Vol.4.Issue.2.2016 (April-June)

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com
Email:editorbomsr@gmail.com

ISSN:2348-0580

RESEARCH ARTICLE

GRAPH OF EQUIVALENCE CLASSES OF A COMMUTATIVE IS-ALGEBRA

SAMY M. MOSTAFA[1], FATEMA F.KAREEM[2]

[1]Department of Mathematics, Faculty of Education, Ain Shams University, Roxy, Cairo, Egypt.Email:samymostafa@yahoo.com

[2]Department of Mathematics, Ibn-Al-Haitham college of Education, University of Baghdad, Iraq.Email:fa_sa20072000@yahoo.com

ABSTRACT

In this paper, we introduce the graph of a commutative IS-algebra X, denoted by $\Gamma(X)$, as the (undirected) graph with all elements of X. Moreover, we study the graph $\Gamma_E(X)$ of equivalence classes of X which is determined by annihilator ideals. Also, several examples are presented.

Key Words. IS- algebra, Annihilator ideal, graph of equivalence classes.

2000 Mathematics Subject Classification. 13A15; 13A99; 05C12.

1. INTRODUCTION

Imai and Is´eki [3] in 1966 introduced the notion of a BCK-algebra. In the same year, Is´eki [4] introduced BCI-algebras as a super class of the class of BCK-algebras.

In 1993, Jun et al. [7] introduced a new class of algebras related to BCI-algebras and semigroups, called a BCI- semigroup /BCI-monoid/BCI-group. In 1998, for the convenience of study, Jun et al. [8] renamed the BCI-semigroup (respectively, BCI-monoid and BCI-group) as the IS-algebra (respectively, IM-algebra and IG-algebra) and studied further properties of these algebras (see [6], [10]). Many authors studied the graph theory in connection with semigroups and rings. For example, Beck [1] associated to any commutative ring R its zero divisors graph G(R), whose vertices are the zero divisors of R, with two vertices a, b jointed by an edge in case a . b = 0. Jun and Lee [5] defined the notion of zero divisors and quasi-ideals in BCI-algebra and show that all zero divisors are quasi-ideal. So, they introduced the concept of associated graph of BCK/BCI- algebra and verified some properties of this graph and proved that if X is a BCK-algebra, then the associated graph of X is connected. Moreover, if X is a BCI-algebra, then the associated graph of it is disconnected. Zahiri and R. A. Borzooei [12] associate a new graph to a BCI-algebra X which is denoted by G(X), this definition is based on branches of X. If X is a BCK-algebras, then this definition and last definition which was introduced by Jun and Lee are the same. S.Mulay [9] introduced the graph of equivalence classes of zero-divisors of a ring R, which is constructed from classes of zero divisors determined by annihilator ideals. Inspired by ideas from Mulay, we study the graph of equivalence classes of a commutative IS-algebra which is constructed from classes determined by annihilator ideals.

2. Preliminaries

In this section, we submit some concepts related to IS-algebra (BCI-semi-groups) and theories from the literature, which are necessary for our discussion.

Definition 2.1 [3]**.** Let X be a set with a binary operation $*$ and a constant 0, then $(X,*,0)$ is called a BCI -algebra, if it satisfies the following axioms. For all $x, y, z \in X$.

(BCI-1) ((x *y) *(x *z)) *(z *y) = 0,

(BCI-2) (x *(x *y)) *y = 0, (BCI-3) x *x = 0,

(BCI-4) x * y = 0 and y *x = 0 imply x = y,

If a BCI -algebra X satisfies the identity 0 * x = 0, for all $x \in X$, then X is called a BCK algebra.

It is known that the class of BCK-algebra is a proper subclass of the class of BCI-algebra.

A binary relation ≤ in X is defined by: x ≤ y if and only if x * y = 0.

In a BCI-algebra $(X,*,0)$, the following properties are satisfied:

(BCI1') (x * z) * (y * z) ≤ x * y,

(BCI2') [x * (x* y)] ≤ y,

(BCI3') x ≤ y implies z * x ≤ z * y,

(BCI4') x ≤ y and y ≤ z imply x ≤ z,

(BCI5') (x * y) * z = (x * z) * y,

(BCI6') x ≤ y implies x * z ≤ y * z,

(BCI7') x * 0 = x.

Definition2.2 [3].

A subset A of a BCI-algebra $(X,*,0)$is called an ideal of X, if for any $x, y \in X$, the following conditions hold:

(i) 0 ∈ A,

(ii) x * y and y ∈ A imply that x ∈ A.

Definition2.3 [8]. An IS-algebra is a nonempty set X with two binary operations $*,\bullet$ and constant 0 such that following axioms are satisfied:

1. $(X,*,\bullet)$ is a BCI-algebra,
2. $(X,\bullet)$ is a semigroup,

The operation $\bullet$ is distributive (on both sides) over the operation$*$, i.e.

3. $x\bullet(y*z)=(x\bullet y)*(x\bullet z) \text{ and } (x*y)\bullet z=(x\bullet z)*(y\bullet z)$, for all $x, y, z \in X$.

Definition2.4 [6]. A non empty subset I of X is called a left (resp. right) I-ideal of X if:

(I_1) I is an ideal of a BCI-algebra X,

(I_2) $x \in X$, $a \in I$ *imply that* $x\bullet a \in I$ (*resp.* $a\bullet x \in I$).

Lemma2.5 [10]. Let X be an IS-algebra. Then for any $x, y, z \in X$, we have:

(*i*) $0\bullet x = x\bullet 0 = 0$

(ii) $x \le y$ *implies that* $x\bullet z \le y\bullet z$ *and* $z\bullet x \le z\bullet y$.

Definition2.6. an IS-algebra X is said to be a commutative IS-algebra if the multiplication is commutative i.e., $x\bullet y = y\bullet x$ *for all x, y in X*.

Example 2.7. Let $X=\{0, a, b, c\}$ be a set. Define $*$ -operation and $\bullet$ -operation by the following tables.

$*$	0	a	b	c
0	0	0	b	b
a	a	0	c	b
b	b	b	0	0
c	c	b	a	0

$*$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then, $(X,*,\bullet,0)$ is a commutative IS-algebra

Definition2.8. Let X be a commutative IS-algebra and A be a subset of X. Then we define $ann(A)=\{x\in X;\ a*(a*(a\bullet x))=0 \ for\ each\ a\in A\}$ and call it the IS-annihilator of A.

If $A=\{a\}$, then we write $ann(a)$ instead of $ann(\{a\})$.

Remark 2.9. Let X be a commutative IS-algebra and

$ann(A)=\{x\in X;a*(a*(a\bullet x))=0 \ for\ each\ a\in A\}$ be an annihilator of $A\subseteq X$.

(i) If x is a zero divisor of X, then $b\bullet x\in ann(A)$ for all $b\in A$.

(ii) If $a\bullet x=0$ for all $x\in X$, then $ann(A)=0$ for all $a\in A$.

Proof. (i) Since x is a zero divisor, we have $a*(a*(a\bullet(b\bullet x))=a*(a*(a\bullet(0))=a*a=0$. Hence, $b\bullet x\in ann(A)$.

(ii) Since $a\bullet x=0$ for all $x\in X$, we have $a*(a*(a\bullet x))=a*(a*0))=a*a=0$.

Hence, $ann(A)=0$.

Theorem 2.10. Let A be a non-empty subset of an IS-algebra X. If x is a zero divisor of X, then $ann(A)$ is an ideal of X.

Proof. For every $a\in A$, since $a*(a*(a\bullet 0))=a*a=0$, we have $0\in ann(A)$.

First, we prove that $ann(A)$ is an ideal of a BCI-algebra, we suppose that $x,(y*x)\in ann(A)$. We obtain from definition that $a*(a*(a\bullet x))=0........(i)$, and which implies that $a*(a*(a\bullet x))\le a\bullet x$.

Also $a*(a*(a\bullet(y*x))=0.................(ii)$

It follows from (BCI 2') and (i) that $0*a\bullet x=0$, and hence

$(a*a\bullet x)*a=(a*a)*a\bullet x=0*a\bullet x=0$. This means that $a=a*a\bullet x$ by (BCI-4).

Similarly we have $a=a*a\bullet(y*x)$ .From these, we have in turn

$a=a*(a\bullet(y*x)=(a*a\bullet x)*a\bullet(y*x)=(a*a\bullet x)*(a\bullet y*a\bullet x)\le a*a\bullet y$ by (BCI 1')

$0=a*a\le(a*a\bullet y)*a=0*a\bullet y$ (By the property $x\le y\Rightarrow x*z\le y*z$)

$0=0*(0*a\bullet y)\le a\bullet y$, which implies that $0*a\bullet y=0$.

It follows that $(a*a\bullet y)*a=(a*a)*a\bullet y=0*a\bullet y=0$ and hence $a=a*a\bullet y$. Thus $y\in ann(A)\ and\ ann(A)$ is an ideal of X. In the second place, if x is a zero divisor, then for all $a,b\in A, x\in X$, we have, $b\bullet x\in ann(A)$ by Remark 2.7 .

Then $ann(A)$ $is\ an\ ideal$ of an IS-algebra X.

Lemma2.11. If $A\ne\phi, B\subseteq X$, then

(I) If $A\subseteq B, then\ ann(B)\subseteq ann(A)$

(II) $ann(A\bigcup B)=ann(A)\bigcap ann(B)$

(III) $ann(A)\bigcup ann(B)\subseteq ann(A\bigcap B)$

Proof. (I) Suppose that

$x \in ann(B)$, *then* $b*(b*(b \bullet x) = 0, \forall\, b \in B$, *but* $A \subseteq B$, *therfore* $b*(b*(b \bullet x) = 0, \forall\, b \in A$ *i.e* $x \in ann(A)$, hence $ann(B) \subseteq ann(A)$.

(II) Since $A \subseteq A \bigcup B$ and $B \subseteq A \bigcup B$, we have by part (I) of Lemma 2.9 that,

$ann(A \bigcup B) \subseteq ann(A), ann(B)$, and hence $ann\,(A \bigcup B) \subseteq ann\;(A) \bigcap ann\,(B) \;\; --(1)$

Conversely, if $x \in ann(A) \bigcap ann(B)$, then $x \in ann(A), ann(B)$, therefore

$a*(a*(a \bullet x)) = 0, \forall a \in A$ and $b*(b*(b \bullet x)) = 0, \forall b \in B$. But if $c \in \left(A \bigcup B\right)$, then

$c*(c*(c \bullet x) = 0 \forall c \in (A \bigcap B)$ *we have* $x \in ann\,(A \bigcup B)$, hence

$ann\;(A) \bigcap ann\,(B) \subseteq ann\,(A \bigcup B)$ --- (2)

From (1) and (2), we have $ann(A \bigcup B) = ann(A) \bigcap ann(B)$.

(III): we have $A \supset A \bigcap B$, $B \supset A \bigcap B$,from(I) $ann(A) \subset ann(A \bigcap B)$ and $ann(B) \subset ann(A \bigcap B)$ which implies that

$$ann(A) \bigcup ann(B) \subseteq ann(A \bigcap B) \quad .$$

Lemma 2.12. If A is a nonempty subset of an IS -algebra X , then

$$ann(A) = \bigcap_{a \in A} ann(a)$$

Proof. Since $A = \bigcup_{a \in A} \{a\}$, we have $ann(A) = ann\{\bigcup_{a \in A} \{a\}\} = \bigcap_{a \in A} ann(a)$.

Definition2.13. Define a relation ~ on X as follows:

$x \sim y$ *if and only if* $ann\,(x) = ann\,(y)\,, \forall x, y \in X$

Lemma2.14. the relation ~ (from Definition2.13) is an equivalence relation on X .

Proof. The reflexivity, symmetry, and transitivity follow very easily from Definition 2.13 showing ~ is an equivalence relation.

3- *A graph of IS-algebra*.

In this section, we introduce the concepts of graph of commutative IS-algebra X and the graph of equivalence classes of X . For a graph G , we denote the set of vertices of G as $V(G)$ and the set of edges as $E(G)$. A graph G is said to be complete if every two distinct vertices are joined by exactly one edge. A graph G is said to be bipartite graph if its vertex set $V(G)$ can be partitioned into disjoint subsets V_1 and V_2 such that, every edge of G joins a vertex of V_1 with a vertex of V_2 . So, G is called a complete bipartite graph if every vertex in one of the bipartition subset is joined to every vertex in the other bipartition subset. Also, a graph G is said to be connected if there is a path between any given pairs of vertices, otherwise the graph is disconnected. For distinct vertices x and y of G, let d(x, y) be the length of the shortest path from x to y and if there is no such path we define d(x, y) = ∞. The diameter of G is $diam(G) = \sup\{d(x, y):$ x and y are distinct vertices of G}. The diameter is 0 if the graph consists of a single vertex and a connected graph with more than one vertex has diameter 1 if and only if it is complete; i.e., each pair of distinct vertices forms an edge. The neighborhood of a vertex x is the set $N(x) = \{y \in V(G) : x - y\}$. In commutative IS-algebra X , it is easy to see that $N(x) = ann(x)$ for all $x \neq 0$. A graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Two graphs G_1 and G_2 are said to be isomorphic if there exists a bijective mapping $f : V(G_1) \to V(G_2)$ such that $x - y \in E(G_1)$ then $f(x) - f(y) \in E(G_2)$. For more details we refer to [11 and 12].

Definition3.1. For an IS-algebra X , the graph of a commutative IS-algebra X , denoted by $\Gamma(X)$ is a graph whose vertices are elements of X and two distinct vertices are adjacent in $\Gamma(X)$ if $x*(x*(x\bullet y)=0$.

Example 3.2. Let $Z_6 = \left\{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4},\bar{5}\right\}$. Define Θ-operation and $\circ$-operation by the following tables:

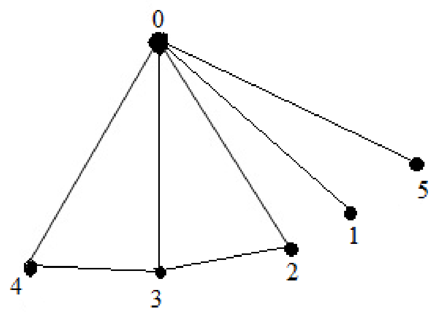
Θ	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{0}$	$\bar{0}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$
$\bar{1}$	$\bar{1}$	$\bar{0}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$
$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{0}$	$\bar{5}$	$\bar{4}$	$\bar{3}$
$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{0}$	$\bar{5}$	$\bar{4}$
$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{0}$	$\bar{5}$
$\bar{5}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{0}$

$\circ$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{4}$	$\bar{0}$	$\bar{2}$	$\bar{4}$
$\bar{3}$	$\bar{0}$	$\bar{3}$	$\bar{0}$	$\bar{3}$	$\bar{0}$	$\bar{3}$
$\bar{4}$	$\bar{0}$	$\bar{4}$	$\bar{2}$	$\bar{0}$	$\bar{4}$	$\bar{2}$
$\bar{5}$	$\bar{0}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$

Then, $(Z_6,\Theta,\circ,0)$ is a commutative IS-algebra. Now we determine the graph of Z_6 as follows.

The set of vertices are $V(\Gamma(Z_6)=\{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4},\bar{5}\}$ and the set of edges are

$E(\Gamma(Z_6)=\{\bar{0}-\bar{1},\bar{0}-\bar{2},\bar{0}-3,\bar{0}-\bar{4},\bar{0}-\bar{5},\bar{2}-\bar{3},\bar{3}-\bar{4}\}$, hence Figure (1) shows the graph of Z_6.



$\Gamma(Z_6)$ Fig.1

Definition3.3. For a commutative IS-algebra X , the graph of equivalence classes of X , denoted by $\Gamma_E(X)$ is a graph whose vertices are the set of equivalence classes $V(\Gamma_E)=\left\{\bar{x};x\sim y,\forall x,y\in X\right\}$ and two distinct vertices $\bar{x},\bar{y}$ are adjacent in $\Gamma_E(X)$ if and only if $x\bullet y=0$.

Example 3.4. Let $X=\{0,a,b,c\}$ be a set. Define $*$-operation and $\bullet$-operation by the following tables.

∗	0	*a*	*b*	*c*
0	0	0	*c*	*b*
a	*a*	0	*c*	*b*
b	*b*	*b*	0	*c*
c	*c*	*c*	*b*	0

∗	0	*a*	*b*	*c*
0	0	0	0	0
a	0	0	0	0
b	0	0	*b*	*c*
c	0	0	*c*	*b*

Then, $(X,*,\bullet,0)$ is a commutative IS-algebra. Now, we determine the graph of X as follows: The set of vertices is $V(X)=\{0,a,b,c,\}$, and the set of edges is $E(X)=\{\{0,a\},\{0,b\},\{0,c\},\{a,b\},\{a,c\}\}$, and the set of vertices of $\Gamma_E(X)$ is $\{[0],[a],[b]\}$ since $ann(0)=X$, $ann(a)=\{0,b,c\}$, $ann(b)=ann(c)=\{0,a\}$, then $E(\Gamma_E(X)=\{\{[0],[a]\},\{[0],[b]\},\{[a],[b]\}\}$. The Figure (2) shows the graph $\Gamma(X)$ and the graph of equivalence classes $\Gamma_E(X)$.

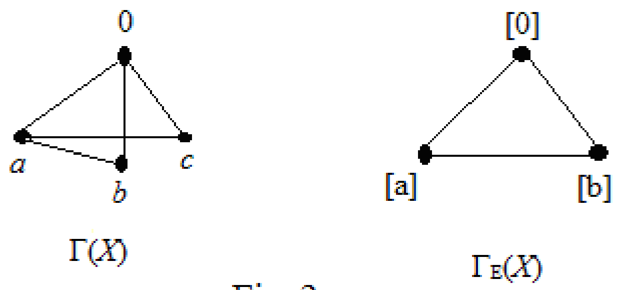


Fig. 2

Theorem3.5. Let X be a commutative IS-algebra. Then $\Gamma_E(X)$ is connected and $diam(\Gamma_E(X))\le 3$.

Proof: Let $\bar{x},\bar{y}\in V(\Gamma_E(X))$ be distinct. We have the following two cases:

Case1: If $x\bullet y=0$. Then $\bar{x},\bar{y}$ are adjacent in $\Gamma_E(X)$ and $d(\bar{x},\bar{y})=1$.

Case2: If $x\bullet y\neq 0$. Then we have the following sub cases:

Sub case1: $x\bullet x=y\bullet y=0$. If $x\bullet y=x$, then $x\bullet y=x\bullet y=(x\bullet y)\bullet y=x\bullet(y\bullet y)=x\bullet 0=0$, which is a contradiction. Thus $x\bullet y\neq x$ and $\bar{x}\bullet\bar{y}\neq\bar{x}$ similarly, $x\bullet y\neq y$. Therefore, $\bar{x}-\bar{x}\bullet\bar{y}-\bar{y}$ is a path of length 2, and so $d(\bar{x},\bar{y})=2$.

Sub case2: $x\bullet x=0$ and $y\bullet y\neq 0$. Then there is $\bar{b}\in V(\Gamma_E(X))\setminus\{\bar{x},\bar{y}\}$ with $b\bullet y=0$. If $b\bullet x=0$, then $\bar{x}-\bar{b}-\bar{y}$ is a path of length 2. If $b\bullet x\neq 0$, then $\bar{x}-\bar{b}\bullet\bar{x}-\bar{y}$ is a path of length 2, in either case $d(\bar{x},\bar{y})=2$.

Sub case3: $y\bullet y=0$ and $x\bullet x\neq 0$. The proof is similar to sub case 2.

Sub case4: $x\bullet x\neq 0$ and $y\bullet y\neq 0$. Then there exist $\bar{a},\bar{b}\in V(\Gamma_E(X))\setminus\{\bar{x},\bar{y}\}$ with $a\bullet x=b\bullet y=0$. If $\bar{a}=\bar{b}$, then $\bar{x}-\bar{a}-\bar{y}$ is a path of length 2. Thus we may assume that $\bar{a}\neq\bar{b}$, if $a\bullet b=0$, then $\bar{x}-\bar{a}-\bar{b}-\bar{y}$ is a path of length 3, and hence $d(\bar{x},\bar{y})\le 3$. If $a\bullet b\neq 0$, then $\bar{x}-\bar{a}\bullet\bar{b}-\bar{y}$ is a path of length 2 so $d(\bar{x},\bar{y})=2$. Hence in all the cases $d(\bar{x},\bar{y})\le 3$, therefore $diam(\Gamma_E(X))\le 3$ and there is a path between every two distinct vertices in $\Gamma_E(X)$, thus it is connected.

Theorem3.6. Let X be a commutative IS-algebra. If $\Gamma_E(X)$ contains a cycle, then $gr\Gamma_E(X)\le 4$.

Proof. Assume that $\Gamma_E(X)$ contains a cycle of length which is greater than four, then $\Gamma_E(X)$ contains a cycle $\bar{x}_0-\bar{x}_1-...-\bar{x}_n-\bar{x}_0$ with $n\ge 4$, then we have two cases.

Case1: If $x_1\bullet x_{n-1}=0$, then we can form the cycle $\bar{x}_0-\bar{x}_1-\bar{x}_{n-1}-\bar{x}_n-\bar{x}_0$ of length 4.

Case2: If $x\bullet x_{n-1}\neq 0$, then we have three sub cases.

Sub case1: If $x_1 \bullet x_{n-1} \neq x_0$ and $x_1 \bullet x_{n-1} \neq x_n$, then $\bar{x}_1 \bullet \bar{x}_{n-1} \neq \bar{x}_0$ and $\bar{x}_1 \bullet \bar{x}_{n-1} \neq \bar{x}_n$ thus, we can form the cycle $\bar{x}_0 - \bar{x}_1 \bullet \bar{x}_{n-1} - \bar{x}_n - \bar{x}_0$ of length 3.

Sub case2: If $x_1 \bullet x_{n-1} = x_0$, then $x_1 \bullet x_{n-1} \bullet x_{n-2} = 0$, so we can form the cycle $\bar{x}_0 - \bar{x}_{n-2} - \bar{x}_{n-1} - \bar{x}_n - \bar{x}_0$ of length 4.

***Lemma* 3.7.** If X is a commutative IS-algebra, then

1) $\Gamma_E(X)$ is a sub graph of $\Gamma(X)$.

2) $ann(0) = 0$, *for all* $x \in X$.

3) If $\Gamma(X)$ is the complete graph then $\Gamma(X) \cong \Gamma_E(X)$.

4) If $\Gamma(X)$ is the complete bipartite graph, then $\Gamma_E(X)$ is an edge.

Proof. (1) and (2) straightforward. (3) Suppose that $V(\Gamma(X)) = \{x_1, x_2, ..., x_n\}$. Since $\Gamma(X)$ is the complete graph, then every pair of its vertices are adjacent. Thus

$ann(x_1) = \{x_2, x_3, ..., x_i\}, i = 2, ..., n$, $ann(x_2) = \{x_1, x_3, ..., x_i\}, i = 1,3, ..., n$ ...

$ann(x_n) = \{x_1, x_2, ..., x_{i-1}\}, i = 1, ..., n$. Then $ann(x_1) \neq ann(x_2) \neq ... \neq ann(x_n)$, therefore every vertex of $\Gamma(X)$ is a equivalence class of $\Gamma_E(X)$, thus the vertices of $\Gamma_E(X)$ are distinct and the same number of vertices of $\Gamma(X)$, then there exist an isomorphism $f : \Gamma(X) \to \Gamma_E(X)$ satisfies $f(x_i) = \bar{x}_i$ for each $i \in \{1,2, ..., n\}$. And the mapping of edges $f : E(\Gamma(X)) \to E(\Gamma_E(X))$, which sends the edge $x_i - x_j$ in $\Gamma(X)$ to the edge $\bar{x}_i - \bar{x}_j$ in $\Gamma_E(X)$ is a well-defined bijection. Thus $\Gamma(X) \cong \Gamma_E(X)$.

(4) Suppose that $\Gamma(X)$ is the complete bipartite graph with vertex set $V(\Gamma(X)) = \{x_1, x_2, ..., x_{r_1}, x_{r_1+1}, ..., x_r\}$. This set can be split into two sets $A = \{x_1, x_2, ..., x_{r_1}\}$ and $B = \{x_{r_1+1}, ..., x_r\}$ such that each vertex of A is joined to each vertex of B by exactly one edge. Thus

$E(\Gamma(X)) = \{x_1 - x_{r_1+1}, x_1 - x_{r_2+1}, ..., x_1 - x_r, x_2 - x_{r_1+1}, ..., x_2 - x_r, ..., x_{r_1} - x_{r_1+1}, x_{r_1} - x_{r_2+1}, ..., x_{r_1} - x_r\}$

, so $ann(x_1) = ann(x_2) = ..., ann(x_{r_1}) = B$ and $ann(x_{r_1+1}) = ann(x_{r_2+2}) = ..., ann(x_r) = A$

Then there are two distinct equivalence classes $\bar{x}_1$ and $\bar{x}_{r_1+1}$ in $\Gamma_E(X)$, which are adjacent. Thus $\Gamma_E(X)$ is an edge.

Theorem3.8. Let X be a commutative IS-algebra.

(a) If $diam(\Gamma(X)) = 0$, then $diam(\Gamma_E(X)) = 0$

(b) If $diam(\Gamma(X)) = 1$, then $diam(\Gamma_E(X)) = 0$ or 1

(c) If $diam(\Gamma(X)) = 2$, then $diam(\Gamma_E(X)) = 0,1$ or 2

(d) If $diam(\Gamma(X)) = 3$, then $diam(\Gamma_E(X)) = 0,1,2$ or 3

Proof: (a) Let $diam(\Gamma(X)) = 0$ i.e. there exist $x \in \Gamma(X)$, which is one vertex. Since $ann(x) = ann(x)$, then $\bar{x} \in \Gamma_E(X)$. Thus $\Gamma_E(X)$ has also one vertex and so $diam(\Gamma_E(X)) = 0$.

(b) if $diam(\Gamma(X)) = 1$, then $\Gamma(X)$ is complete graph with more than one vertex. Thus there exist two vertices $x, y \in \Gamma(X)$, such that $x - y$ is a path of length 1 connecting x and y. Now, either $ann(x) = ann(y)$, then $\Gamma_E(X)$ has one vertex. Thus $diam(\Gamma_E(X)) = 0$ or $ann(x) \neq ann(y)$ and $x \bullet y = 0$, by Definition3.3, then there exist an edge connecting x and y so $diam(\Gamma_E(X)) = 1$.

(c) If $diam(\Gamma(X)) = 2$, then there exist three vertices $x, y, z \in \Gamma(X)$, such that $x - y - z$ is a path of length 2 . Now, if $ann(x) = ann(y) = ann(z)$ then there exist one equivalent class contains these points, thus $\Gamma_E(X)$ has one vertex and so $diam(\Gamma_E(X)) = 0$. If $ann(x) \neq ann(y)$ and $ann(x) = ann(z)$ then $\Gamma_E(X)$ have two vertices $\bar{x}$ and $\bar{y}$ such that $\bar{x} - \bar{y}$ is a path of length 1, thus $diam(\Gamma_E(X)) = 1$. If $ann(x) \neq ann(y) \neq ann(z)$ then $\Gamma_E(X)$ have three vertices $\bar{x}$, $\bar{y}$ and $\bar{z}$ such that $\bar{x} - \bar{y} - \bar{z}$ is a path of length 2, thus $diam(\Gamma_E(X)) = 2$.

(d) If $diam(\Gamma(X)) = 3$, then there exist four vertices $x, y, z, l \in \Gamma(X)$, such that $x - y - z - l$ is a path of length 3. Now, if $ann(x) = ann(y) = ann(z) = ann(l)$ then $\Gamma_E(X)$ has one vertex. Thus $diam(\Gamma_E(X)) = 0$. If $ann(x) = ann(y), ann(z) = ann(l)$ then $\Gamma_E(X)$ have two vertices $\bar{x}$ and $\bar{z}$ such that $\bar{x} - \bar{z}$ is a path of length 1, thus $diam(\Gamma_E(X)) = 1$. If $ann(x) \neq ann(y)$ and $ann(z) = ann(l)$, then $\Gamma_E(X)$ have three vertices $\bar{x}$, $\bar{y}$ and $\bar{z}$ such that $\bar{x} - \bar{y} - \bar{z}$ is a path of length 2, thus $diam(\Gamma_E(X)) = 2$. Finally, if $ann(x) \neq ann(y) \neq ann(z) \neq ann(l)$, then $\Gamma_E(X)$ have four vertices $\bar{x}$, $\bar{y}$, $\bar{z}$ and $\bar{l}$ such that $\bar{x} - \bar{y} - \bar{z} - \bar{l}$ is a path of length 3, thus $diam(\Gamma_E(X)) = 3$.

Theorem3.9. Let X and Y be two commutative IS-algebras. If $\Gamma(X) \cong \Gamma(Y)$, then $\Gamma_E(X) \cong \Gamma_E(Y)$.

Proof: clear.

The converse of this theorem is false as illustrated in Example 3.8. We have that $\Gamma_E(Z_6) \cong \Gamma_E(Z_{10})$ but $\Gamma(Z_6) \not\cong \Gamma(Z_{10})$.

Example 3.10. Figure (2) displays the zero divisor graphs and the equivalence class graphs of Z_6 and Z_{10}.

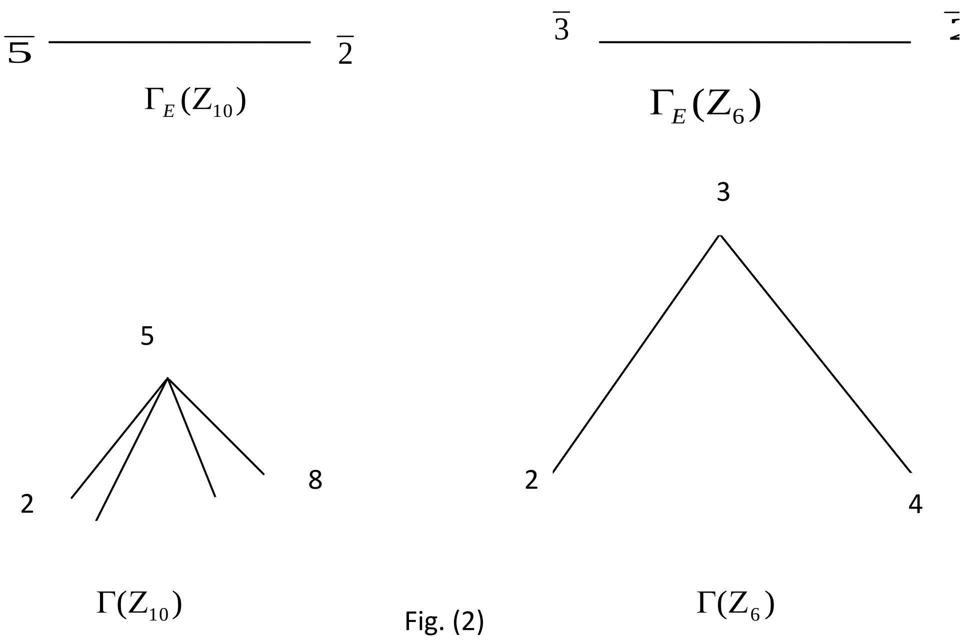


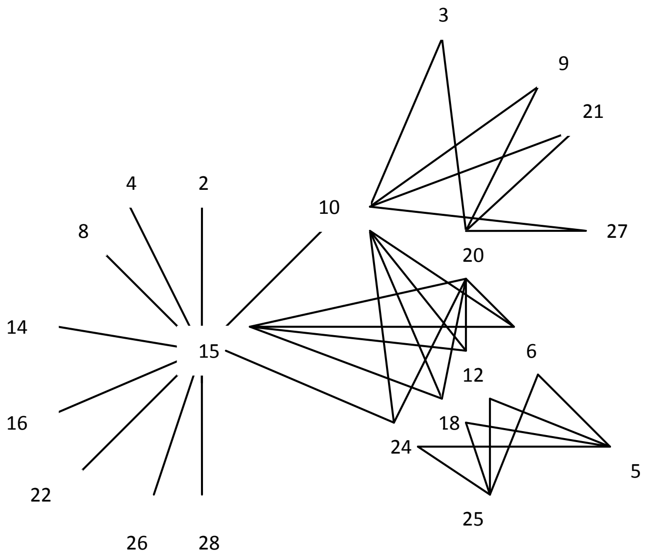
Fig. (2)

***Theorem*3.11.** Let $\Gamma_E(X)$ be the associated graph of equivalence classes of IS algebra X . For any distinct vertices $\bar{x}, \bar{y} \in \Gamma_E(X)$, if $\bar{x}$ *and* $\bar{y}$ are connected by an edge, then $ann(x) \neq ann(y)$.

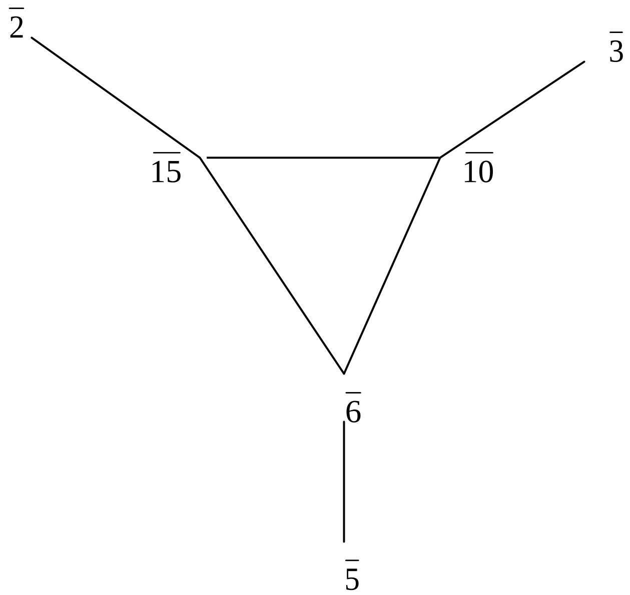
Proof: Suppose that $ann(x) = ann(y)$, then $x \sim y$ and hence $\bar{x} = \bar{y}$, which is a contradiction. Therefore, $ann(x)$ *and* $ann(y)$ are distinct annihilator ideal of X .

The following example shows that the converse of Theorem 3.11 may not be true.

Example 3.12. Let $(Z_{30}, \Theta, \circ, 0)$ be an IS-algebra, Figure (3) shows the difference between the zerodivisor graph and the equivalence class graph.



$\Gamma(Z_{30})$



$\Gamma_E(Z_{30})$

Fig.(3)

In $\Gamma_E(Z_{30})$, the vertices $\bar{3}$ *and* $\bar{5}$ are distinct annihilators, but no edge joint between them.

Acknowledgement

The authors are thankful to the referees for a careful reading of the paper and for valuable comments and suggestions.

References

[1]. I. Beck, Coloring of commutative rings, J. Algebra, 116 (1988) 208-226.

[2]. Y.Imai and K.Iseki, On axiom system of propositional calculi, XIV, Japan Acad. (1966) 19-22

[3]. K.Iseki, algebra related with a propositional calculus, Proc.Japan.Acad.42 (1966) 351-366.

[4]. Y.B. Jun and K. J.Lee, graphs based on BCK\BCI-algebra, Int. J. Math. Math. Sci. 1 (2011) Article ID616981.

[5]. Y.B.Jun, Roh E.H. and X.L.Xin, I-ideals generated by a set in IS-algebras, Bull Kcrean Math.Soc., 35 (1998) 615-624.

[6]. Y.B.Jun, S.M.Hong, BCI-semigroups, Honam math. (1993).

[7]. Y.B.Jun, X.I.Xin, and E.H.Roh, Aclass of algebras related to BCI-algebras and semi groups Soochow J. Math., 4 (24) (1998) 309-321.MR200b.06056.

[8]. S. M. Mostafa, A. E. Radwan, F. A. Ibrahem and F. F.Kareem, The graph of a commutative KU-algebra, Algebra Letters 1 (2015) 1-18.

[9]. S.B.Mulay, cycles and symmetries of zerodivisors, Comm. Algebra, 7(30)(2002) 3533-3558.

[10]. E.H.Roh, Y.B.Jun, and W.H.shim, some ideals in IS algebras, Sc. Math.3(2) (1999) 315-320 electronic CMP 1718 275.

[11]. D. Weber, zero-divisor graphs and lattices of finite commutative rings, Rose-Hulman undergraduate math. J. 1(12) Spring (2011).

[12]. R.J.Wilson, Introduction to Graph Theory. Oliver and Boyed, Edinburgh (1972).

[13]. O. Zahiri and R. A. Borzooei, Graph of BCI-algebras, Int. J. Math. Math. Sci. 1(2012) Article ID126835.