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EXPANSIVE TYPE FIXED POINT RESULTS IN G_b-METRIC SPACES

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ABSTRACT

Aghajani et. al. [2] introduced G_b-metric space and established common fixed point of generalized weak contractive mapping in partially ordered G_b-metric spaces. In the present paper, we prove some fixed point theorems for onto mappings satisfying various expansive type conditions in the setting of a generalized *b*-metric space. The presented theorems extend, generalize and improve many existing results in the literature.

Keywords: G_b-metric spaces, onto mapping, expansive mapping, and fixed point.

1. INTRODUCTION

The fixed point theorems in metric spaces are playing major role to construct methods in mathematics to solve problems in applied mathematics and sciences. So the attraction of metric spaces to a large numbers of mathematicians is understandable. Some generalizations of the notion of a metric space have been proposed by some authors.

In 1992, Dhage [5] introduced the concept of a D-metric space. Mustafa and Sims [22, 24] have shown that most of the results concerning Dhage's D-metric spaces are invalid. Therefore, they introduced an improved version of the generalized metric space structure, which they called G-metric spaces. Aghajani et. al. [2] introduced G_b-metric space and established common fixed point of generalized weak contractive mapping in partially ordered G_b-metric spaces. The study of expansive mappings is very interesting research area of fixed point theory. The study of expansive mappings is a very interesting research area in fixed point theory. In 1984, Wang et.al [19] introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces. In 1992, Daffer and Kaneko [8] defined an expanding condition for a pair of mappings and proved some common fixed point theorems for two mappings in complete metric spaces. Aage and Salunke [1] introduced several meaningful fixed point theorems for one expanding mapping.

Daheriya et al. [9] proved some fixed point theorems for Expansive Type Mapping in dislocated metric space.

In the present paper, we prove some fixed point theorems for self-mappings satisfying expansive condition in G_b-metric spaces. These results improve and generalized some important known results.

2. PRELIMINARIES

Following definitions and fundamental results are required for our further use.

Definition 2.1 [2] Let X be a non-empty set and $s \geq 1$ be a given real number. Suppose that a mapping$G: X \times X \times X \to R^+$ satisfies:

(GB1). $G(x, y, z) = 0$ if $x = y = z$,

(GB2). $0 < G(x, x, y), \forall\, x, y \in X$ with $x \neq y$,

(GB3). $G(x, x, y) \leq G(x, y, z), \forall\, x, y, z \in X$ with $y \neq z$,

(GB4). $G(x, y, z) = G\{p(x, y, z)\}$ (Symmetry),

(GB5). $G(x, y, z) \leq s(G(x, a, a) + G(a, y, z)), \forall\, x, y, z \in X$ (Rectangle inequality).

Then the pair (X, G) is called a generalized G_b-metric space or, more specifically, a G_b-metric space. Obverse that if $s = 1$ the ordinary rectangle inequality in a generalized metric space is satisfied; however, it does not hold true when $s > 1$. Thus the class of G_b-metric spaces are effectively larger than that of ordinary G- metric spaces. That is, every G-metric space is a G_b-metric space, but the converse need not be true. Therefore, it is obvious that G_b-metric spaces generalize G_b-metric spaces.

Example 2.2[2] Let (X, G) be a G-metric space, and $G_*(x, y, z) = G^p(x, y, z)$, where $p > 1$ is a real number. Note that G_* is a G_b- metric with $s = 2^{p-1}$. In [], it is prove that (X, G_*) is not necessarily a G-metric space

Example 2.3[2] Let $\mathrm{X} = \mathbb{R}$ and $d(x, y) = |x - y|^2$. We know that (X, d) is a b-metric space with $s = 2$. Let $\mathrm{G(x, y, z)} = d(x, y) + d(y, z) + d(z, x) \forall\, \mathrm{x, y, z} \in \mathrm{X}$, then (X, G) is not a G_b- metric space. If we define $\mathrm{G(x, y, z)} = \max\{d(x, y), d(y, z), d(z, x)\}\ \forall\, \mathrm{x, y, z} \in \mathrm{X}$.Then (X, G) is a G_b-metric space with $s = 2$.

Definition 2.4 [2] Let (X, G) be a G_b-metric space. A sequence $\{x_n\}_{n=1}^{\infty}$in X is said to be:

1) a G_b- Cauchy sequence if, for each $\epsilon > 0$ there exists $n_0 \in N$ such that for all $n, m, l > n_0$, $G(x_n, x_m, x_l) < \epsilon$.
2) a G_b- convergent sequence if, for each $\epsilon > 0$ there exists $n_0 \in N$ such that for all $n, m > n_0$, $G(x_n, x_m, x) < \epsilon$.for some fixed x in X. Here x is called G_b-limit of $\{x_n\}_{n=1}^{\infty}$ and is denoted by $G_b - \lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 2.5 [2] A G_b- metric space X is said to be G_b-complete metric space, if every G_b- Cauchy sequence in X is G_b- convergent in X.

Proposition 2.6[2] Let (X, G) be a G_b-metric space. Then the following are equivalent:

(1). $\{x_n\}_{n=1}^{\infty}$ is G_b-Cauchy in X,

(2). For every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $G(x_n, x_m, x_m) < \epsilon$.

Proposition 2.7 [2] Let (X, G) be a G_b- metric space. Then the function $G(x, y, z)$ is not jointly continuous in all three variables.

3. MAIN RESULT

We begin with following some lemmas.

Lemma 3.1 Let $(X, \mathrm{G}, \mathrm{s})$ be a G_b-metric space with the coefficient $s \geq 1$ and let $\{\mathrm{x_n}\}_{\mathrm{n}=1}^{\infty}$ be a sequence in X. If $\{\mathrm{x_n}\}_{\mathrm{n}=1}^{\infty}$ converges to x and also $\{\mathrm{x_n}\}_{\mathrm{n}=1}^{\infty}$ converges to y, then $x = y$. That is, the limit of $\{\mathrm{x_n}\}_{\mathrm{n}=1}^{\infty}$ is unique.

Proof: Since $x_n \to x$ and $x_n \to y$ as $n \to +\infty$, that is, $\lim_{n\to+\infty} G(x_n, x, x) = 0$ and $\lim_{n\to+\infty} G(x_n, y, y) = 0$. By using rectangle inequality, we have

$$G(x, y, y) \leq s[G(x, x_n, x_n) + G(x_n, y, y)]$$

By taking limit as $n \to +\infty$, we get $G(x, y, y) = 0$ and so $x = y$.

Lemma 3.2 Let (X, G, s) be a G_b-metric space with the coefficient $s \geq 1$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. If $\{x_n\}_{n=1}^{\infty}$ converges to x. Then

$$\frac{1}{s} G(x, y, y) \leq \lim_{n\to+\infty} G(x_n, y, y) \leq sG(x, y, y) \tag{3.1}$$

$\forall\, y \in X$.

Proof From rectangle inequality, we have

$$\begin{aligned} \frac{1}{s} G(x, y, y) - \lim_{n\to+\infty} G(x, x_n, x_n) & \\ \leq \lim_{n\to+\infty} G(x_n, y, y) & \\ = \lim_{n\to+\infty} G(y, y, x_n) & \\ \leq s(G(y, y, x) + \lim_{n\to+\infty} G(x, x, x_n)) & \end{aligned} \tag{3.2}$$

and so

$$\frac{1}{s} G(x, y, y) \leq \lim_{n\to+\infty} G(x_n, y, y) \leq sG(x, y, y)$$

$\forall\, y \in X$.

Lemma 3.3 Let (X, G, s) be a G_b-metric space with the coefficient $s \geq 1$ and let $\{x_k\}_{k=0}^{n} \subset X$. Then

$$\begin{aligned} G(x_0, x_n, x_n) \leq sG(x_0, x_1, x_1) + s^2 G(x_2, x_3, x_3) + \cdots + s^{n-1} G(x_{n-2}, x_{n-1}, x_{n-1}) \\ + s^{n-1} G(x_{n-1}, x_n, x_n) \end{aligned} \tag{3.3}$$

From Lemma 3.3, we deduce the following result.

Lemma 3.4 Let (X, G, s) be a G_b-metric metric space with the coefficient $s \geq 1$. Let $\{x_n\}_{n=1}^{\infty}$be a sequence of points of Xsuch that

$$G(x_n, x_{n+1}, x_{n+1}) \leq \lambda G(x_{n-1}, x_n, x_n) \tag{3.4}$$

where$\lambda \in \left[0, \frac{1}{s}\right)$ and $n = 1, 2, \ldots$. Then $\{x_n\}_{n=1}^{\infty}$ is a G_b-Cauchy sequence in (X, G, s).

Proof Let $m > n$. It follows that

$$\begin{aligned} G(x_n, x_m, x_m) &\leq s\{G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_m, x_m)\} \\ &\leq sG(x_n, x_{n+1}, x_{n+1}) + s^2\{G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_m, x_m)\} \\ &\leq sG(x_n, x_{n+1}, x_{n+1}) + s^2 G(x_{n+1}, x_{n+1}, x_{n+2}) + \cdots \\ &\quad + s^{m-n}\big(G(x_{m-2}, x_{m-1}, x_{m-1}) + G(x_{m-1}, x_m, x_m)\big) \\ &\leq sk^n G(x_0, x_1, x_1) + s^2 k^{n+1} G(x_0, x_1, x_1) + \cdots \\ &\quad + s^{m-n} k^{m-2} G(x_0, x_1, x_1) + s^{m-n} k^{m-1} G(x_0, x_1, x_1) \\ &= \{sk^n + s^2 k^{n+1} + \cdots + s^{m-n} k^{m-2} + s^{m-n} k^{m-1}\} G(x_0, x_1, x_1) \\ &= sk^n\{1 + (sk)^2 + \cdots \ldots \ldots ..\} G(x_0, x_1, x_1) \\ &\leq \frac{sk^n}{1 - sk} G(x_0, x_1, x_1) \end{aligned} \tag{3.5}$$

It is noted that $s\lambda < 1$. Assume that $G(x_0, x_1, x_1) > 0$. By taking limit as $m, n \to +\infty$ in above inequality we get

$$\lim_{n,m\to+\infty} G(x_n, x_m, x_m) = 0. \tag{3.6}$$

For $n, m, l \in \mathbb{N}$, (G_b) implies that

$$G(x_n, x_m, x_l) \leq s\big(G(x_n, x_m, x_m) + G(x_l, x_m, x_m)\big) \tag{3.7}$$

Taking limit as $n, m, l \to +\infty$, we get $G(x_n, x_m, x_l) \to 0$. So (x_n) is a G_b-Cauchy sequence. Also, if $G(x_0, x_1, x_1) = 0$, then $G(x_n, x_m, x_m) = 0$ for all $m > n$ and hence $\{x_n\}_{n=1}^{\infty}$ is a G_b-Cauchy sequence in X.

Now, our first main results as follows.

Theorem 3.5 Let (X, G) be a complete G_b-metric space with the coefficient $s \geq 1$. Assume that the mapping $T : X \to X$ is onto and satisfies the condition

$$G(Tx, Ty, Tz) \geq aG(x, y, z) + bG(x, x, Tx) + cG(y, y, Ty) + dG(z, z, Tz) \quad (3.8)$$

where a, b, c, d are non-negative constants with $a + sb + c + d > s$. Then T has a fixed point in X.

Proof: Let $x_0 \in X$ be arbitrary. Since T is onto, there is an element $x_1 \in X$ satisfying $x_1 \in T^{-1}(x_0)$. By the same way, we can find $x_n \in T^{-1}(x_{n-1})$ for $n = 2,3,4, \ldots$ If $x_{m-1} = x_m$ for some m, then $x_m \in T^{-1}(x_{m-1})$ implies $Tx_m = x_{m-1} = x_m$ and so x_m is a fixed point of T. Without loss of generality, we can suppose that $x_n \neq x_{n-1}$ for every n. From (3.8), we have

$$\begin{aligned} G(x_{n-1}, x_n, x_n) &= G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\ &\geq aG(x_n, x_{n+1}, x_{n+1}) + bG(x_n, x_n, Tx_n) + cG(x_{n+1}, x_{n+1}, Tx_{n+1}) \\ &\quad + dG(x_{n+1}, x_{n+1}, Tx_{n+1}) \\ &= aG(x_n, x_{n+1}, x_{n+1}) + bG(x_n, x_n, x_{n-1}) + cG(x_{n+1}, x_{n+1}, x_n) \\ &\quad + dG(x_{n+1}, x_{n+1}, x_n) \end{aligned}$$

So, it must be the case that

$$(1 - b)G(x_{n-1}, x_n, x_n) \geq (a + c + d)G(x_n, x_{n+1}, x_{n+1}) \quad (3.9)$$

If $a + c + d = 0$, then $b \leq 1$, which is contradiction, since $a + sb + c + d > s$. Hence $a + c + d \neq 0$ and from

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{1-b}{a+c+d} G(x_{n-1}, x_n, x_n) \quad (3.10)$$

where $0 < \frac{1-b}{a+c+d} < \frac{1}{s}$.

Let $k = \frac{1-b}{a+c+d}$. Then $0 < k < 1$ and

$$G(x_n, x_{n+1}, x_{n+1}) \leq kG(x_{n-1}, x_n, x_n) \quad (3.11)$$

By Lemma 3.4, $\{x_n\}_{n=1}^{\infty}$ is a G_b-Cauchy sequence. By completeness of (X, G), there exists $x^* \in X$ such that $x_n \to x^*$. Now T is onto mapping. So there exists a point $p \in X$ such that $p \in T^{-1}(x^*)$ and so $x^* = Tp$. Consider from (3.8), we have

$$\begin{aligned} G(x_n, x^*, x^*) &= G(Tx_{n+1}, Tp, Tp) \\ &\geq aG(x_{n+1}, p, p) + bG(x_{n+1}, x_{n+1}, Tx_{n+1}) + cG(p, p, Tp) \\ &\quad + dG(p, p, Tp) \\ &\geq aG(x_{n+1}, p, p) + bG(x_{n+1}, x_{n+1}, x_n) + cG(p, p, x^*) \\ &\quad + dG(p, p, x^*) \end{aligned} \quad (3.12)$$

Taking the limit as $n \to +\infty$, we have

$$0 \geq aG(x^*, p, p) + bG(x^*, x^*, x^*) + cG(p, p, x^*) + dG(p, p, x^*)$$

So,

$$0 \geq (a + c + d)G(p, p, x^*). \quad (3.13)$$

which implies that $G(p, p, x^*) = 0$, since $a + c + d \neq 0$. Therefore $p = x^*$ and hence $Tx^* = x^*$.

Theorem 3.6 Let (X, G) be a complete G_b-metric space with the coefficient $s \geq 1$, and let $T : X \to X$ be onto G_b- continuous mapping satisfying the condition

$$G\big(T(x), T^2(x), T^2(x)\big) \geq aG\big(x, T(x), T(x)\big) \quad (3.14)$$

for all $x \in X$, where $a > s$.Then T has a fixed point in X.

Proof: Let $x_0 \in X$ be arbitrary. Since T is onto, there is an element $x_1 \in X$ satisfying $x_1 \in T^{-1}(x_0)$. By the same way, we can find $x_n \in T^{-1}(x_{n-1})$ for $n = 2,3,4, \ldots$ If $x_{m-1} = x_m$ for some m, then $x_m \in T^{-1}(x_{m-1})$ implies $Tx_m = x_{m-1} = x_m$ and so x_m is a fixed point of T. Without loss of generality, we can suppose that $x_n \neq x_{n-1}$ for every n. From (3.14), we have

$$G\big(T(x_{n+1}), T^2(x_{n+1}), T^2(x_{n+1})\big) \geq aG\big(x_{n+1}, T(x_{n+1}), T(x_{n+1})\big)$$

So,

$$G(x_n, x_{n-1}, x_{n-1}) \geq aG(x_{n+1}, x_n, x_n)$$

this implies that

$$G(x_{n+1}, x_n, x_n) \leq kG(x_n, x_{n-1}, x_{n-1}) \tag{3.15}$$

where $k = \frac{1}{a} < \frac{1}{s}$. By repeated application of (3.15), we have

$$G(x_{n+1}, x_n, x_n) \leq k^n G(x_1, x_0, x_0) \tag{3.16}$$

Then for all $n, m \in \mathbb{N}$; $n < m$, we have by repeated use of the rectangle inequality and (3.16) that

$$\begin{aligned} G(x_n, x_m, x_m) &\leq s\{G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_m, x_m)\} \\ &\leq sG(x_n, x_{n+1}, x_{n+1}) + s^2\{G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_m, x_m)\} \\ &\leq sG(x_n, x_{n+1}, x_{n+1}) + s^2 G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots \\ &\quad + s^{m-n}\big(G(x_{m-2}, x_{m-1}, x_{m-1}) + G(x_{m-1}, x_m, x_m)\big) \\ &\leq 2s^2 G(x_n, x_n, x_{n+1}) + 2s^3 G(x_{n+1}, x_{n+1}, x_{n+2}) + \cdots \\ &\quad + 2s^{m-n+1}\big(G(x_{m-2}, x_{m-2}, x_{m-1}) + G(x_{m-1}, x_{m-1}, x_m)\big) \\ &\leq 2s^2 k^n G(x_1, x_0, x_0) + 2s^3 k^{n+1} G(x_1, x_0, x_0) + \cdots \\ &\quad + 2s^{m-n+1} k^{m-2} G(x_1, x_0, x_0) + 2s^{m-n+1} k^{m-1} G(x_1, x_0, x_0) \\ &= 2s\{sk^n + s^2 k^{n+1} + \cdots + s^{m-n} k^{m-2} + s^{m-n} k^{m-1}\} G(x_1, x_0, x_0) \\ &= 2s^2 k^n \{1 + (sk)^2 + \cdots \ldots \ldots ..\} G(x_1, x_0, x_0) \\ &\leq \frac{2s^2 k^n}{1 - sk} G(x_1, x_0, x_0) \end{aligned} \tag{3.17}$$

Then $lim\, G(x_n, x_m, x_m) = 0$, as $n, m \to \infty$, since $lim \frac{2s^2 k^n}{1-sk} G(x_1, x_0, x_0) = 0$, as $n, m \to \infty$. For $n, m, l \in \mathbb{N}$, (G_b) implies that

$$G(x_n, x_m, x_l) \leq s\big(G(x_n, x_m, x_m) + G(x_l, x_m, x_m)\big)$$

Taking limit as $n, m, l \to \infty$, we get $G(x_n, x_m, x_l) \to 0$. So (x_n) is a G_b-Cauchy sequence. By completeness of (X, G), there exists $x^* \in X$ such that $x_n \to x^*$. By the G_b-continuity of T, we have

$$T(x_n) = x_{n-1} \to T(x^*)$$

this implies that $T(x^*) = x^*$.

As an application of Theorem 3.6, we have the following results.

Theorem 3.7 Let (X, G) be a complete G_b-metric space with the coefficient $s \geq 1$, and let $T : X \to X$ be onto G_b- continuous mapping satisfying the condition

$$\begin{aligned} &G\big(T(x), T(y), T(z)\big) \\ &\quad \geq a \min\big\{G(x, y, z), G\big(x, T(x), T(x)\big), G\big(y, T(y), T(y)\big), G\big(y, T(y), T(y)\big)\big\} \end{aligned} \tag{3.18}$$

for all $x \in X$, where $a > s$.Then T has a fixed point in X.

Proof: Replacing y and z by $T(x)$ in (3.18), we obtain

$$G\big(T(x), T^2(x), T^2(x)\big) \geq a \min\big\{G\big(x, T(x), T(x)\big), G\big(T(x), T^2(x), T^2(x)\big)\big\} \tag{3.19}$$

Without loss of generality, we may assume that $T(x) \neq T^2(x)$. For, otherwise, T has a fixed point. Then $T(x) \neq T^2(x)$ and condition (3.19) imply that

$$G\big(T(x), T^2(x), T^2(x)\big) \geq aG\big(x, T(x), T(x)\big)$$

which is Condition (3.14). Hence the result follows from Theorem 3.6.

Theorem 3.8 Let (X, G) be a complete G_b-metric space with the coefficient $s \geq 1$, and let $S, T : X \to X$ be onto G_b- continuous. If there exists a with

$$\begin{aligned} &min\, \big\{G\big(S(x), T(y), T(y)\big), G\big(T(y), S(x), S(x)\big)\big\} \\ &\qquad \geq a\{G(S(x), x, x) + G(\,T(y), y, y)\} \end{aligned} \tag{3.20}$$

for all $x \in X$, where $(1 + s)a > s$.Then T has a fixed point in X.

Proof: Let $x_0 \in X$ be arbitrary. Since S is onto, there is an element $x_1 \in X$ satisfying $x_1 \in S^{-1}(x_0)$. Since T is also onto, there is an element $x_2 \in X$ satisfying $x_2 \in T^{-1}(x_1)$. Proceeding in the same way, we can find $x_{2n+1} \in S^{-1}(x_{2n})$ and $x_{2n+2} \in T^{-1}(x_{2n+1})$ for $n = 1,2,3,4, \ldots.$ Therefore $Sx_{2n+1} = x_{2n}$ and $Tx_{2n+2} = x_{2n+1}$. Now, if $n = 2m$, from (3.20), we have

$$G(x_{n-1}, x_n, x_n) = G(x_{2m-1}, x_{2m}, x_{2m})$$
$$= G\big(T(x_{2m}), S(x_{2m+1}), S(x_{2m+1})\big)$$
$$= min\,\{G\big(S(x_{2m+1}), T(x_{2m}), T(x_{2m})\big), G\big(T(x_{2m}), S(x_{2m+1}), S(x_{2m+1})\big)\}$$
$$\geq a\,\{G(S(x_{2m+1}), x_{2m+1}, x_{2m+1}) + G(T(x_{2m}), x_{2m}, x_{2m})\}$$
$$= a\,\{G(x_{2m}, x_{2m+1}, x_{2m+1}) + G(x_{2m-1}, x_{2m}, x_{2m})\}$$
$$= a\,\{G(x_n, x_{n+1}, x_{n+1}) + G(x_{n-1}, x_n, x_n)\} \quad (3.21)$$

Therefore,

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{1-a}{a} G(x_{n-1}, x_n, x_n) \quad (3.22)$$

If $n = 2m + 1$, then by the same argument used in above, we obtain

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{1-a}{a} G(x_{n-1}, x_n, x_n) \quad (3.23)$$

Thus for any positive integer n,

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{1-a}{a} G(x_{n-1}, x_n, x_n) \quad (3.24)$$

Let $k = \frac{1-a}{a} < \frac{1}{s}$. Hence

$$G(x_n, x_{n+1}, x_{n+1}) \leq kG(x_{n-1}, x_n, x_n) \quad (3.25)$$

By Lemma 3.4, $\{\mathrm{x_n}\}_{\mathrm{n=1}}^{\infty}$ is a G_b-Cauchy sequence. By completeness of (X, G), there exists $x^* \in X$ such that $x_n \to x^*$. By the G_b-continuity of S and T, we have

$$Sx_{2n+1} = x_{2n} \to S(x^*),$$
$$Tx_{2n+2} = x_{2n+1} \to T(x^*) \quad (3.26)$$

as $n \to \infty$. This implies that $S(x^*) = x^*$ and $T(x^*) = x^*$, which means that x^* is a common fixed point of S and T.

Now, motivated by the work in [13], we give the following.

Let $\Psi_{\mathcal{B}}^{L}$ denote the class of those function $\mathcal{B}: (0, \infty) \to (L^2, \infty)$ which satisfy the condition $\mathcal{B}(t_n) \to (L^2)^+ \Rightarrow t_n \to 0$, where $L > 0$.

Theorem 3.9 Let (X, G, s) be a complete G_b-metric space. Assume that the mapping $T: X \to X$ is surjection and satisfies

$$G(Tx, Ty, Tz) \geq \mathcal{B}\big(G(x, y, z)\big)G(x, y, z) \quad (3.26)$$

$\forall x, y, z \in X$, where $\mathcal{B} \in \Psi_{\mathcal{B}}^{\mathrm{S}}$. Then T has a fixed point.

Proof Let $x_0 \in X$. Since T is surjection, choose $x_1 \in X$ such that $Tx_1 = x_0$. Inductively, we can define a sequence $\{x_n\}_{n=1}^{\infty} \in X$ such that

$$x_n = Tx_{n+1},\ \forall\, n \in \mathbb{N} \cup \{0\}. \quad (3.27)$$

In case $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then it is clear that x_{n_0} is a fixed point of T. Now assume that $x_n \neq x_{n-1}$ for all n.Consider

$$G(x_{n-1}, x_n, x_n) = G(Tx_n, Tx_{n+1}, Tx_{n+1}) \quad (3.28)$$

Now by (3.26) and definition of the sequence

$$G(x_{n-1}, x_n, x_n) = G(Tx_n, Tx_{n+1}, Tx_{n+1})$$
$$\geq \mathcal{B}\big(G(x_n, x_{n+1}, x_{n+1})\big)G(x_n, x_{n+1}, x_{n+1})$$
$$\geq s^2 G(x_n, x_{n+1}, x_{n+1})$$
$$\geq G(x_n, x_{n+1}, x_{n+1}) \quad (3.29)$$

Thus the sequence $\{G(x_n, x_{n+1}, x_{n+1})\}_{n=1}^{\infty}$ is a decreasing sequence in $\mathbb{R}^+$ and so there exists $r \geq 0$ such that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = r \quad (3.31)$$

Let us prove that $r = 0$. Suppose to the contrary that $r > 0$. By (3.26) we can deduce that

$$s^2 \frac{G(x_{n-1}, x_n, x_n)}{G(x_n, x_{n+1}, x_{n+1})} \geq \frac{G(x_{n-1}, x_n, x_n)}{G(x_n, x_{n+1}, x_{n+1})}$$

$$\geq \mathcal{B}\big(G(x_n, x_{n+1}, x_{n+1})\big) \geq s^2 \tag{3.32}$$

By taking limit as $n \to +\infty$ in the above inequality, we have

$$\lim_{n\to+\infty} \mathcal{B}\big(G(x_n, x_{n+1}, x_{n+1})\big) = s^2 \tag{3.33}$$

Hence by definition of $\mathcal{B}$, we have

$$r = \lim_{n\to+\infty} G(x_n, x_{n+1}, x_{n+1}) = 0 \tag{3.34}$$

which is a contradiction. That is $r = 0$. Now, we shall show that

$$\lim_{n,m\to+\infty} \sup G(x_n, x_m, x_m) = 0 \tag{3.35}$$

Suppose to the contrary that $\lim_{n,m\to\infty} \sup G(x_n, x_m, x_m) > 0$.

By (3.26), we have

$$G(x_n, x_m, x_m) = G(Tx_{n+1}, Tx_{m+1}, Tx_{m+1})$$
$$\geq \mathcal{B}\big(G(x_{n+1}, x_{m+1}, x_{m+1})\big)G(x_{n+1}, x_{m+1}, x_{m+1})$$

That is,

$$\frac{G(x_n,x_m,x_m)}{\mathcal{B}\big(G(x_{n+1},x_{m+1},x_{m+1})\big)} \geq G(x_{n+1}, x_{m+1}, x_{m+1}) \tag{3.36}$$

By triangular inequality, we have

$$G(x_n, x_m, x_m) \leq sG(x_n, x_{n+1}, x_{n+1}) + s^2G(x_{n+1}, x_{m+1}, x_{m+1})$$
$$+s^2G(x_{m+1}, x_m, x_m)$$
$$\leq sG(x_n, x_{n+1}, x_{n+1}) + s^2 \frac{G(x_n,x_m,x_m)}{\mathcal{B}\big(G(x_{n+1},x_{m+1},x_{m+1})\big)}$$
$$+s^2G(x_{m+1}, x_m, x_m) \tag{3.37}$$

Therefore,

$$G(x_n, x_m, x_m) \leq \left(1 - \frac{s^2}{\mathcal{B}\big(G(x_{n+1},x_{m+1},x_{m+1})\big)}\right)^{-1}$$
$$\big(sG(x_n, x_{n+1}, x_{n+1}) + s^2G(x_{m+1}, x_m, x_m)\big) \tag{3.38}$$

By taking limit as $n, m \to +\infty$ in the above inequality, since $\lim_{n,m\to+\infty} \sup G(x_n, x_m, x_m) > 0$ and $r = 0 = \lim_{n\to+\infty} G(x_n, x_{n+1}, x_{n+1})$, then we obtain

$$\lim_{n,m\to+\infty} \left(1 - \frac{s^2}{\mathcal{B}\big(G(x_{n+1},x_{m+1},x_{m+1})\big)}\right)^{-1} = +\infty \tag{3.39}$$

which implies that

$$\lim_{m,n\to+\infty} \sup \mathcal{B}\big(G(x_{n+1}, x_{m+1}, x_{m+1})\big) = (s^2)^+ \tag{3.40}$$

and so by definition of $\mathcal{B}$, we have

$$\lim_{m,n\to+\infty} \sup \mathcal{B}\big(G(x_{n+1}, x_{m+1}, x_{m+1})\big) = 0 \tag{3.41}$$

which is a contradiction. Hence,

$$\lim_{m,n\to+\infty} \sup \mathcal{B}\big(G(x_n, x_m, x_m)\big) = 0 \tag{3.42}$$

Since $\lim_{m,n\to+\infty} \sup G(x_n, x_m, x_m) = 0$. So, $\{x_n\}_{n=1}^{\infty}$ is a G_b-Cauchy sequence. Since X is a complete G_b-metric space, the sequence $\{x_n\}_{n=1}^{\infty}$ in X G_b-converges to $x^\star \in X$. so that

$$\lim_{n\to+\infty} G(x_n, x^\star, x^\star) = 0 \tag{3.43}$$

As T is surjective, so there exists $p \in X$ such that $x^\star = Tp$. Let us prove that $x^\star = p$. Suppose to the contrary that $x^\star \neq p$.Then by (3.26), we have

$$G(x_n, x^\star, x^\star) = G(Tx_{n+1}, Tp, Tp)$$
$$\geq \mathcal{B}\left(G(x_{n+1}, p, p)\right)G(x_{n+1}, p, p) \tag{3.44}$$

By Taking limit as $n \to +\infty$ in the above inequality and applying Lemma 3.2, we obtain

$$0 = \lim_{n\to+\infty} G(x_n, x^\star, x^\star)$$
$$\geq \lim_{n\to+\infty} \mathcal{B}\left(G(x_{n+1}, p, p)\right) \lim_{n\to\infty} G(x_{n+1}, p, p)$$
$$\geq \frac{1}{s} \lim_{n\to+\infty} \mathcal{B}\left(G(x_{n+1}, x^\star, x^\star)\right) G(x^\star, p, p) \tag{3.45}$$

and hence,

$$\lim_{n\to+\infty}\mathcal{B}\left(G(x_{n+1},x^\star,x^\star)\right)=0 \quad (3.46)$$

which is a contradiction. Indeed,

$$\lim_{n\to+\infty}\mathcal{B}\big(G(x_{n+1},x_n,x_n)\big)\geq s^2.$$

Since $\mathcal{B}(t) > s^2$ for all $t \in [0, \infty)$, therefore $x^\star = p$. Hence $x^\star = Tp = Tx^\star$.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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