_5), (v_5, v_2), (v_5, v_5)\}$ then $L_m^2(V(H)) = \{v_2, v_3, v_4\}$ and $L_m^2(V(K)) = \{v_2, v_3, v_5\}$ But, $H \cap K = (V(H) \cap V(K), E(H) \cap E(K))$: $V(H) \cap V(K) = \{v_2, v_3\}$, $E(H) \cap E(K) = \{(v_2, v_3)\}$ such that $L_m^2(V(H) \cap V(K)) = \{v_3\}$ and so $L_m^2(V(H) \cap V(K)) \neq L_m^2(V(H)) \cap L_m^2(V(H))$.

(U_1) if $H = (V(H), E(H))$: $V(H) = \{v_3, v_4\}$, $E(H) = \{(v_3, v_4), (v_4, v_4)\}$, then $L_m^2(V(H)) = \{v_4\}$. Therefore, $V(H) \nsubseteq U_m^2(V(H))$.

(U_8) if $H = (V(H), E(H))$: $V(H) = \{v_4, v_5\}$, $E(H) = \{(v_4, v_4), (v_5, v_5)\}$, then $U_m^2(V(H)) = \{v_2\}$, $U_m^2(U_m^2(V(H))) = \phi$. Therefore, $U_m^2(V(H)) \neq U_m^2(U_m^2(V(H)))$.

(U_9) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_2, v_3\}$, $E(H) = \{(v_1, v_1), (v_2, v_3)\}$, then $U_m^2(V(H)) = \{v_1, v_4, v_5\}$, $L_m^2(U_m^2(V(H))) = \{v_1, v_2, v_3, v_4\}$. Therefore, $U_m^2(V(H)) \neq L_m^2(U_m^2(V(H)))$.

(U_{10}) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_5\}$, $E(H) = \{(v_1, v_1), (v_5, v_5)\}$, then $L_m^2(V(H) = \{v_1, v_2\}$, $U_m^2(L_m^2(V(H)) = \{v_1, v_4\}$. Therefore, $V(H) \nsupseteq U_m^2(L_m^2(V(H))$.

(U_{11}) if $H = (V(H), E(H))$: $V(H) = \{v_2, v_3, v_4\}$, $E(H) = \{(v_2, v_3), (v_2, v_4), (v_3, v_4), (v_4, v_4)\}$, then $U_m^2(V(H)) = \{v_3, v_4, v_5\}$, $U_m^2(U_m^2(V(H))) = \{v_2, v_4, v_5\}$. Therefore, $U_m^2(V(H)) \nsupseteq U_m^2(U_m^2(V(H)))$.

(U_{12}) let $H = (V(H), E(H))$: $V(H) = \{v_2, v_5\}$, $E(H) = \{(v_5, v_2), (v_5, v_5)\}$ and $K = (V(K), E(K))$: $V(K) = \{v_3, v_4\}$, $E(K) = \{(v_3, v_4), (v_2, v_4)\}$, then $U_m^2(V(H)) = \{v_3, v_5\}$ and $U_m^2(V(K)) = \{v_4\}$ But, $H \cup K = (V(H) \cup V(K), E(H) \cup E(K))$: $V(H) \cup V(K) = \{v_2, v_3, v_4, v_5\}$, $E(H) \cup E(K) = \{(v_2, v_3), (v_2, v_4), (v_3, v_4), (v_4, v_4), (v_5, v_2), (v_5, v_5)\}$ such that $U_m^2(V(H) \cup V(K)) = \{v_2, v_3, v_4, v_5\}$ and so $U_m^2(V(H) \cup V(K)) \neq U_m^2(V(H)) \cup U_m^2(V(K))$.

(LU) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_3, v_4\}$, $E(H) = \{(v_1, v_1), (v_3, v_4), (v_4, v_1), (v_4, v_4)\}$, then $L_m^2(V(H)) = \{v_1, v_2, v_4\}$, $U_m^2(V(H)) = \{v_1, v_4\}$. Therefore, $L_m^2(V(H)) \neq U_m^2(V(H))$.

Remark 2.3.7. Let $G = (V(G), E(G))$ be a non-empty finite serial graph. If $V(G) \neq \bigcup_{v \in V(G)} OD(v)$, then the properties ($L_1$), ($L_3$), ($L_8$), ($L_9$),($L_{10}$),($L_{11}$), ($L_{12}$),($U_1$),($U_2$),($U_8$),($U_9$),($U_{10}$),($U_{11}$),($U_{12}$) and ($LU$) are not necessarily true for every $H, K \subseteq G$.

The next example illustrates this remark.

Example 2.3.8. Let $G = (V(G), E(G))$: $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{(v_1, v_1), (v_1, v_3), (v_2, v_3), (v_2, v_4), (v_3, v_4), (v_3, v_5), (v_4, v_1), (v_4, v_4), (v_5, v_5)\}$.

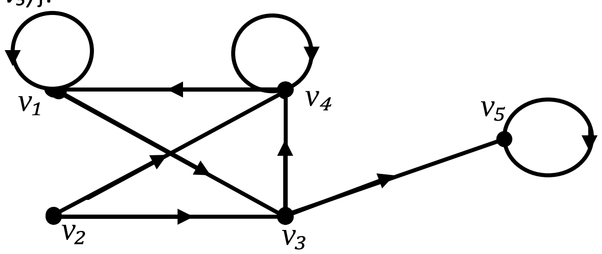

Figure 2.3.2 : Graph *G* given in Example 2.3.8.

We get

$OD(v_1) = \{v_1, v_3\}$, $OD(v_2) = \{v_3, v_4\}$, $OD(v_3) = \{v_4, v_5\}$, $OD(v_4) = \{v_1, v_4\}$, $OD(v_5) = \{v_5\}$

Also we have

$ID(v_1) = \{v_1, v_4\}$, $ID(v_2) = \phi$, $ID(v_3) = \{v_1, v_2\}$, $ID(v_4) = \{v_2, v_3, v_4\}$, $ID(v_5) = \{v_3, v_5\}$

Then we obtain

$MDS(v_1) = \{\{v_1, v_3\}, \{v_1, v_4\}\}$, $MDS(v_2) = \{\{v_3, v_4\}, \phi\}$, $MDS(v_3) = \{\{v_4, v_5\}, \{v_1, v_2\}\}$, $MDS(v_4) = \{\{v_1, v_4\}, \{v_2, v_3, v_4\}\}$, $MDS(v_5) = \{\{v_5\}, \{v_3, v_5\}\}$.

Consequently, we have

(L_1) if $H = (V(H), E(H))$: $V(H) = \{v_5\}$, $E(H) = \{(v_5, v_5)\}$, then $L^2_m(V(H)) = \{v_2, v_5\}$. Therefore, $L^2_m(V(H)) \not\subseteq V(H)$.

(L_3) if $H = (V(H), E(H))$: $V(H) = \phi$, $E(H) = \phi$, then $L^2_m(V(H)) = \{v_2\}$. Therefore, $L^2_m(\phi) \neq \phi$.

(L_8) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_3\}$, $E(H) = \{(v_1, v_1), (v_1, v_3)\}$, then $L^2_m(V(H)) = \{v_1, v_2\}$, $L^2_m(L^2_m(V(H)) = \{v_2, v_3\}$. Therefore, $L^2_m(V(H)) \neq L^2_m(L^2_m(V(H))$.

(L_9) if $H = (V(H), E(H))$: $V(H) = \{v_3, v_5\}$, $E(H) = \{(v_3, v_5), (v_5, v_5)\}$, then $L^2_m(V(H)) = \{v_2, v_5\}$, $U^2_m(L^2_m(V(H)) = \{v_3, v_5\}$. Therefore, $L^2_m(V(H)) \neq U^2_m(L^2_m(V(H))$.

(L_{10}) if $H = (V(H), E(H))$: $V(H) = \{v_4, v_5\}$, $E(H) = \{(v_4, v_4), (v_5, v_5)\}$, then $U^2_m(V(H) = \{v_4, v_5\}$, $L^2_m(U^2_m(V(H)) = \{v_2, v_5\}$. Therefore, $V(H) \not\subseteq L^2_m(U^2_m(V(H))$.

(L_{11}) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_2, v_5\}$, $E(H) = \{(v_1, v_1), (v_5, v_5)\}$, then $L^2_m(V(H)) = \{v_2, v_3, v_5\}$, $L^2_m(L^2_m(V(H)) = \{v_2, v_5\}$. Therefore, $L^2_m(V(H)) \not\subseteq L^2_m(L^2_m(V(H))$.

(L_{12}) let$H = (V(H), E(H))$: $V(H) = \{v_1, v_4, v_5\}$, $E(H) = \{(v_1, v_1), (v_4, v_1), (v_4, v_4), (v_5, v_5)\}$and $K = (V(K), E(K))$: $V(K) = \{v_2, v_3, v_4\}$, $E(K) = \{(v_2, v_3), (v_2, v_4), (v_3, v_4), (v_4, v_4)\}$then $L^2_m(V(H)) = V(G)$ and $L^2_m(V(K)) = \{v_2, v_4\}$ But, $H\cap K = (V(H)\cap V(K), E(H)\cap E(K))$: $V(H)\cap V(K) = \{v_4\}$, $E(H)\cap E(K) = \{(v_4, v_4)\}$ such that $L^2_m(V(H) \cap V(K)) = \{v_2\}$ and so $L^2_m(V(H) \cap V(K)) \neq L^2_m(V(H)) \cap L^2_m(V(H))$.

(U_1) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_2\}$, $E(H) = \{(v_1, v_1)\}$, then $U^2_m(V(H)) = \{v_1, v_4\}$. Therefore, $V(H) \not\subseteq U^2_m(V(H))$.

(U_2) if $H = (V(H), E(H))$: $V(H) = V(G)$, $E(H) = \{(v_1, v_1), (v_1, v_3), (v_2, v_3), (v_2, v_4), (v_3, v_4), (v_3, v_5), (v_4, v_1), (v_4, v_4), (v_5, v_5)\}$, then $U^2_m(V(H)) = \{v_1, v_3, v_4, v_5\}$. Therefore, $U^2_m(V(G)) \neq V(G)$.

(U_8) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_2, v_3\}$, $E(H) = \{(v_1, v_1), (v_1, v_3), (v_2, v_3)\}$, then $U^2_m(V(H)) = \{v_1, v_4\}$, $U^2_m(U^2_m(V(H))) = \{v_1, v_3, v_4\}$. Therefore, $U^2_m(V(H)) \neq U^2_m(U^2_m(V(H)))$.

(U_9) if $H = (V(H), E(H))$: $V(H) = \{v_2, v_4, v_5\}$, $E(H) = \{(v_2, v_4), (v_4, v_4), (v_5, v_5)\}$, then $U^2_m(V(H)) = \{v_3, v_4, v_5\}$, $L^2_m(U^2_m(V(H))) = \{v_2, v_3, v_5\}$. Therefore, $U^2_m(V(H)) \neq L^2_m(U^2_m(V(H)))$.

(U_{10}) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_4, v_5\}$, $E(H) = \{(v_1, v_1), (v_4, v_1), (v_4, v_4), (v_5, v_5)\}$, then $L^2_m(V(H) = V(G)$, $U^2_m(L^2_m(V(H)) = \{v_1, v_3, v_4, v_5\}$. Therefore, $V(H) \not\supseteq U^2_m(L^2_m(V(H))$.

(U_{11}) if $H = (V(H), E(H))$: $V(H) = \{v_2, v_4\}$, $E(H) = \{(v_2, v_4), (v_4, v_4)\}$, then $U^2_m(V(H)) = \{v_3, v_4\}$, $U^2_m(U^2_m(V(H))) = \{v_1, v_4\}$. Therefore, $U^2_m(V(H)) \not\supseteq U^2_m(U^2_m(V(H)))$.

(U_{12}) let$H = (V(H), E(H))$: $V(H) = \{v_3, v_5\}$, $E(H) = \{(v_3, v_5), (v_5, v_5)\}$and $K = (V(K), E(K))$: $V(K) = \{v_4, v_5\}$, $E(K) = \{(v_4, v_4), (v_5, v_5)\}$,then $U^2_m(V(H)) = \{v_5\}$ and $U^2_m(V(K)) = \{v_4, v_5\}$ But, $H\cup K = (V(H)\cup V(K), E(H)\cup E(K))$: $V(H)\cup V(K) = \{v_3, v_4, v_5\}$, $E(H)\cup E(K) = \{(v_3, v_4), (v_3, v_5), (v_4, v_4), (v_5, v_5)\}$ such that $U^2_m(V(H) \cup V(K)) = \{v_1, v_4, v_5\}$ and so $U^2_m(V(H) \cup V(K)) \neq U^2_m(V(H)) \cup U^2_m(V(K))$.

(LU) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_2, v_4, v_5\}$, $E(H) = \{(v_1, v_1), (v_2, v_4), (v_4, v_1), (v_4, v_4), (v_5, v_5)\}$, then $L^2_m(V(H)) = V(G)$, $U^2_m(V(H)) = \{v_1, v_3, v_4, v_5\}$. Therefore, $L^2_m(V(H)) \neq U^2_m(V(H))$.

Proposition 2.3.9. Let $G = (V(G), E(G))$ be a non-empty finite reflexive graph, then the following properties holds for every $H \subseteq G$.

(L_1) $L^2_m(V(H)) \subseteq V(H)$,

(L_3) $L^2_m(\phi) = \phi$,
(U_1) $V(H) \subseteq U^2_m(V(H))$,
(U_2) $U^2_m(V(G)) = V(G)$ and
(LU) $L^2_m(V(H)) \subseteq U^2_m(V(H))$.

Proof.

(L_1) Since G is reflexive graph, then $v \in MD(v)$ for all $v \in V(G)$. Now, let $v \in L^2_m(V(H)) \Longrightarrow \exists MD(v)$, $MD(v) \subseteq V(H)$. But $v \in MD(v)$ for all $v \in V(G)$, so, $v \in V(H)$. Therefore, $L^2_m(V(H)) \subseteq V(H)$.

(L_3) Since any non-empty finite reflexive graph is serial with $V(G) \neq \cup_{v \in V(G)} OD(v)$, then the proof ($L_3$) is immediately derived from Proposition (2.3.4).

(U_1) let $v \in V(H)$ and since $v \in MD(v)$ for all $v \in V(G)$ then for all $MD(v)$, $MD(v) \cap V(H) \neq \phi$. So, $v \in U^2_m(V(H))$ and hence $V(H) \subseteq U^2_m(V(H))$.

(U_2) Since any non-empty finite reflexive graph is serial with $V(G) \neq \cup_{v \in V(G)} OD(v)$, then the proof ($U_2$) is immediately derived from Proposition (2.3.4).

(LU) By Using (L_1) and (U_1) we have $L^2_m(V(H)) \subseteq U^2_m(V(H))$.

Remark 2.3.10, Let G = ($V(G)$, $E(G)$) be a non-empty finite reflexive graph, then the properties (L_8),(L_9),(L_{10}),(L_{11}),(L_{12}),(U_8),(U_9),(U_{10}), (U_{11}) and (U_{12}) are not true in general for every $H, K \subseteq G$.

The following example illustrates this remark.

Example 2.3.11. Let G = ($V(G)$, $E(G)$): $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{(v_1, v_1), (v_1, v_3), (v_2, v_2), (v_2, v_4), (v_3, v_3), (v_3, v_4), (v_4, v_1), (v_4, v_4), (v_5, v_2), (v_5, v_5)\}$.

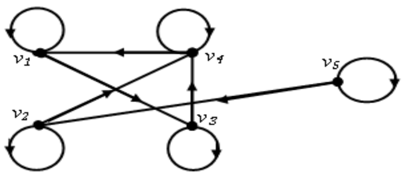

Figure 2.3.3 : Graph G given in Example 2.3.11.

We get

$OD(v_1) = \{v_1, v_3\}$, $OD(v_2) = \{v_2, v_4\}$, $OD(v_3) = \{v_3, v_4\}$, $OD(v_4) = \{v_1, v_4\}$, $OD(v_5) = \{v_2, v_5\}$

Also we have

$ID(v_1) = \{v_1, v_4\}$, $ID(v_2) = \{v_2, v_5\}$, $ID(v_3) = \{v_1, v_3\}$, $ID(v_4) = \{v_2, v_3, v_4\}$, $ID(v_5) = \{v_5\}$

Then we obtain

$MDS(v_1) = \{\{v_1, v_3\}, \{v_1, v_4\}\}$, $MDS(v_2) = \{\{v_2, v_4\}, \{v_2, v_5\}\}$, $MDS(v_3) = \{\{v_3, v_4\}, \{v_1, v_3\}\}$, $MDS(v_4) = \{\{v_1, v_4\}, \{v_2, v_3, v_4\}\}$, $MDS(v_5) = \{\{v_2, v_5\}, \{v_5\}\}$.

Therefore, we have

(L_8) if H = ($V(H)$, $E(H)$): $V(H) = \{v_2, v_4\}$, $E(H) = \{(v_2, v_2), (v_2, v_4), (v_4, v_4)\}$, then $L^2_m(V(H)) = \{v_2\}$, $L^2_m(L^2_m(V(H)) = \phi$. Therefore, $L^2_m(V(H)) \neq L^2_m(L^2_m(V(H))$.

(L_9) if H = ($V(H)$, $E(H)$): $V(H) = \{v_1, v_3\}$, $E(H) = \{(v_1, v_1), (v_1, v_3), (v_3, v_3)\}$, then $L^2_m(V(H)) = \{v_1, v_3\}$, $U^2_m(L^2_m(V(H)) = \{v_1, v_3, v_4\}$. Therefore, $L^2_m(V(H)) \neq U^2_m(L^2_m(V(H))$.

(L_{10}) if H = ($V(H)$, $E(H)$): $V(H) = \{v_3, v_5\}$, $E(H) = \{(v_3, v_3), (v_5, v_5)\}$, then $U^2_m(V(H) = \{v_3, v_5\}$, $L^2_m(U^2_m(V(H)) = \{v_5\}$. Therefore, $V(H) \nsubseteq L^2_m(U^2_m(V(H))$.

(L_{11}) if H = ($V(H)$, $E(H)$): $V(H) = \{v_3, v_4\}$, $E(H) = \{(v_3, v_3), (v_3, v_4), (v_4, v_4)\}$, then $L^2_m(V(H)) = \{v_3\}$, $L^2_m(L^2_m(V(H)) = \phi$. Therefore, $L^2_m(V(H)) \nsubseteq L^2_m(L^2_m(V(H))$.

(L_{12}) let H = ($V(H)$, $E(H)$): $V(H) = \{v_1, v_2, v_3\}$, $E(H) = \{(v_1, v_1), (v_1, v_3), (v_2, v_2), (v_3, v_3)\}$ and K = ($V(K)$, $E(K)$): $V(K) = \{v_1, v_2, v_4\}$, $E(K) = \{(v_1, v_1), (v_2, v_2), (v_2, v_4), (v_4, v_4)\}$ then $L^2_m(V(H)) = \{v_1, v_3\}$ and $L^2_m(V(K)) = \{v_1, v_2,$

v_4} But, $H \cap K$ = ($V(H) \cap V(K), E(H) \cap E(K)$): $V(H) \cap V(K)$ = {v_1, v_2}, $E(H) \cap E(K)$ = {(v_1, v_1), (v_2, v_2)} such that $L_m^2(V(H) \cap V(K)) = \phi$ and so $L_m^2(V(H) \cap V(K)) \neq L_m^2(V(H)) \cap L_m^2(V(H))$.

(U_8) if H = ($V(H)$, $E(H)$): $V(H)$ = {v_1, v_2, v_5}, $E(H)$ = {(v_1, v_1), (v_2, v_2), (v_2, v_5), (v_5, v_5)}, then $U_m^2(V(H))$ = {v_1, v_5}, $U_m^2(U_m^2(V(H)))$ = {v_5}. Therefore, $U_m^2(V(H)) \neq U_m^2(U_m^2(V(H)))$.

(U_9) if H = ($V(H)$, $E(H)$): $V(H)$ = {v_1, v_5}, $E(H)$ = {(v_1, v_1), (v_5, v_5)}, then $U_m^2(V(H))$ = {v_1, v_5}, $L_m^2(U_m^2(V(H)))$ = {v_5}. Therefore, $U_m^2(V(H)) \neq L_m^2(U_m^2(V(H)))$.

(U_{10}) if H = ($V(H)$, $E(H)$): $V(H)$ = {v_2, v_3, v_4, v_5}, $E(H)$ = {(v_2, v_2), (v_2, v_4), (v_3, v_3), (v_3, v_4), (v_4, v_4), (v_5, v_2), (v_5, v_5)}, then $L_m^2(V(H)$ = {v_2, v_3, v_4, v_5}, $U_m^2(L_m^2(V(H)) = V(G)$. Therefore, $V(H) \not\supseteq U_m^2(L_m^2(V(H))$.

(U_{11}) if H = ($V(H)$, $E(H)$): $V(H)$ = {v_2, v_3, v_5}, $E(H)$ = {(v_2, v_2), (v_3, v_3), (v_5, v_2), (v_5, v_5)}, then $U_m^2(V(H))$ = {v_1, v_2, v_3, v_4}, $U_m^2(U_m^2(V(H))) = V(G)$. Therefore, $U_m^2(V(H)) \not\supseteq U_m^2(U_m^2(V(H)))$.

(U_{12}) let H = ($V(H)$, $E(H)$): $V(H)$ = {v_1, v_4}, $E(H)$ = {(v_1, v_1), (v_4, v_1), (v_4, v_4)} and K = ($V(K)$, $E(K)$): $V(K)$ = {v_1, v_5}, $E(K)$ = {(v_1, v_1), (v_5, v_5)}, then $U_m^2(V(H))$ = {v_1, v_3, v_4} and $U_m^2(V(K))$ = {v_1, v_5} But, $H \cup K$ = ($V(H) \cup V(K), E(H) \cup E(K)$): $V(H) \cup V(K)$ = {v_1, v_4, v_5}, $E(H) \cup E(K)$ = {(v_1, v_1), (v_4, v_1), (v_4, v_4), (v_5, v_5)} such that $U_m^2(V(H) \cup V(K))$ = {v_1, v_4, v_5} and so $U_m^2(V(H) \cup V(K)) \neq U_m^2(V(H)) \cup U_m^2(V(K))$.

Proposition 2.3.12. Let G = ($V(G)$, $E(G)$) be a non-empty finite symmetric graph, then the following properties holds for every $H, K \subseteq G$.

(L_{10}) $V(H) \subseteq L_m^2(U_m^2(V(H))$,

(L_{12}) $L_m^2(V(H) \cap V(K)) = L_m^2(V(H)) \cap L_m^2(V(K))$,

(U_{10}) $V(H) \supseteq U_m^2(L_m^2(V(H))$ and

(U_{12}) $U_m^2(V(H) \cup V(K)) = U_m^2(V(H)) \cup U_m^2(V(K))$.

Proof.

(L_{10}) since G is a symmetric graph, then $G = G^{-1}$ and consequently we have $MD(v) = OD(v) = ID(v)$ for all $v \in V(G)$. Now, let $v \in V(H)$. Since G is symmetric, then for all $x \in MD(v)$ we have $v \in MD(x)$. Hence $MD(x) \cap V(H) \neq \phi$ because $v \in V(H)$. Thus by Definition (2.2.1), we get $x \in U_m^2(V(H)$ for all $x \in MD(v)$ which implies $MD(v) \subseteq U_m^2(V(H)$. So, by Definition (2.2.1), we have $v \in L_m^2(U_m^2(V(H))$. Therefore, $V(H) \subseteq L_m^2(U_m^2(V(H))$.

(L_{12}) since G is a symmetric graph, then $MD(v) = OD(v) = ID(v)$ for all $v \in V(G)$. Now, let $v \in L_m^2(V(H)) \cap L_m^2(V(K)) \Longrightarrow v \in L_m^2(V(H)) \wedge v \in L_m^2(V(K)) \Longrightarrow \exists MD(v)$: $MD(v) \subseteq V(H) \wedge MD(v) \subseteq V(K) \Longrightarrow \exists MD(v)$: $MD(v) \subseteq V(H) \cap V(K) \Longrightarrow v \in L_m^2(V(H) \cap V(K))$. Hence $L_m^2(V(H)) \cap L_m^2(V(K)) \subseteq L_m^2(V(H) \cap V(K))$. By using (L_5) in Proposition (2.3.1), we have $L_m^2(V(H) \cap V(K)) \subseteq L_m^2(V(H)) \cap L_m^2(V(K))$ so $L_m^2(V(H) \cap V(K)) = L_m^2(V(H)) \cap L_m^2(V(K))$.

(U_{10}) let $v \in U_m^2(L_m^2(V(H))$, then $\forall MD(v)$, $MD(v) \cap L_m^2(V(H)) \neq \phi$. Hence there exists $u \in MD(v)$ such that $u \in L_m^2(V(H)$. Since G is a symmetric graph, then $u \in MD(v)$ implies $v \in MD(u)$. Now since $u \in L_m^2(V(H))$ then by Definition (2.2.1), we have $MD(u) \subseteq V(H)$. But $v \in MD(u)$ and so $v \in V(H)$. Consequently, $V(H) \supseteq U_m^2(L_m^2(V(H))$.

(U_{12}) Firsty, we need to show that $U_m^2(V(H) \cup V(K)) \subseteq U_m^2(V(H)) \cup U_m^2(V(K))$. Now, let $v \notin U_m^2(V(H)) \cup U_m^2(V(K)) \Longrightarrow v \notin U_m^2(V(H)) \wedge v \notin U_m^2(V(K))$. Then, by Definition (2.2.1), we obtain $MD(v) \cap V(H) = \phi \wedge MD(v) \cap V(K) = \phi$. So, $MD(v) \cap (V(H) \cap V(K)) = \phi$ and hence $v \notin U_m^2(V(H) \cup V(K))$. By using ($U_6$) in Proposition (2.3.1), we have $U_m^2(V(H) \cup V(K)) \supseteq U_m^2(V(H)) \cup U_m^2(V(H))$. Therefore, $U_m^2(V(H) \cup V(K)) = U_m^2(V(H)) \cup U_m^2(V(K))$.

Remark 2.3.13. Let G = ($V(G)$, $E(G)$) be a non-empty finite symmetric graph, then the properties (L_1),(L_3),(L_8),(L_9),(L_{11}),(U_1),(U_2),(U_8),(U_9),(U_{11}) and (LU) are not true in general for every $H \subseteq G$.

The following example shows this remark.

Example 2.3.14. Let $G = (V(G), E(G))$: $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{(v_1, v_2), (v_1, v_4), (v_2, v_1), (v_2, v_5), (v_3, v_3), (v_4, v_1), (v_4, v_4), (v_5, v_2), (v_5, v_5)\}$

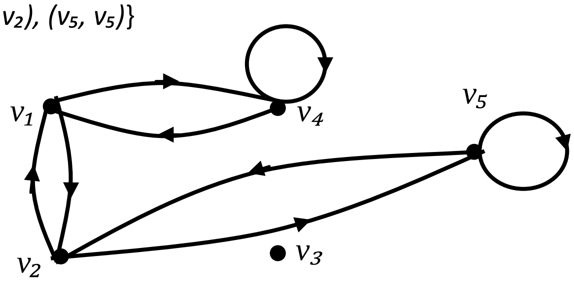

Figure 2.3.4 : Graph *G* given in Example 2.3.14.

We get

$OD(v_1) = \{v_2, v_4\}$, $OD(v_2) = \{v_1, v_5\}$, $OD(v_3) = \phi$, $OD(v_4) = \{v_1, v_4\}$, $OD(v_5) = \{v_2, v_5\}$

Also we have

$ID(v_1) = \{v_2, v_4\}$, $ID(v_2) = \{v_1, v_5\}$, $ID(v_3) = \phi$, $ID(v_4) = \{v_1, v_4\}$, $ID(v_5) = \{v_2, v_5\}$

Then we obtain

$MDS(v_1) = \{\{v_2, v_4\}\}$, $MDS(v_2) = \{\{v_2, v_5\}\}$, $MDS(v_3) = \{\phi\}$, $MDS(v_4) = \{\{v_1, v_4\}\}$, $MDS(v_5) = \{\{v_2, v_5\}\}$.

Therefore, we have

(L_1) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_2\}$, $E(H) = \{(v_1, v_2), (v_2, v_1)\}$, then $L^2_m(V(H)) = \{v_3\}$. Therefore, $L^2_m(V(H)) \not\subseteq V(H)$.

(L_3) if $H = (V(H), E(H))$: $V(H) = \phi$, $E(H) = \phi$, then $L^2_m(V(H)) = \{v_3\}$. Therefore, $L^2_m(\phi) \neq \phi$.

(L_8) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_4\}$, $E(H) = \{(v_1, v_4), (v_4, v_1), (v_4, v_4)\}$, then $L^2_m(V(H)) = \{v_3, v_4\}$, $L^2_m(L^2_m(V(H)) = \{v_3\}$. Therefore, $L^2_m(V(H)) \neq L^2_m(L^2_m(V(H))$.

(L_9) if $H = (V(H), E(H))$: $V(H) = \{v_3, v_5\}$, $E(H) = \{(v_5, v_5)\}$, then $L^2_m(V(H)) = \{v_3\}$, $U^2_m(L^2_m(V(H)) = \phi$. Therefore, $L^2_m(V(H)) \neq U^2_m(L^2_m(V(H))$.

(L_{11}) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_3, v_5\}$, $E(H) = \{(v_5, v_5)\}$, then $L^2_m(V(H)) = \{v_2, v_3\}$, $L^2_m(L^2_m(V(H)) = \{v_3\}$. Therefore, $L^2_m(V(H)) \not\subseteq L^2_m(L^2_m(V(H))$.

(U_1) if $H = (V(H), E(H))$: $V(H) = \{v_1\}$, $E(H) = \phi$, then $U^2_m(V(H)) = \{v_2, v_4\}$. Therefore, $V(H) \not\subseteq U^2_m(V(H))$.

(U_2) if $H = (V(H), E(H))$: $V(H) = V(G)$, $E(H) = \{(v_1, v_2), (v_1, v_4), (v_2, v_1), (v_2, v_5), (v_4, v_1), (v_4, v_4), (v_5, v_2), (v_5, v_5)\}$, then $U^2_m(V(H)) = \{v_1, v_2, v_4, v_5\}$. Therefore, $U^2_m(V(G)) \neq V(G)$.

(U_8) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_3\}$, $E(H) = \phi$, then $U^2_m(V(H)) = \{v_2, v_4\}$, $U^2_m(U^2_m(V(H))) = \{v_1, v_4, v_5\}$. Therefore, $U^2_m(V(H)) \neq U^2_m(U^2_m(V(H)))$.

(U_9) if $H = (V(H), E(H))$: $V(H) = \{v_4, v_5\}$, $E(H) = \{(v_4, v_4), (v_5, v_5)\}$, then $U^2_m(V(H)) = \{v_1, v_2, v_4, v_5\}$, $L^2_m(U^2_m(V(H))) = V(G)$. Therefore, $U^2_m(V(H)) \neq L^2_m(U^2_m(V(H)))$.

(U_{11}) if $H = (V(H), E(H))$: $V(H) = \{v_3\}$, $E(H) = \phi$, then $U^2_m(V(H)) = \phi$, $U^2_m(U^2_m(V(H))) = \{v_3\}$. Therefore, $U^2_m(V(H)) \not\supseteq U^2_m(U^2_m(V(H)))$.

(LU) if $H = (V(H), E(H))$: $V(H) = \{v_2, v_3, v_4\}$, $E(H) = \{(v_4, v_4)\}$, then $L^2_m(V(H)) = \{v_1, v_3\}$, $U^2_m(V(H)) = \{v_1, v_4, v_5\}$. Therefore, $L^2_m(V(H)) \neq U^2_m(V(H))$.

Lemma 2.3.15. Let $G = (V(G), E(G))$ be a non-empty finite transitive graph and $MD(v)$ be a mixed degree set of a vertex $v \in V(G)$, then for each $u \in MD(v)$ there exists $MD(u)$ such that $MD(u) \subseteq MD(v)$.

Proof. Firsly, if $MD(v) = OD(v)$ then we want to show that for each $x \in OD(v)$ we have $OD(x) \subseteq OD(v)$. Now, let $x \in OD(v)$ and $y \in OD(x)$. Since *G* is a transitive graph, $y \in OD(x)$ and $x \in OD(v)$ then $y \in OD(v)$. So, $OD(x) \subseteq OD(v)$ for all $x \in OD(v)$. On the other hand, if $MD(v) = ID(v)$ then we need to show that for each $x \in ID(v)$ we get $ID(x) \subseteq ID(v)$. So, let $x \in ID(v)$ and $z \in ID(x)$. Since *G* is transitive graph then $G^{-1}$is also transitive. Since $R^{-1}$is transitive, $z \in ID(x)$ and $x \in ID(v)$ then $z \in ID(v)$. Hence, $ID(x)$

$\subseteq ID(v)$ for all $x \in ID(v)$. Consequently, for each $x \in MD(v)$ there exists $MD(x)$ such that $MD(x) \subseteq MD(v)$.

Proposition 2.3.16. Let $G = (V(G), E(G))$ be a non-empty finite transitive graph, then the following properties holds for every $H \subseteq G$.

(L_{11}) $L^2_m(V(H)) \subseteq L^2_m(L^2_m(V(H))$,

(U_{11}) $U^2_m(V(H)) \supseteq U^2_m(U^2_m(V(H))$.

Proof.

(L_{11}) let $v \in L^2_m(V(H))$, $\exists MD(v)$, $MD(v) \subseteq V(H)$. Now, let $u \in MD(v)$ by Lemma (4.3.15), there exist $MD(u)$ such that $MD(u) \subseteq MD(v)$. Thus, $\exists MD(u)$, $MD(u) \subseteq V(H) \Longrightarrow u \in L^2_m(V(H))$. So, $MD(v) \subseteq L^2_m(V(H))$. Therefore, $\exists MD(v)$, $MD(v) \subseteq L^2_m(V(H)) \Longrightarrow v \in L^2_m(L^2_m(V(H))$. Hence, $L^2_m(V(H)) \subseteq L^2_m(L^2_m(V(H))$.

(U_{11}) let $v \in U^2_m(U^2_m(V(H)) \Longrightarrow \forall MD(v)$, $MD(v) \cap U^2_m(V(H)) \neq \phi \Longrightarrow \forall$ $MD(v)$ there exists u such that $u \in MD(v)$ and $u \in U^2_m(V(H))$. Now, since $u \in U^2_m(V(H))$ then for all $MD(u)$, $MD(u) \cap V(H) \neq \phi$ since $u \in MD(v)$, then by Lemma(4.3.15), there exists $MD(u)$ such that $MD(u) \subseteq MD(v)$. But, for all $MD(u)$, $MD(u) \cap V(H) \neq \phi$. Consequently, we have for all $MD(v)$, $MD(v) \cap V(H) \neq \phi$. Hence $v \in U^2_m(V(H))$. Therefore $U^2_m(V(H))$ $U^2_m(U^2_m(V(H))$.

Remark 2.3.17.. Let $G = (V(G), E(G))$ be a non-empty finite transitive graph, then the properties (L_1), (L_1), (L_8), (L_9), (L_{10}), (L_{12}), (U_1), (U_1), (U_8), (U_9), (U_{10}), (U_{12}) and (LU) are not true in general for every $H, K \subseteq G$.

The following two examples illustrate this remark.

Example 2.3.18. Let $G = (V(G), E(G))$: $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{(v_1, v_1), (v_2, v_4), (v_3, v_2), (v_3, v_3), (v_3, v_4), (v_4, v_4), (v_5, v_1), (v_5, v_5)\}$.

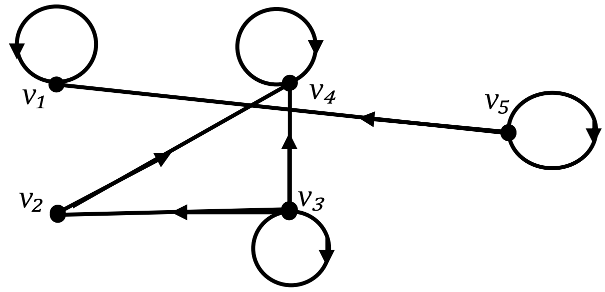

Figure 2.3.5 : Graph G given in Example 2.3.18.

We get

$OD(v_1) = \{v_1\}$, $OD(v_2) = \{v_4\}$, $OD(v_3) = \{v_2, v_3, v_4\}$, $OD(v_4) = \{v_4\}$, $OD(v_5) = \{v_1, v_5\}$

Also we have

$ID(v_1) = \{v_1, v_5\}$, $ID(v_2) = \{v_3\}$, $ID(v_3) = \{v_3\}$, $ID(v_4) = \{v_2, v_3, v_4\}$, $ID(v_5) = \{v_5\}$

Then we obtain

$MDS(v_1) = \{\{v_1\}, \{v_1, v_5\}\}$, $MDS(v_2) = \{\{v_4\}, \{v_3\}\}$, $MDS(v_3) = \{\{v_2, v_3, v_4\}, \{v_3\}\}$, $MDS(v_4) = \{\{v_4\}, \{v_2, v_3, v_4\}\}$, $MDS(v_5) = \{\{v_1, v_5\}, \{v_5\}\}$.

Example 2.3.19. Let $G = (V(G), E(G))$: $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{(v_1, v_1), (v_2, v_4), (v_2, v_5), (v_4, v_5)\}$.

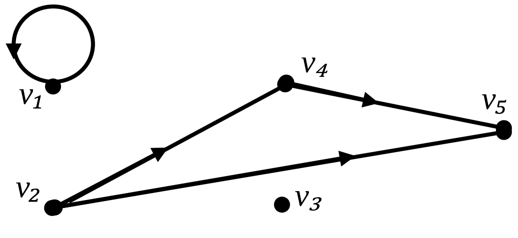

Figure 2.3.6 : Graph *G* given in Example 2.3.19.

We get

$OD(v_1) = \{v_1\}$, $OD(v_2) = \{v_4, v_5\}$, $OD(v_3) = \phi$, $OD(v_4) = \{v_5\}$, $OD(v_5) = \phi$

Also we have

$ID(v_1) = \{v_1\}$, $ID(v_2) = \phi$, $ID(v_3) = \phi$, $ID(v_4) = \{v_2, v_3, v_4\}$, $ID(v_5) = \{v_5\}$

Then we obtain

$MDS(v_1) = \{\{v_1\}\}$, $MDS(v_2) = \{\{v_2, v_4\}, \phi\}$, $MDS(v_3) = \{\phi\}$, $MDS(v_4) = \{\{v_5\}, \{v_2, v_3, v_4\}\}$, $MDS(v_5) = \{\phi, \{v_5\}\}$.

Consequently, we have

(L_1) if $H = (V(H), E(H))$: $V(H) = \{v_3\}$, $E(H) = \{(v_3, v_3)\}$, then $L^2_m(V(H)) = \{v_2, v_3\}$. Therefore, $L^2_m(V(H)) \not\subseteq V(H)$.

(L_3) if $H = (V(H), E(H))$: $V(H) = \phi$, $E(H) = \phi$, then $L^2_m(V(H)) = \{v_2, v_3, v_5\}$. Therefore, $L^2_m(\phi) \neq \phi$.

(L_8) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_3\}$, $E(H) = \{(v_1, v_1)\}$, then $L^2_m(V(H)) = \{v_1, v_2, v_3, v_5\}$, $L^2_m(L^2_m(V(H)) = V(G)$. Therefore, $L^2_m(V(H)) \neq L^2_m(L^2_m(V(H))$.

(L_9) if $H = (V(H), E(H))$: $V(H) = \{v_2, v_4\}$, $E(H) = \{(v_2, v_4)\}$, then $L^2_m(V(H)) = \{v_2, v_3, v_4, v_5\}$, $U^2_m(L^2_m(V(H)) = \{v_4\}$. Therefore, $L^2_m(V(H)) \neq U^2_m(L^2_m(V(H))$.

(L_{10}) if $H = (V(H), E(H))$: $V(H) = \{v_2, v_3, v_5\}$, $E(H) = \{(v_2, v_5)\}$, then $U^2_m(V(H) = \{v_4\}$, $L^2_m(U^2_m(V(H)) = \phi$. Therefore, $V(H) \not\subseteq L^2_m(U^2_m(V(H))$.

(L_{12}) let$H = (V(H), E(H))$: $V(H) = \{v_2, v_5\}$, $E(H) = \{(v_2, v_5)\}$and $K = (V(K), E(K))$: $V(K) = \{v_3, v_5\}$, $E(K) = \phi$, then $L^2_m(V(H)) = \{v_2, v_3, v_4, v_5\}$and $L^2_m(V(K)) = \{v_2, v_3, v_4, v_5\}$ But, $H \cap K = (V(H) \cap V(K), E(H) \cap E(K))$: $V(H) \cap V(K) = \{v_5\}$, $E(H) \cap E(K) = \phi$ such that $L^2_m(V(H) \cap V(K)) = \{v_2, v_3, v_5\}$ and so $L^2_m(V(H) \cap V(K)) \neq L^2_m(V(H)) \cap L^2_m(V(H))$.

(U_1) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_2\}$, $E(H) = \{(v_1, v_1)\}$, then $U^2_m(V(H)) = \{v_1\}$. Therefore, $V(H) \not\subseteq U^2_m(V(H))$.

(U_2) if $H = (V(H), E(H))$: $V(H) = V(G)$, $E(H) = \{(v_1, v_1), (v_2, v_4), (v_2, v_5), (v_4, v_5)\}$, then $U^2_m(V(H)) = \{v_1, v_4\}$. Therefore, $U^2_m(V(G)) \neq V(G)$.

(U_8) if $H = (V(H), E(H))$: $V(H) = \{v_2, v_3, v_5\}$, $E(H) = \{(v_2, v_5)\}$, then $U^2_m(V(H)) = \{v_4\}$, $U^2_m(U^2_m(V(H))) = \phi$. Therefore, $U^2_m(V(H)) \neq U^2_m(U^2_m(V(H)))$.

(U_9) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_2, v_5\}$, $E(H) = \{(v_1, v_1), (v_2, v_5)\}$, then $U^2_m(V(H)) = \{v_1, v_4\}$, $L^2_m(U^2_m(V(H))) = \{v_1, v_2, v_3, v_5\}$. Therefore, $U^2_m(V(H)) \neq L^2_m(U^2_m(V(H)))$.

(U_{10}) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_5\}$, $E(H) = \{(v_1, v_1)\}$, then $L^2_m(V(H) = V(G)$, $U^2_m(L^2_m(V(H)) = \{v_1, v_4\}$. Therefore, $V(H) \not\supseteq U^2_m(L^2_m(V(H))$.

(U_{12}) let$H = (V(H), E(H))$: $V(H) = \{v_1, v_5\}$, $E(H) = \{(v_1, v_1), (v_5, v_1), (v_5, v_5)\}$and $K = (V(K), E(K))$: $V(K) = \{v_2, v_3\}$, $E(K) = \{(v_3, v_2), (v_3, v_3)\}$,then $U^2_m(V(H)) = \{v_1, v_5\}$ and $U^2_m(V(K)) = \{v_3\}$ But, $H \cup K = (V(H) \cup V(K), E(H) \cup E(K))$: $V(H) \cup V(K) = \{v_1, v_2, v_3, v_5\}$, $E(H) \cup E(K) = \{(v_1, v_1), (v_3, v_2), (v_3, v_3), (v_5, v_1), (v_5, v_5)\}$ such that $U^2_m(V(H) \cup V(K)) = \{v_1, v_2, v_3, v_5\}$ and so $U^2_m(V(H) \cup V(K)) \neq U^2_m(V(H)) \cup U^2_m(V(K))$.

(LU) if $H = (V(H), E(H))$: $V(H) = \{v_2, v_3, v_4, v_5\}$, $E(H) = \{(v_2, v_4), (v_2, v_5), (v_4, v_5)\}$, then $L^2_m(V(H)) = \{v_2, v_3, v_4, v_5\}$, $U^2_m(V(H)) = \{v_4\}$. Therefore, $L^2_m(V(H)) \neq U^2_m(V(H))$.

Proposition 2.3.20.. Let $G = (V(G), E(G))$ be a non-empty finite tolerance graph, then the properties (L_1), (L_3), (L_{10}), (L_{12}), (U_1), (U_2), (U_{10}), (U_{12}) and (LU) hold for every $H, K \subseteq G$.

Proof. By using Propositions (2.3.9) and (2.3.12) the proof is obvious.

Remark 2.3.21. Let $G = (V(G), E(G))$ be a non-empty finite tolerance graph, then the properties(L_8), (L_9), (L_{11}), (U_8), (U_9) and (U_{11}) are not true in general for every $H \subseteq G$.

The following example shows this remark.

Example 2.3.21. Let G = ($V(G)$, $E(G)$): $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{(v_1, v_1), (v_1, v_2), (v_2, v_2), (v_2, v_2), (v_2, v_3), (v_3, v_2), (v_3, v_3), (v_4, v_4), (v_5, v_5)\}$.

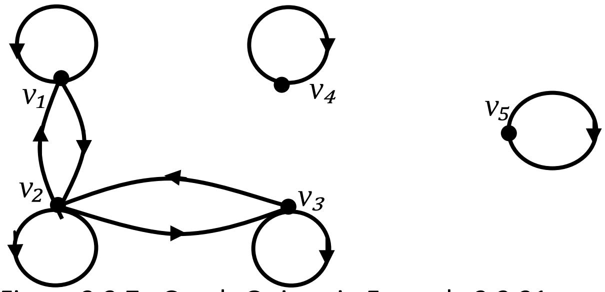

Figure 2.3.7 : Graph G given in Example 2.3.21.

We get

$OD(v_1) = \{v_1, v_2\}$, $OD(v_2) = \{v_1, v_2, v_3\}$, $OD(v_3) = \{v_2, v_3\}$, $OD(v_4) = \{v_4\}$, $OD(v_5) = \{v_5\}$

Also we have

$ID(v_1) = \{v_1, v_2\}$, $ID(v_2) = \{v_1, v_2, v_3\}$, $ID(v_3) = \{v_2, v_3\}$, $ID(v_4) = \{v_4\}$, $ID(v_5) = \{v_5\}$

Then we obtain

$MDS(v_1) = \{\{v_1, v_2\}\}$, $MDS(v_2) = \{\{v_1, v_2, v_3\}\}$, $MDS(v_3) = \{\{v_2, v_3\}\}$, $MDS(v_4) = \{\{v_4\}\}$, $MDS(v_5) = \{\{v_5\}\}$

Therefore, we have

(L_8) if H = ($V(H)$, $E(H)$): $V(H) = \{v_1, v_2, v_4\}$, $E(H) = \{(v_1, v_1), (v_1, v_2), (v_2, v_1), (v_2, v_2), (v_4, v_4)\}$, then $L^2_m(V(H)) = \{v_1, v_4\}$, $L^2_m(L^2_m(V(H)) = \{v_4\}$. Therefore, $L^2_m(V(H)) \neq L^2_m(L^2_m(V(H))$.

(L_9) if H = ($V(H)$, $E(H)$): $V(H) = \{v_1, v_2, v_5\}$, $E(H) = \{(v_1, v_1), (v_1, v_2), (v_2, v_1), (v_2, v_2), (v_5, v_5)\}$, then $L^2_m(V(H)) = \{v_1, v_5\}$, $U^2_m(L^2_m(V(H)) = \{v_1, v_2, v_5\}$. Therefore, $L^2_m(V(H)) \neq U^2_m(L^2_m(V(H))$.

(L_{11}) if H = ($V(H)$, $E(H)$): $V(H) = \{v_2, v_3, v_5\}$, $E(H) = \{(v_2, v_2), (v_2, v_3), (v_3, v_2), (v_3, v_3), (v_5, v_5)\}$,then $L^2_m(V(H)) = \{v_3, v_5\}$, $L^2_m(L^2_m(V(H)) = \{v_5\}$. Therefore, $L^2_m(V(H)) \nsubseteq L^2_m(L^2_m(V(H))$.

(U_8) if H = ($V(H)$, $E(H)$): $V(H) = \{v_1, v_4\}$, $E(H) = \{(v_1, v_1), (v_4, v_4)\}$, then $U^2_m(V(H)) = \{v_1, v_2, v_4\}$, $U^2_m(U^2_m(V(H))) = \{v_1, v_2, v_3, v_4\}$. Therefore, $U^2_m(V(H)) \neq U^2_m(U^2_m(V(H)))$.

(U_9) if H = ($V(H)$, $E(H)$): $V(H) = \{v_1, v_5\}$, $E(H) = \{(v_1, v_1), (v_5, v_5)\}$, then $U^2_m(V(H)) = \{v_1, v_2, v_5\}$, $L^2_m(U^2_m(V(H))) = \{v_1, v_5\}$. Therefore, $U^2_m(V(H)) \neq L^2_m(U^2_m(V(H)))$.

(U_{11}) if H = ($V(H)$, $E(H)$): $V(H) = \{v_1, v_4, v_5\}$, $E(H) = \{(v_1, v_1), (v_4, v_4), (v_5, v_5)\}$, then $U^2_m(V(H)) = \{v_1, v_2, v_4\}$, $U^2_m(U^2_m(V(H))) = \{v_1, v_2, v_3, v_4\}$. Therefore, $U^2_m(V(H)) \nsupseteq U^2_m(U^2_m(V(H)))$.

Proposition 2.3.23. . Let G = ($V(G)$, $E(G)$) be a non-empty finite dominance graph, then the properties (L_1), (L_3), (L_8), (L_{11}), (U_1), (U_2), (U_8), (U_{11}) and (LU) holds for every $H \subseteq G$.

Proof. By using Propositions (2.3.9) and (2.3.16) the proof is obvious.

Remark 2.3.24. . Let G = ($V(G)$, $E(G)$) be a non-empty finite dominance graph, then the properties (L_9), (L_{10}), (L_{12}), (U_9), (U_{10}) and (U_{12}) are not true in general for every $H, K \subseteq G$.

The following example illustrates this remark.

Example 2.3.25. Let G = ($V(G)$, $E(G)$): $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{(v_1, v_1), (v_2, v_2), (v_2, v_4), (v_2, v_5), (v_3, v_3), (v_4, v_4), (v_4, v_5), (v_5, v_5)\}$.

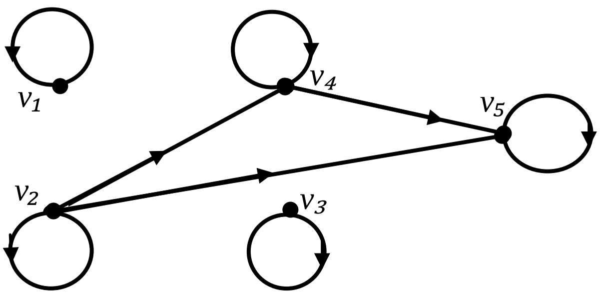

Figure 2.3.8 : Graph *G* given in Example 2.3.25.

We get

$OD(v_1) = \{v_1\}$, $OD(v_2) = \{v_2, v_4, v_5\}$, $OD(v_3) = \{v_3\}$, $OD(v_4) = \{v_4, v_5\}$, $OD(v_5) = \{v_5\}$

Also we have

$ID(v_1) = \{v_1\}$, $ID(v_2) = \{v_2\}$, $ID(v_3) = \{v_3\}$, $ID(v_4) = \{v_2, v_4\}$, $ID(v_5) = \{v_2, v_4, v_5\}$

Then we obtain

$MDS(v_1) = \{\{v_1\}\}$, $MDS(v_2) = \{\{v_2, v_4, v_5\}, \{v_2\}\}$, $MDS(v_3) = \{\{v_3\}\}$, $MDS(v_4) = \{\{v_4, v_5\}, \{v_2, v_4\}\}$, $MDS(v_5) = \{\{v_5\}, \{v_2, v_4, v_5\}\}$.

(L_9) if $H = (V(H), E(H))$: $V(H) = \{v_2, v_5\}$, $E(H) = \{(v_2, v_2), (v_2, v_5), (v_5, v_5)\}$, then $L^2_m(V(H)) = \{v_2, v_5\}$, $U^2_m(L^2_m(V(H)) = \{v_2, v_4, v_5\}$. Therefore, $L^2_m(V(H)) \neq U^2_m(L^2_m(V(H))$.

(L_{10}) if $H = (V(H), E(H))$: $V(H) = \{v_3, v_4\}$, $E(H) = \{(v_3, v_3), (v_4, v_4)\}$, then $U^2_m(V(H) = \{v_3, v_4\}$, $L^2_m(U^2_m(V(H)) = \{v_3\}$. Therefore, $V(H) \not\subseteq L^2_m(U^2_m(V(H))$.

(L_{12}) let$H = (V(H), E(H))$: $V(H) = \{v_1, v_2, v_4\}$, $E(H) = \{(v_1, v_1), (v_2, v_2), (v_2, v_4), (v_4, v_4)\}$and $K = (V(K), E(K))$: $V(K) = \{v_1, v_4, v_5\}$, $E(K) = \{(v_1, v_1), (v_4, v_4), (v_4, v_5), (v_5, v_5)\}$, then $L^2_m(V(H)) = \{v_1, v_2, v_4\}$and $L^2_m(V(K)) = \{v_1, v_4, v_5\}$ But, $H \cap K = (V(H) \cap V(K), E(H) \cap E(K))$: $V(H) \cap V(K) = \{v_1, v_4\}$, $E(H) \cap E(K) = \{(v_1, v_1), (v_4, v_4)\}$such that $L^2_m(V(H) \cap V(K)) = \{v_1\}$ and so $L^2_m(V(H) \cap V(K)) \neq L^2_m(V(H)) \cap L^2_m(V(H))$.

(U_9) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_3, v_4\}$, $E(H) = \{(v_1, v_1), (v_3, v_3), (v_4, v_4)\}$, then $U^2_m(V(H)) = \{v_1, v_3, v_4\}$, $L^2_m(U^2_m(V(H))) = \{v_1, v_3\}$. Therefore, $U^2_m(V(H)) \neq L^2_m(U^2_m(V(H)))$.

(U_{10}) if $H = (V(H), E(H))$: $V(H) = \{v_1, v_2, v_3, v_5\}$, $E(H) = \{(v_1, v_1), (v_2, v_2), (v_2, v_5), (v_3, v_3), (v_5, v_5)\}$, then $L^2_m(V(H) = \{v_1, v_2, v_3, v_5\}$, $U^2_m(L^2_m(V(H)) = V(G)$. Therefore, $V(H) \not\supseteq U^2_m(L^2_m(V(H))$.

(U_{12}) let$H = (V(H), E(H))$: $V(H) = \{v_2\}$, $E(H) = \{(v_2, v_2)\}$and $K = (V(K), E(K))$: $V(K) = \{v_5\}$, $E(K) = \{(v_5, v_5)\}$,then $U^2_m(V(H)) = \{v_2\}$ and $U^2_m(V(K)) = \{v_5\}$ But, $H \cup K = (V(H) \cup V(K), E(H) \cup E(K))$: $V(H) \cup V(K) = \{v_2, v_5\}$, $E(H) \cup E(K) = \{(v_2, v_2), (v_2, v_5), (v_5, v_5)\}$ such that $U^2_m(V(H) \cup V(K)) = \{v_2, v_4, v_5\}$ and so $U^2_m(V(H) \cup V(K)) \neq U^2_m(V(H)) \cup U^2_m(V(K))$.

Proposition 2.3.26.. Let $G = (V(G), E(G))$ be a non-empty finite equivalence graph, then the properties (L_1), (L_3), (L_8), (L_9), (L_{10}), (L_{11}), (L_{12}), (U_1), (U_2), (U_8), (U_9), (U_{10}), (U_{11}), (U_{12}) and (LU) holds for every $H, K \subseteq G$.

Proof. By using Propositions (2.3.9), (2.3.12) and (2.3.16) the proof is obvious.

Table (2.3.1)

Prop.	Arbt.	Ser.1	Ser.2	Refl.	Sym.	Trans.	Tole.	Dom.	Eque.
L_1				*			*	*	*
L_2	*	*	*	*	*	*	*	*	*
L_3		*	*	*			*	*	*
L_4	*	*	*	*	*	*	*	*	*

L_5	*	*	*	*	*	*	*	*	*
L_6	*	*	*	*	*	*	*	*	*
L_7	*	*	*	*	*	*	*	*	*
L_8								*	*
L_9									*
L_{10}					*		*		*
L_{11}						*		*	*
L_{12}				*	*		*		*
U_1				*			*	*	*
U_2		*	*	*			*	*	*
U_3	*	*	*	*	*	*	*	*	*
U_4	*	*	*	*	*	*	*	*	*
U_5	*	*	*	*	*	*	*	*	*
U_6	*	*	*	*	*	*	*	*	*
U_7	*	*	*	*	*	*	*	*	*
U_8								*	*
U_9									*
U_{10}					*		*		*
U_{11}						*		*	*
U_{12}					*		*		*
LU				*			*	*	*

In Table(2.3.1), we summarize the previous results for the properties of approximation operatorsL_m^2 and $U_m^2$when the graph G is arbitrary, serial with $V(G) \neq \cup_{v \in V(G)} OD(v)$,serial with $V(G) = \cup_{v \in V(G)} OD(v)$, reflexive, symmetric, transitive, tolerance, dominance and equivalence respectively.
A star (*) indicate that property is satisfies. The first column contains the list of properties. The next ninth columns assign the properties which are satisfied for the above mention Kinds of G. Also, in table (2.3.1), ser.1 mean serial with $V(G) \neq \cup_{v \in V(G)} OD(v)$ and ser.2 mean serial with $V(G) = \cup_{v \in V(G)} OD(v)$.

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