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RESEARCH ARTICLE

DERIVATION OF EULERIAN INTEGRALS OF H-FUNCTIONS

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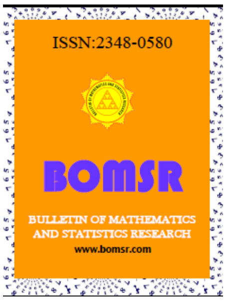

ABSTRACT

The principle object of the present paper is to determine various key formulas for the fractional integration of the multivariable H-function (some preliminary concepts of Gama and beta functions). Each of the general Eulerian integral formulas (acquired in this paper) are appeared to yield fascinating new results for different groups of summed up hyper geometric functions of several variables.

Key Words: Eulerian Beta integrals, Multivariable H-function and Derivation of integral formula.

1. Preliminary concepts

In the theory of Gamma and Beta functions, it is well known that the Eulerian Beta integrals

(1.1) $$B(\alpha,\beta):=\int_0^1 t^{\alpha-1}(1-t)^{\alpha-1}dt=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

$$(\Re(\alpha)>0;\quad \Re(\beta)>0)$$

Can be rewritten (by a simple change of the variable of integration) in its equivalent form, putting t=t-a and (1-t) = {1-(1-b)} we get the following equation

(1.2) $$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}dt=(b-a)^{\alpha-\beta-1}B(\alpha,\beta),$$

$$(\Re(\alpha)>0;\quad \Re(\beta)>0;\qquad a\neq b).$$

Since

(1.3) $$(ut+v)^{\gamma}=(au+v)^{\gamma}\sum_{l=0}^{\infty}\frac{(-\gamma)l}{l\,!}\left\{-\frac{(t-a)u}{au+v}\right\}^{l},\qquad (|(t-a)u|<|(au+v)|;\quad t\in[a,b]),$$

$$(\lambda)_{\mu}=\Gamma(\lambda+\mu)/\Gamma(\lambda)$$

Where,
we readily find that (cf., e.g., [1, p.301, Entry 2.2.6.1]) on using the above formula we get

(1.4) $\int_a^b (t-a)(b-t)(ut+v)^{\gamma}\, dt = (b-a)^{\alpha-\beta-1}(au+v)^{\gamma} \sum_{l=0}^{\propto} \cdot \frac{(-\gamma)l}{l!}$

$\{\frac{(t-a)u}{au+v}\}^{l}$, $(|(t-a)u| < |(au+v)|;\ t\in [a, b])$,

(1.5) $$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}\, dt = (b-a)^{\alpha-\beta-1}(au+v)^{\gamma}\ B(\alpha,\beta), \cdot {}_2F_1\left[\alpha,-\gamma;\ \alpha+\beta;-\frac{(b-a)u}{au+v}\right],$$

$$\left(\Re(\alpha)>0;\quad \Re(\beta)>0;\ \left|\arg\left(\frac{bu+v}{au+v}\right)\right| \le \pi-\in (0<\in<\pi);\ \mathbf{b}\ne a\right),$$

Where denotes, as usual, a generalized hyper geometric function for p numerator and q denominator parameters, and the argument condition emerges from the analytic continuation of the Gaussian hyper geometric function ${}_2F_1$occurring on the right-hand side of (1.5).
Now substitute

$\gamma=-\alpha-\beta$, $u=\lambda-\mu$ and $v=(1+\mu)b-(1+\lambda)a$

In (1.5) we get

(1.6) $\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}[(\lambda-\mu)\, t+(1+\mu)\, b-(1+\lambda)\, a]^{-\alpha-\beta}=$

$(b-a)^{\alpha+\beta-1}[a\,(\lambda-\mu)+(1+\mu)\, b-(1+\lambda)\, a]^{-\alpha-\beta}\ B(\alpha,\beta){}_2F_1[\alpha,-\gamma,\alpha+\beta;-\frac{(b-a)u}{au+v}]$

(1.7) $\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}[(\lambda t-\mu t)+b+b\mu-a-\lambda a]^{-\alpha-\beta}=$

$(b-a)^{\alpha+\beta-1}[a(\lambda-\mu)+(1+\mu)\, b-(1+\lambda)\, a]^{-\alpha-\beta}\ B(\alpha,\beta)\ {}_2F_1[\alpha,-\gamma,\alpha+\beta;-\frac{(b-a)u}{au+v}]$

(1.8) $\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}[(\lambda(t-a)+\mu(b-t)]^{-\alpha-\beta}= \frac{(1+\lambda)^{-\alpha}(1+\mu)^{-\beta}}{(a-b)}\mathrm{B}(\alpha,\beta)$

(1.9) $\int_a^b \cdot\frac{(t-a)^{\alpha-1}(b-t)^{\beta-1}}{\{(b-a)+\lambda(t-a)+\mu(b-t)\}^{\alpha+\beta}}=\frac{(1+\lambda)^{-\alpha}(1+\mu)^{-\beta}}{(a-b)}\mathrm{B}(\alpha,\beta)$

In terms of the new parametersλ and μ, the special case $\gamma=-\alpha-\beta,$ of (1.5) would yield (cf., e.g., [2, P.287, Entry 3,198])

(1.10) $$\int_a^b \frac{(t-a)^{\alpha-1}(b-t)^{\beta-1}}{\{b-a+\lambda(t-a)+\mu(b-t)\}^{\lambda+\mu}}\, dt = \frac{(1+\lambda)^{-\alpha}(1+\mu)^{-\beta}}{b-a}\ B(\alpha,\beta),$$

Making use of Raina and Srivastava [3] addressed the problem of closed-form evaluation of the following general Eulerian integral:

(1.11) $$I(x) := \int_a^b \frac{(t-a)^{\lambda}(b-t)^{\mu}}{\{f(t)\}^{\lambda+\mu+2}}\ H_{p,q}^{m,n}\left[z\{g(t)\}^{\upsilon}\ \middle|\ \begin{matrix}(a_j, A_j)_{1,p}\\ (b_j, B_j)_{1,q}\end{matrix}\right] dt\ ,$$

Where

(1.12) $$f(t) := b-a+\rho(t-a)+\sigma(\eta-t),$$

(1.13) $$g(t) := \frac{(t-a)^{\gamma}(b-t)^{\delta}\{f(t)\}^{1-\gamma-\delta}}{(b-a)\beta+(\beta\rho+\alpha-\beta)(t-a)+\beta\sigma(b-t)},$$

And $H_{p,q}^{m,n}[z\,|\,\dots]$ denote the familiar H-function of Fox [4, p.408], defined by (see also, [5, Chapter 2])

(1.14) $$H_{p,q}^{m,n}\left[z \;\middle|\; \begin{matrix} (a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,q} \end{matrix} \right] := \frac{1}{2\pi i}\int_{\varsigma} \theta(\zeta)\, z^{\varsigma}\, d\zeta,$$

$$\left(i := \sqrt{-1};\; z \in C \setminus \{0\};\quad z^{\varsigma} = \exp\{\zeta\,[\log|z| + i\arg(x)]\} \right)$$

Where log |z| represents the natural logarithm of |z| and arg (z) is not necessarily the principal value. Here, for convenience,

(1.15) $$\theta(\zeta) := \frac{\prod_{j=1}^{m}\Gamma(b_j - B_j\zeta)\prod_{j=1}^{n}\Gamma(1 - a_j + A_j\zeta)}{\prod_{j=m+1}^{q}\Gamma(1 + b_j + B_j\zeta)\prod_{j=n+1}^{p}\Gamma(a_j - A_j\zeta)};$$

an empty product is interpreted (as usual) as 1; the integers m, n, p, q satisfy the inequalities $0 \le n \le p$ and $1 \le m \le q;$

the coefficient $A_j > 0 \quad (j = 1, \dots\dots, p)$ and $B_j > 0 \quad (j = 1, \dots\dots, q)$

and the complex parameters $a_j\ (j = 1, \dots\dots, p)$ and $b_j\ (j = 1, \dots\dots, q)$ are so constrained that no poles of the integrated in (1.14) coincide, and L is a suitable contour of the Mellin-Barnes type (in the complex ζ- plane) which separates the poles of one product from those of the other. Furthermore, if we let

(1.16) $$\Omega := \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j - \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0,$$

then the integral in (1.14) converges absolutely and defines the H-function, analytic in the sector;

(1.17) $$|\arg(z)| < \frac{1}{2}\Omega\pi,$$

the point z=0 being tacitly excluded. In fact, according to Braaksma [6, p.278], the H-function makes sense and defines and analytic function of z also when either

(1.18) $$A := \sum_{j=1}^{p} A_j - \sum_{j=1}^{q} B_j < 0 \text{ and } 0 < |z| < \infty$$

or

(1.19) A=0 and $$0 < |z| < R := \prod_{j=1}^{p} A_j^{-A_j} \prod_{j=1}^{q} B_j^{B_j}$$

Recently, Saxena and Nishimoto [7] made use of the integral formula (1.5) in order to evaluate the following Eulerian integrals in terms of an H-function of two variables:

(1.20) $$:\Im_{\pm s\delta}(x)\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}$$
$$.\,H_{p,q}^{m,n}\left[z(ut+v)^{\pm\delta} \;\middle|\; \begin{matrix} (a_j, A_j)_{1,p} \\ (b_j, B_j)_{1,q} \end{matrix} \right] dt.$$

They also considered a number of interesting special cases of their integral formulas involving (1.20). In each case, however, their result was expressed in terms of an H-function of two variables. The present paper has stemmed essentially from our attempt to express the integrals in (1.20), and indeed also those that are contained in (1.20), in terms of special functions of similar or lesser complexity. Thus, in general, we aim at expressing an Eulerian integral of the type (1.20), involving an H-function of r variables, in terms of an H-function of r variables.

2. Eulerian integrals of the multivariable H-Functions

on using the formulla (1.3) in the most famous H-function of several variables we have

(2.1) $$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}.H[z_1(ut+v)^{-\rho_1},........,z_r(ut+v)^{\rho_r}]dt$$

$$=(b-a)^{\alpha+\beta-1}(au+v)^{\gamma}B(\alpha,\beta)\sum_{l=0}^{\infty}\frac{(\alpha)_l}{l!(\alpha+\beta)l}\left\{-\frac{(b-a)u}{au+v}\right\}.H^{0,n+1;m_2,n_1;......m_r,n_r}_{p+1,q+1;\,p_1q_1;.....;p_r,q_r}$$

$$\left[\begin{matrix} z_1(au+v)^{-\rho_1} \\ \vdots \\ z_r(au+v)^{-\rho_r}\end{matrix}\;\middle|\;\begin{matrix}(1+\gamma-l;\rho_1,......,\rho_r),\ (a_j;\alpha'_j,......\alpha_j^{(r)})_{1,p}: \\ (1+\gamma;\rho_1,......,\rho_r),\ (b_j;\beta'_j,......\beta_j^{(r)})_{1,q}:\end{matrix}\right.$$

$$\left.\begin{matrix}(c'_j,\gamma'_j)_{1,p_1};.....;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (c'_j,\delta'_j)_{1,q_1};.....;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix}\right]$$

provided (in addition to the appropriate convergence and existence conditions) that

$$\min\{\rho_1,......,\rho_r\}>0;\qquad \left|\frac{(b-a)u}{au+v}<1;\right|\qquad b\neq a;$$

and $\min\{\Re(\alpha),\Re(\beta)\}>0.$

Furthermore, if we employ the notation,

$$H^*[z_1,....,z_r]=H[z_1,....,z_r]|_{n=0},$$

we also obtain the following companion of the integral formula (1.1):

(2.2) $$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}.H^*[z_1(ut+v)^{-\rho_1},........,z_r(ut+v)^{\rho_r}]dt$$

$$=(b-a)^{\alpha+\beta-1}(au+v)^{\gamma}B(\alpha,\beta)\sum_{l=0}^{\infty}\frac{(\alpha)_l}{l!(\alpha+\beta)l}\left\{-\frac{(b-a)u}{au+v}\right\}.H^{0,1;n_1,m_1;......n_r,m_r}_{p+1,q+1;\,p_1,q_1;.....;p_r,q_r}$$

$$\left[\begin{matrix} z_1^{-1}(au+v)^{-\rho_1} \\ \vdots \\ z_r^{-1}(au+v)^{-\rho_r}\end{matrix}\;\middle|\;\begin{matrix}(1+\gamma-l;\rho_1,......,\rho_r),\ (1-b_j;\beta'_j,......\beta_j^{(r)})_{1,p}: \\ (1+\gamma;\rho_1,......,\rho_r),\ (1-a_j;\alpha'_j,......\alpha_j^{(r)})_{1,q}:\end{matrix}\right.$$

$$\left.\begin{matrix}(1-d'_j,\delta'_j)_{1,q_1};.....;(1-d_j^{(r)},\delta_j^{(r)})_{1,q_r} \\ (1-c'_j,\gamma'_j)_{1,p_1};.....;(1-c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\end{matrix}\right]$$

provided (in addition to the appropriate convergence and existence conditions) that

$$\min\{\rho_1,......,\rho_r\} > 0; \quad \left|\frac{(b-a)u}{au+v} < 1;\right| \quad b \neq a;$$

and $[\min\{\Re(\alpha),\ \Re(\beta)\} > 0.\quad]$

3. Derivation of the integral formula

For a simple and direct proof of the integral formula (2), we first replace the multivariable H-function occurring on the left-hand side by its Mellin-Barnes contour integral [5, p. 251, equation (C.1) J, collect the powers of(ut + v), and apply the binomial expansion (1.3) with, of course, g replaced by

$$\gamma - \sum_{k=1}^{r} \rho_k \xi_k ,$$

where $\xi_1 ,----,\xi_r$ denote the variables of the aforementioned Mellin-Barnes contour integral. We then make use of the Eulerian integral (1.2) and interpret the resulting Mellin-Barnes contour integral as an H -function of the r variables:

$$\frac{z_1}{(au+v)^{\rho_1}},.....,\frac{z_r}{(au+v)^{\rho_r}}$$

We are thus led finally to the integral formula (2).

The (sufficient) conditions of validity of the integral formula (3), which we stated already with (2), would follow by appealing to the principle of analytic continuation.

Our proof of the integral formula (2) is much akin to that of (3), which we have outlined above. Indeed, in the proof of (2.2), we apply the binomial expansion (1.3) with, replaced by

(3.1) $$\gamma - \sum_{k=1}^{r} \rho_k \xi_k ,$$

And then set

(3.2) $$\xi_{1=-}\zeta_{k.}, \quad (k=1,.....,r),$$

with a view to interpreting the resulting Mellin-Barnes contour integral as an H-function of the r variables

$$\frac{1}{z_1(au+v)^{\rho_1}},.....,\frac{1}{z_r(au+v)^{\rho_r}}$$

The details may be omitted.

Each of the integral formulas (2) and (3) can be put in a much more general setting. As a matter of fact, if we employ the binomial expansions (1.3) and

(3.3) $$(yt+z)^{\delta} = (by+z)^{\delta} \sum_{m=0}^{\infty} \frac{(-\delta)_m}{m!} \left\{\frac{(b-t)y}{by+z}\right\},$$

$$(|(b-t)y\ |<|by+z|; \quad t \in [a,b]),$$

Simultaneously, we shall similarly obtain the following (symmetrical) generalizations of the integral formulas (2) and (3):

(3.4) $\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\delta}$

$$.H[z_1(ut+v)^{-\rho_1}(yt+z)^{-\sigma_1},........,z_r(ut+v)^{\rho_r}(yt+z)^{-\sigma_1}]dt$$

$$=(b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\delta},\sum_{l,m=0}^{\infty}\frac{B(\alpha+l,\beta+m)}{l\,!m!}\left\{-\frac{(b-a)u}{au+v}\right\}^{l}\left\{-\frac{(b-a)y}{by+z}\right\}^{m}$$

$$.\,H^{0,n+2;m_1,n_1;......m_r,n_r}_{p+2,q+2;\,p_1,q_1;.....;p_r,q_r}\left[\begin{matrix} z_1(au+v)^{-\rho_1}(by+z)^{-\sigma_1} \\ \vdots \\ z_r(au+v)^{-\rho_r}(by+z)^{-\sigma_r}\end{matrix}\right|$$

$$\begin{matrix}(1+\gamma-l;\rho_1,......,\rho_r),\ (1+\delta-m;\sigma_1,....,\sigma_r),(a_j;\alpha_j',......\alpha_j^{(r)})_{1,p}: \\ (1+\gamma;\rho_1,......,\rho_r),\ (1+\delta;\sigma_1,....,\sigma_r),(b_j;\beta_j',......\beta_j^{(r)})_{1,q}:\end{matrix}$$

$$\left.\begin{matrix}(c_j',\gamma_j')_{1,p_1};.....;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (d_j',\delta_j')_{1,q_1};.....;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix}\right]$$

provided (in addition to the appropriate convergence and existence conditions) that

$$\min_{1\le k\le r}\{\rho_k,\sigma_k\}>0;\quad \max\left\{\left|\frac{(b-a)u}{au+v}\right|,\left|\frac{(b-a)y}{by+z}\right|\right\}<1;\quad b\neq a;$$

and $\min\{\Re(\alpha),\ \Re(\beta)\}>0;$

(3.5) $.H^{*}[z_1(ut+v)^{-\rho_1}(yt+z)^{-\sigma_1},........,z_r(ut+v)^{\rho_r}(yt+z)^{-\sigma_1}]dt$

$$=(b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\delta},\sum_{l,m=0}^{\infty}\frac{B(\alpha+l,\beta+m)}{l\,!m!}\left\{-\frac{(b-a)u}{au+v}\right\}^{l}\left\{-\frac{(b-a)y}{by+z}\right\}^{m}$$

$$.\,H^{0.2,n_1;m_1,n_1;......m_r,n_r}_{q+2,\,p+2;\,q_1,p_1;.....;q_r,p_r}\left[\begin{matrix} z_1^{-1}(au+v)^{-\rho_1}(by+z)^{-\sigma_1} \\ \vdots \\ z_r^{-1}(au+v)^{-\rho_r}(by+z)^{-\sigma_r}\end{matrix}\right|$$

$$\begin{matrix}(1+\gamma-l;\rho_1,......,\rho_r),\ (1+\delta-m;\sigma_1,....,\sigma_r),(1-b_j;\beta_j',......\beta_j^{(r)})_{1,q}: \\ (1+\gamma;\rho_1,......,\rho_r),\ (1+\delta;\sigma_1,....,\sigma_r),(1-a_j;\alpha_j',......\alpha_j^{(r)})_{1,p}:\end{matrix}$$

$$\left.\begin{matrix}(1-d_j',\delta_j')_{1,q_1};.....;(1-d_j^{(r)},\delta_j^{(r)})_{1,q_r} \\ (1-c_j',\gamma_j')_{1,q_1};.....;(1-c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\end{matrix}\right]$$

provided (in addition to the appropriate convergence and existence conditions) that

$$\min_{1\le k\le r}\{\rho_k,\sigma_k\}>0;\quad \max\left\{\left|\frac{(b-a)u}{au+v}\right|,\left|\frac{(b-a)y}{by+z}\right|\right\}<1;\quad b\neq a;$$

and

$$\min\{\Re(\alpha),\ \Re(\beta)\} > 0.$$

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