Vol.4.Issue.3.2016 (July-Sept.,)

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com

Email:editorbomsr@gmail.com

ISSN:2348-0580

RESEARCH ARTICLE

OPTIMAL CONVEX COMBINATION BOUNDS OF SQUARE-ROOT AND QUADRATIC MEANS FOR NEUMAN-SÁNDOR MEAN

MA HONGYAN[1], LIU CHUNRONG[2]

[1,2]College of Mathematics and Information Science, Hebei University, Baoding, 071002, P. R. China

Ma Hongyan

ABSTRACT

In this paper, we present the least value α and the greatest value β such that the double inequality

$\alpha N(a,b)+(1-\alpha)Q(a,b)<M(a,b)<\beta N(a,b)+(1-\beta)Q(a,b)$

holds for all $a,b>0$ with $a\neq b$. Here $N(a,b)$, $Q(a,b)$ and $M(a,b)$ denote the square-root, quadratic and Neuman-Sándor means of two positive numbers a and b, respectively.

Keywords: Inequality, Neuman-Sándor mean, square-root mean, quadratic mean

1. INTRODUCTION

For $a,b>0$ with $a\neq b$, the Neuman-Sándor mean $M(a,b)$ [1] was defined by

$$M(a,b)=\frac{a-b}{2\sinh^{-1}[(a-b)/(a+b)]}, \tag{1.1}$$

where $\sinh^{-1}x=\log(x+\sqrt{1+x^2})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the subject of intensive research. In particular, many remarkable inequalities for $M(a,b)$ can be found in the literature [1, 2]. Let $H(a,b)=(2ab)/(a+b)$, $G(a,b)=\sqrt{ab}$, $L(a,b)=(a-b)/(\log a-\log b)$, $N(a,b)=\left[(\sqrt{a}+\sqrt{b})/2\right]^2$, $P(a,b)=(a-b)\big/(4\tan^{-1}\sqrt{a/b}-\pi)$, $A(a,b)=(a+b)/2$, $T(a,b)=(a-b)\big/[2\tan^{-1}(a-b)/(a+b)]$, $Q(a,b)=\sqrt{(a^2+b^2)/2}$ and $C(a,b)=(a^2+b^2)\big/(a+b)$ be the harmonic, geometric, logarithmic, square-root, first Seiffert, arithmetic, second Seiffert, quadratic and contra-harmonic mean of a and b, respectively. Then

$$\min\{a,b\} < H(a,b) < G(a,b) < L(a,b) < N(a,b) < P(a,b) \\ < A(a,b) < M(a,b) < T(a,b) < Q(a,b) < C(a,b) < \max(a,b) \quad (1.2)$$

hold for all $a,b > 0$ with $a \neq b$.

In [3], Neuman proved that the double inequalities

$$aQ(a,b) + (1-a)A(a,b) < M(a,b) < bQ(a,b) + (1-b)A(a,b) \quad (1.3)$$

and

$$l\,C(a,b) + (1-l)A(a,b) < M(a,b) < mC(a,b) + (1-m)A(a,b) \quad (1.4)$$

hold for all $a,b > 0$ with $a \neq b$ if and only if

$a \le [1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})] = 0.3249\ldots$, b

$\ge 1/3$, $l \le [1-\log(1+\sqrt{2})]/\log(1+\sqrt{2})$ and $m \ge 1/6$.

In [4], Li etc showed that the double inequality

$$L_{p_0}(a,b) < M(a,b) < L_2(a,b) \quad (1.5)$$

holds for all $a,b > 0$ with $a \neq b$, where

$L_p(a,b) = [(a^{p+1} - b^{p+1})/(p+1)(a-b)]^{1/p}$ $(p \neq -1, 0)$, $L_0(a,b)$

$= 1/e[(a^a)/b^b]^{1/(a-b)}$ and $L_{-1}(a,b) = (a-b)/(\log a - \log b)$ is the p-th generalized logarithmic mean of a and b, and $p_0 = 1.843\ldots$ is the unique solution of the equation $(p+1)^{1/p} = \log(1+\sqrt{2})$.

In [5], Chu etc proved that the double inequalities

$$a_1 L(a,b) + (1-a_1)Q(a,b) < M(a,b) < b_1 L(a,b) + (1-b_1)Q(a,b) \quad (1.6)$$

and

$$a_2 L(a,b) + (1-a_2)C(a,b) < M(a,b) < b_2 L(a,b) + (1-b_2)C(a,b) \quad (1.7)$$

hold for all $a,b > 0$ with $a \neq b$ if and only if

$a_1 \ge 2/5$, $b_1 \le 1 - 1/[\sqrt{2}\log(1+\sqrt{2})] = 0.1977\ldots$, $a_2 \ge 5/8$

and $b_2 \le 1 - 1/[2\log(1+\sqrt{2})] = 0.4327\ldots$.

In [6], Liu etc attested that the double inequality

$$aG(a,b) + (1-a)Q(a,b) < M(a,b) < bG(a,b) + (1-b)Q(a,b) \quad (1.8)$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $a \ge 1/3$ and $b \le 1 - 1/[\sqrt{2}\log(1+\sqrt{2})] = 0.1977\ldots$

The main purpose of this paper is to found the least value a and the greatest value b such that the double inequality

$$aN(a,b) + (1-a)Q(a,b) < M(a,b) < bN(a,b) + (1-b)Q(a,b)$$

holds for all $a,b > 0$ with $a \neq b$.

2. MAIN RESULTS

THEOREM 2.1. The double inequality

$$aN(a,b) + (1-a)Q(a,b) < M(a,b) < bN(a,b) + (1-b)Q(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha \geq 4/9$ and

$$\beta \leq \left[2(4+\sqrt{2})\right]/7\cdot\left[1-1/\left(\sqrt{2}\log(1+\sqrt{2})\right)\right]$$
$$=0.3058\cdots.$$

Proof. Without loss of generality, we assume that $a>b>0$. Let $x=(a-b)/(a+b)\in(0,1)$, $\lambda=\left[2(4+\sqrt{2})\right]/7\cdot\left[1-1/\left(\sqrt{2}\log(1+\sqrt{2})\right)\right]$ and $p\in\{4/9,\lambda\}$. Then

$$\frac{N(a,b)}{A(a,b)}=\frac{1}{2}(1+\sqrt{1-x^2}),\quad \frac{M(a,b)}{A(a,b)}=\frac{x}{\sinh^{-1}x},\quad \frac{Q(a,b)}{A(a,b)}=\sqrt{1+x^2}. \tag{2.1}$$

and

$$\frac{pN(a,b)+(1-p)Q(a,b)-M(a,b)}{A(a,b)}=\frac{p(1+\sqrt{1-x^2})+2(1-p)\sqrt{1+x^2}}{2\log(x+\sqrt{1+x^2})}D_p(x), \tag{2.2}$$

where

$$D_p(x)=\log(x+\sqrt{1+x^2})-\frac{2x}{p(1+\sqrt{1-x^2})+2(1-p)\sqrt{1+x^2}}. \tag{2.3}$$

Equation (2.3) leads to

$$\lim_{x\to 0^+}D_p(x)=0, \tag{2.4}$$

$$\lim_{x\to 1^-}D_p(x)=\log(1+\sqrt{2})-\frac{2}{p+2(1-p)\sqrt{2}}, \tag{2.5}$$

and

$$D_p'(x)=\frac{1}{\left[p(1+\sqrt{1-x^2})+2(1-p)\sqrt{1+x^2}\right]^2}F_p(x), \tag{2.6}$$

where

$$F_p(x)=\frac{(3p^2-8p+4)x^2+2p(3p-2)}{\sqrt{1+x^2}}+\frac{2p^2(1-x^2)}{\sqrt{1-x^4}}$$
$$+\frac{4p(p-1)x^2+2p(1-2p)}{\sqrt{1-x^2}}+2p(1-2p). \tag{2.7}$$

Let $x=\sqrt{t}$, $t\in(0,1)$, then

$$F_p(x)=\frac{(3p^2-8p+4)t+2p(3p-2)}{\sqrt{1+t}}+\frac{2p^2(1-t)}{\sqrt{1-t^2}}$$
$$+\frac{4p(p-1)t+2p(1-2p)}{\sqrt{1-t}}+2p(1-2p) \tag{2.8}$$
$$=G_p(t).$$

Now we distinguish between two cases:

Case 1. $p=4/9$. (2.8) yields

$$G_{4/9}(t)=\frac{4(7t-4)}{27\sqrt{1+t}}+\frac{32}{81(1+t)}\sqrt{1-t^2}+\frac{8(1-10t)}{81\sqrt{1-t}}+\frac{8}{81}. \quad (2.9)$$

(2.9) is rewritten into

$$G_{4/9}(t)=-\frac{16(1-t)}{27\sqrt{1+t}}+\frac{4t}{9\sqrt{1+t}}+\frac{32}{81(1+t)}\sqrt{1-t^2}+\frac{8\sqrt{1-t}}{81}-\frac{8t}{9\sqrt{1-t}}+\frac{8}{81}. \quad (2.10)$$

Because the Taylor series of the following functions $\frac{1}{\sqrt{1+t}},\frac{1}{\sqrt{1-t}},\sqrt{1-t},\sqrt{1-t^2}$ and $\sqrt{1+t}$ are

$$\frac{1}{\sqrt{1+t}}=1+\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}t^n,\ t\in(-1,1), \quad (2.11)$$

$$\frac{1}{\sqrt{1-t}}=1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}t^n,\ t\in(-1,1), \quad (2.12)$$

$$\sqrt{1-t}=1-\frac{1}{2}t-\sum_{n=2}^{\infty}\frac{(2n-3)!!}{(2n)!!}t^n,\ t\in(-1,1), \quad (2.13)$$

$$\sqrt{1-t^2}=1-\frac{1}{2}t^2-\sum_{n=2}^{\infty}\frac{(2n-3)!!}{(2n)!!}t^{2n},\ t\in(-1,1), \quad (2.14)$$

and

$$\sqrt{1+t}=1+\frac{1}{2}t+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(2n-3)!!}{(2n)!!}t^n,\ t\in(-1,1), \quad (2.15)$$

respectively, inequalities

$$1-\frac{1}{2}t+\frac{3}{8}t^2-\frac{5}{16}t^3<\frac{1}{\sqrt{1+t}}<1-\frac{1}{2}t+\frac{3}{8}t^2, \quad (2.16)$$

$$\frac{1}{\sqrt{1-t}}>1+\frac{1}{2}t+\frac{3}{8}t^2, \quad (2.17)$$

$$\sqrt{1-t}<1-\frac{1}{2}-\frac{1}{8}t^2, \quad (2.18)$$

$$\sqrt{1-t^2}<1-\frac{1}{2}t^2-\frac{1}{8}t^4, \quad (2.19)$$

and

$$\sqrt{1+t}>1+\frac{1}{2}t-\frac{1}{8}t^2 \quad (2.20)$$

hold for all $t\in(0,1)$. Making use of the inequalities (2.16)- (2.19) for (2.10) cause the conclusion that

$$\begin{aligned} G_{4/9}(t) &< -\frac{16(1-t)}{27}\left(1-\frac{1}{2}t+\frac{3}{8}t^2-\frac{5}{16}t^3\right)+\frac{4t}{9}\left(1-\frac{1}{2}t+\frac{3}{8}t^2\right)+\frac{32}{81(1+t)}\Big(1 \\ &\quad -\frac{1}{2}t^2-\frac{1}{8}t^4\Big)+\frac{8}{81}\left(1-\frac{1}{2}-\frac{1}{8}t^2\right)-\frac{8t}{9}\left(1+\frac{1}{2}t+\frac{3}{8}t^2\right)+\frac{8}{81} \\ &= -\frac{t^2}{162(1+t)}\left[30t^3+(1+t)(156-t)+6\right] \\ &< 0 \end{aligned} \tag{2.21}$$

for $t \in (0,1)$. From (2.6), (2.8) and (2.21) we clearly see that $D_{4/9}(x)$ is strictly decreasing in $x \in (0,1)$. It follows from (2.2) and (2.4) together with the monotonicity of $D_{4/9}(x)$ that

$$\frac{4}{9}N(a,b)+\frac{5}{9}Q(a,b) < M(a,b). \tag{2.22}$$

Case 2. $p=\lambda=\left[2(4+\sqrt{2})\right]\Big/7\cdot\left[1-1\Big/\left(\sqrt{2}\log(1+\sqrt{2})\right)\right]$. (2.8) leads to

$$\begin{aligned} G_\lambda(t) &= \frac{(3\lambda^2-8\lambda+4)t+2\lambda(3\lambda-2)}{\sqrt{1+t}}+\frac{2\lambda^2(1-t)}{\sqrt{1-t^2}} \\ &\quad +\frac{4\lambda(\lambda-1)t+2\lambda(1-2\lambda)}{\sqrt{1-t}}+2\lambda(1-2\lambda). \end{aligned} \tag{2.23}$$

Simple computations yield

$$\lim_{t\to 0^+} G_\lambda(t)=0, \tag{2.24}$$

$$\lim_{t\to 1^-} G_\lambda(t)=-\infty, \tag{2.25}$$

$$G_\lambda'(t)=\frac{(3\lambda^2-8\lambda+4)t+4(2-3\lambda)}{2(1+t)^{3/2}}+\frac{2\lambda^2(t-1)}{(1-t^2)^{3/2}}+\frac{2\lambda(1-\lambda)t+\lambda(2\lambda-3)}{(1-t)^{3/2}}, \tag{2.26}$$

$$\lim_{t\to 0^+} G_\lambda'(t)=2(2-3\lambda)>0, \tag{2.27}$$

$$\lim_{t\to 1^-} G_\lambda'(t)=-\infty, \tag{2.28}$$

$$G_\lambda''(t)=\frac{1}{4(1-t^2)^{5/2}}H(t), \tag{2.29}$$

where

$$\begin{aligned} H(t) &= \left[4\lambda(1-\lambda)t+2\lambda(2\lambda-5)\right](1+t)^2\sqrt{1+t}+8\lambda^2(2t^2-3t+1) \\ &\quad -\frac{\left[(3\lambda^2-8\lambda+4)t-2(3\lambda^2+10\lambda-8)\right](1-t)^3}{\sqrt{1-t}}. \end{aligned} \tag{2.30}$$

Making use of (2.17) and (2.20) for (2.27) leads to

$$H(t) < [4\lambda(1-\lambda)t + 2\lambda(2\lambda-5)](1+t)^2(1+\frac{1}{2}t-\frac{1}{8}t^2) + 8\lambda^2(2t^2-3t+1)$$
$$-[(3\lambda^2-8\lambda+4)t-2(3\lambda^2+10\lambda-8)](1-t)^3(1+\frac{1}{2}t+\frac{3}{8}t^2) \tag{2.31}$$
$$=H_1(t).$$

Again making use of the transform $t = 1/u$, $u \in (1,+\infty)$ for the function $H_1(t)$, we have

$$H_1(t) = \frac{1}{8u^6}H_2(u), \tag{2.32}$$

where

$$H_2(u) = 16(9\lambda^2+5\lambda-8)u^6 - 72(4\lambda^2+7\lambda-4)u^5 + 2(129\lambda^2+35\lambda-80)u^4 - (127\lambda^2$$
$$-60\lambda-20)u^3 + 3(11\lambda^2+26\lambda-20)u^2 - (29\lambda^2+24\lambda-28)u + 3(3\lambda^2-8\lambda+4). \tag{2.33}$$

Simple calculations lead to

$$\lim_{u\to 1^+} H_2(u) = -264\lambda < 0, \tag{2.34}$$

$$H_2'(u) = 96(9\lambda^2+5\lambda-8)u^5 - 360(4\lambda^2+7\lambda-4)u^4 + 8(129\lambda^2+35\lambda-80)u^3$$
$$-3(127\lambda^2-60\lambda-20)u^2 + 6(11\lambda^2+26\lambda-20)u - (29\lambda^2+24\lambda-28), \tag{2.35}$$

$$\lim_{u\to 1^+} H_2'(u) = -8\lambda(181-14\lambda) < 0, \tag{2.36}$$

$$H_2''(u) = 6[80(9\lambda^2+5\lambda-8)u^4 - 240(4\lambda^2+7\lambda-4)u^3 + 4(129\lambda^2+35\lambda$$
$$-80)u^2 - (127\lambda^2-60\lambda-20)u + (11\lambda^2+26\lambda-20)], \tag{2.37}$$

$$\lim_{u\to 1^+} H_2''(u) = -12\lambda(527-80\lambda) < 0, \tag{2.38}$$

$$H_2'''(u) = 6[320(9\lambda^2+5\lambda-8)u^3 - 720(4\lambda^2+7\lambda-4)u^2$$
$$+8(129\lambda^2+35\lambda-80)u - (127\lambda^2-60\lambda-20)], \tag{2.39}$$

$$\lim_{u\to 1^+} H_2'''(u) = -30(60+620\lambda-181\lambda^2) < 0, \tag{2.40}$$

$$H_2^{(4)}(u) = 48[120(9\lambda^2+5\lambda-8)u^2 - 180(4\lambda^2+7\lambda-4)u + (129\lambda^2+35\lambda-80)], \tag{2.41}$$

$$\lim_{u\to 1^+} H_2^{(4)}(u) = -48(320+625\lambda-489\lambda^2) < 0, \tag{2.42}$$

and

$$H_2^{(5)}(u) = -2880[4(8-5\lambda-9\lambda^2)(u-1) + (20+\lambda-24\lambda^2)] < 0 \tag{2.43}$$

for $u > 1$. From (2.43) we clearly see that $H_2^{(4)}(u)$ is strictly decreasing in $(1,+\infty)$. It follows from (2.29), (2.31), (2.32), (2.34), (2.36), (2.38), (2.40) and (2.42) together with the monotonicity of $H_2^{(4)}(u)$ that $G_\lambda''(t) < 0$ for $t \in (0,1)$, hence $G_\lambda'(t)$ is strictly decreasing in $(0,1)$. From (2.27) and (2.28) together with the monotonicity of $G_\lambda'(t)$ we know that there exists $\mu_1 \in (0,1)$ such that

$G_\lambda'(t) > 0$ for $t \in (0,\mu_1)$ and $G_\lambda'(t) < 0$ for $t \in (\mu_1,1)$, thus $G_\lambda(t)$ is strictly increasing in $(0,\mu_1)$ and strictly decreasing in $(\mu_1,1)$. From (2.6), (2.8), (2.24) and (2.25) together with the monotonicity of $G_\lambda(t)$ we confirm that there exists $\mu_2 \in (0,1)$ such that $D_\lambda'(t) > 0$ for $t \in (0,\mu_2)$ and $D_\lambda'(t) < 0$ for $t \in (\mu_2,1)$, thus $D_\lambda(t)$ is strictly increasing in $(0,\mu_2)$ and strictly decreasing in $(\mu_2,1)$.

Notice that (2.5) becomes

$$\lim_{x \to 1^-} D_\lambda(x) = 0. \tag{2.44}$$

Therefore the inequality

$$M(a,b) < \lambda N(a,b) + (1-\lambda)Q(a,b) \tag{2.45}$$

follows from (2.2), (2.4) and (2.44) together with the monotonicity of $D_\lambda(t)$.

Finally we prove that $4/9N(a,b) + 5/9Q(a,b)$ is the best possible lower convex combination bound and $\lambda N(a,b) + (1-\lambda)Q(a,b)$ is the best possible upper convex combination bound of the square-root and quadratic means for the Nueman-Sándor mean.

Equations (2.1) lead to

$$\frac{Q(a,b) - M(a,b)}{Q(a,b) - N(a,b)} = \frac{\sqrt{1+x^2} - \dfrac{x}{\sinh^{-1} x}}{\sqrt{1+x^2} - \dfrac{1}{2}(1+\sqrt{1-x^2})} = B(x). \tag{2.46}$$

From (2.46) one has

$$\lim_{x \to 0^+} B(x) = \frac{4}{9}, \tag{2.47}$$

and

$$\lim_{x \to 1^-} B(x) = \lambda. \tag{2.48}$$

If $\alpha < 4/9$, then (2.46) and (2.47) lead to the conclusion that there exists $0 < \delta_1 < 1$ such that $\alpha N(a,b) + (1-\alpha)Q(a,b) > M(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (0,\delta_1)$.

If $\beta > \lambda$, then (2.46) and (2.48) lead to the conclusion that there exists $0 < \delta_2 < 1$ such that $M(a,b) > \beta N(a,b) + (1-\beta)Q(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (1-\delta_2,1)$.

Acknowledgment: This work was supported by the Natural Science Foundation of Hebei Province (F2015201089).

3. REFERENCES

[1]. E. Neuman and J. Sándor, *On the Schwab-Borchardt mean,* Math. Pannon. 14, 2(2003), 253-266.

[2]. E. Neuman and J. Sándor, *On the Schwab-Borchardt mean II*, Math. Pannon. 17, 1 (2006), 49-59.

[3]. Y.-M. Li, B.-Y. Long and Y.-M. Chu, *Sharp bounds for the Neuman-Sándor mean in terms of generalized logarithmic mean,* J. Math. Inequal. 6, 4 (2012), 567-577.

[4]. E. Neuman, *A note on a certain bivariate mean,* J. Math. Inequal. 6, 4 (2012), 637-643.

[5]. Y.-M. Chu, T.-H. Zhao and B.-Y. Liu, *Optimal bound for Neuman-S_andor mean in terms of the convex combination of logarithmic and quadratic or contra-harmonic means,* J. Math. Inequal. 8, 2 (2014), 201-217.

[6]. C.-R. Liu and J. Wang, *Optimal Bounds for Neuman-Sándor Mean in Terms of the Convex Combination of Geometric and Quadratic Means,* Mathematica Aeterna, 5, 1 (2015), 83-94.