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UPPER EDGE GEODETIC DOMINATION AND UPPER CONNECTED EDGE GEODETIC DOMINATION NUMBER OF A GRAPH

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ABSTRACT

In this paper the concept of upper edge geodetic domination number (UEGD number) and upper connected edge geodetic domination number (UCEGD number) of a graph is studied. An edge geodetic domination set (EGD set) *S* in a connected graph is minimal EGD set if no proper subset of *S* is an edge geodetic domination set. The maximum cardinality of all the minimal edge geodetic domination set is called UEGD number. An EGD set *S* in a connected graph is minimal CEGD set if no proper subset of S is a CEGD set. The maximum cardinality of all the minimal connected edge geodetic domination set is called UCEGD number. Here the UEGD number and UCEGD number of certain graphs are identified. Also for two positive integers *p* and *q* there exist some connected graph with EGD number *p* and UEGD number *q*. Similarly for two positive integers *p* and *q* there exist some connected graph with CEGD number *p* and UCEGD number *q*.

Keywords : Geodetic domination number, edge geodetic domination number, upper edge geodetic domination number, upper connected edge geodetic domination number.

AMS subject Classification: 05C12, 05C05

1. INTRODUCTION

By a graph $G = (V, E)$ we consider a finite undirected graph without loops or multiple edges. The order and size of a graph are denoted by *p* and *q* respectively. For the basic graph theoretic notations and terminology we refer to Buckley and Harary [4].For vertices *u* and *v* in a connected graph *G*, the distance $d(u, v)$ is the length of a shortest u–v path in G. A u–v path of length $d(u, v)$ is called au–v geodesic. A geodetic set of *G* is a set $S \subseteq V(G)$ such that every edge of *G* is contained in a geodesic joining some pair of vertices in S. The edge geodetic number $g_e(G)$ of G is the minimum order of its edge geodetic sets.

The neighbourhood of a vertex v is the set $N(v)$ consisting of all vertices which are adjacent with v. A vertex v is an extreme vertex if the sub graph induced by its neighbourhood is complete. A vertex v in a connected graph G is a cut-vertex of G, if $G-v$ is disconnected. A vertex v in a connected graph G is said to be a semi-extreme vertex if $\Delta(<N(v)>)=|N(v)|-1$. A graph G is said to be semi-extreme graph if every vertex of G is a semi-extreme vertex. An acyclic connected graph is called a tree [4].

A dominating set in a graph G is a subset of vertices of G such that every vertex outside the subset has neighbour in it. The size of a minimum dominating set in a graph G is called the domination number of G and is denoted by $\gamma(G)$. A geodetic domination set of G is a subset of $V(G)$ which is both geodetic and dominating set of G. The minimum cardinality of a geodetic domination set is denoted by $\gamma_{g_e}(G)$.A detailed study of geodetic domination set is available in [9]. An edge geodetic set of G is a subset $S \subset V(G)$ such that every edge of G is contained in a geodesic joining some pair of vertices in S. The edge geodetic number $g_e(G)$ of G is the minimum order of its edge geodetic sets. Edge geodetic set of a connected graph is studied in [12].

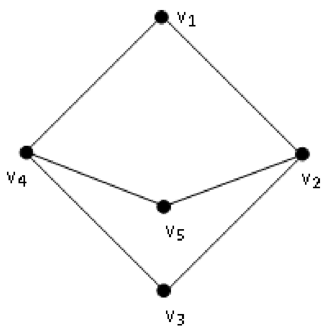
A set of vertices of G is said to be edge geodetic domination set or EGD set if it is both edge geodetic set and a domination set of G. The minimum cardinality among all the EGD sets of G is called edge geodetic domination number and is denoted by $\gamma_{g_e}(G)$. A detailed study of EGD sets is available in [3]. A set S of vertices of a graph connected edge geodetic domination set (abbreviated as CEGD set) if it is (i) an edge geodetic set of G (ii) a domination set of G and (iii) the induced sub graph of S ,$<S>$ is connected. The minimum cardinality among all the CEGD set of G is called CEGD number and is denoted by $\gamma_{g_{ce}}(G)$.

2 UPPER EDGE GEODETIC DOMINATION NUMBER OF A GRAPH

2.1 Definition: An edge geodetic domination set S in a connected graph G is called a minimal edge geodetic domination set if no proper subset of S is an edge geodetic domination set of G.

2.2 Definition: The upper edge geodetic domination number (abbreviated as UEGD number) is the maximum cardinality among all minimal edge geodetic domination sets and is denoted by $\gamma_{g_e}{}^{+}(G)$.

2.3 Example: Consider the graph given in figure 01. Here $S_1 = \{v_2,v_4\}$ is an EGD set and is minimum. Therefore $\gamma_{g_e}(G) = 2$. Now $S_2 = \{v_1,v_2, v_3, v_5\}$ is also an EGD set any of its proper subset is not an EGD set. Thus S_2 is a minimal EGD set. Also note that any EGD set having more than four vertices is not a minimal EGD set. Thus UEGD number of G is 4. That is $\gamma_{g_e}{}^{+}(G) = 4$.



G

Figure:01

2.4 Remark: Every minimum EGD set is also a minimal EGD set. The converse is not true. As in the above example S_2 is a minimal EGD set but not a minimum EGD set.

2.5 Theorem: Let G be a connected graph of order p. Then $2 \leq \gamma_{g_e}(G) \leq \gamma_{g_e}^{+}(G)$.

Proof: Any EGD set contains at least two vertices. Therefore $2 \geq \gamma_{g_e}(G)$. Since every minimum EGD set is also a minimal EGD set, $\gamma_{g_e}^{+}(G) \geq \gamma_{g_e}(G)$. Any EGD set contains at most p vertices so that $p \geq \gamma_{g_e}^{+}(G)$. Hence

$2 \leq \gamma_{g_e}(G) \leq \gamma_{g_e}^{+}(G)$.

2.6 Theorem: For any connected graph G, $\gamma_{g_e}(G) = p$ if and only if $\gamma_{g_e}^{+}(G) = p$.

Proof: If $\gamma_{g_e}(G)=p$. By Theorem 2.5, for any connected graph G, $\gamma_{g_e}^{+}(G) \geq \gamma_{g_e}(G)$ which implies $\gamma_{g_e}^{+}(G) = p$. Conversely, let $\gamma_{g_e}^{+}(G) = p$. That is $S= V(G)$ is the unique minimal EGD set and so it is a minimum EGD set. Hence $\gamma_{g_e}(G) = p$.

2.7 Theorem: Let G be the complete graph K_p. Then $\gamma_{g_e}^{+}(G) = p$.

Proof: For complete graph K_p, any EGD set contains all the vertices. That is $\gamma_{g_e}(G) = p$. Hence the result follows from Theorem 2.6.

2.8 Theorem: Let G be a connected graph of order p with $\gamma_{g_e}(G) = p-1$. Then $\gamma_{g_e}^{+}(G) = p-1$.

Proof: Given $\gamma_{g_e}(G) = p-1$. Hence by Theorem 2.5,$\gamma_{g_e}^{+}(G) \geq p-1$. Therefore $\gamma_{g_e}^{+}(G)$ is either $p-1$ or p. If

$\gamma_{g_e}^{+}(G) = p$ then by Theorem 2.6, $\gamma_{g_e}(G)=p$ which is a contradiction. Therefore $\gamma_{g_e}^{+}(G) = p-1$.

2.9 Theorem: Each semi-extreme vertex belongs to every minimal EGD set.

Proof**:** By Theorem 2.5of [1], every semi-extreme vertex belongs every EGD set. Since minimal EGD set is itself an EGD set, the result follows.

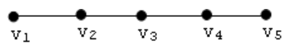
2.10 Theorem: For a semi-extreme graph G of order p, $\gamma_{g_e}^{+}(G) = p$.

Proof: For a semi-extreme vertex and by Theorem 2.9, it belongs to every minimal EGD set.

2.11 Theorem: Each extreme vertex of G belongs to every minimal EGD set of G.

Proof: Since every extreme vertex is a semi-extreme vertex, the result follows by Theorem 2.9.

2.12 Remarks: The set of all extreme vertices need not form a minimal EGD set. Consider the path graph P_5.



P_5

Figure: 02

Here the set of extreme vertices $S = \{v_1,v_5\}$ is only an edge geodetic set but not dominating set.

2.13 Theorem: For complete bipartite graph $G = K_{m,n}$,

$$\gamma_{g_e}^{+}(G) = \begin{cases} 2, & if \;\; m = n = 1 \\ n, & if \; n \geq 2, m = 1 \\ \max\{m,n\}, & if \; m,n \geq 2 \end{cases}$$

Proof**:** Case (i): Let $m = n= 1$. Then $K_{m,n} = K_2$. Hence by Theorem 2.7, $\gamma_{g_e}^{+}(G) = 2$. Case (ii): Let $n \geq 2$,$m = 1$. Then $K_{m,n}$ is a rooted tree with n end vertices and these extreme vertices belongs to every EGD set and so every minimal EGD set. Case (iii): Let $m,n \geq 2$. Without loss of generality assume that $m \leq n$. Let $V_1 = \{u_1,u_2,\dots,u_m\}$ and $V_2=\{w_1,w_2,\dots,w_n\}$ be a partition of G. Take $S = V_2$. We prove S is a minimal EGD set. Any edge $u_iw_j(1 \leq i \leq m , 1 \leq j \leq n)$ lies in the geodetic path $w_iu_jw_k$ for $k \neq i$ so that S is an edge geodetic set. Also these n vertices dominate G. Thus S is an EGD set.

Next, let $S_1 \subset S$ strictly. Then there is at least one vertex $w_j \in S$and does not belongs to S_1. Then the edge w_ju_i does not lie on any geodetic path joining a pair of vertices of S_1. Thus S_1 is not an EGD set. Hence S is a minimal EGD set. Therefore $\gamma_{g_e}{}^{+}(G) \geq |S| = |V_2| = n$.

Let S_2 be any minimal EGD set such that$S_2 \geq n+1$. Since each u_iw_j $(1 \leq i \leq m,\ 1 \leq j \leq n)$ lies on the geodetic path $u_iw_ju_k$ it follows that V_1 is an EGD set. Hence S_2 cannot contain V_1. Similarly since V_2 is a minimal EGD set, V_2 does not lie in S_2. Thus there exist some vertex u_i in V_1 and w_j in V_2 both does not lie in S_2.Therefore the edge u_iw_j does not lie on any geodetic path joining any pair of vertices in S_2. That is,S_2 is not an EGD set which is a contradiction. Therefore$S_2 \leq n$. Hence the maximum cardinality of any minimal EGD set is n. Therefore $\gamma_{g_e}{}^{+}(G)=n=\max\{m,n\}$.

2.14 Theorem: Let G be a connected graph and v be a cut-vertex of G. If S is a minimal EGD set of G, then every component of $G-v$ contains some vertices of S.

Proof: By Theorem 2.8 of [1], if v is a cut-vertex, every component of $G-v$ contains some vertices of any EGD set. Since every minimal EGD set is also an EGD set the result follows.

3 UPPER CONNECTED EDGE GEODETIC DOMINATION NUMBER OF A GRAPH

3.1 Definition: A connected edge geodetic domination set (CEGD) S in a connected graph G is called a minimal connected edge geodetic domination set if no proper subset of S is CEGD set of G.

The upper connected edge geodetic domination number (abbreviated as UCEGD number) is the maximum cardinality among all minimal CEGD sets and is denoted by $\gamma_{g_{ce}}{}^{+}(G)$.

3.2 Example: Consider the following graph (figure 03)

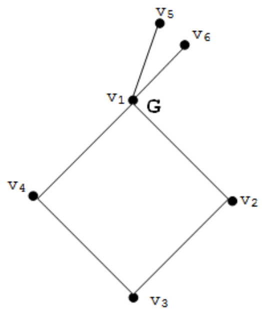


Figure 3

Here $S=\{v_3,v_5,v_6\}$ is a minimal EGD set. Therefore $\gamma_{g_e}{}^{+}(G)=3$. But it is not connected. Here $\{v_1,v_2,v_3,v_5,v_6\}$ is a minimal CEGD set. Therefore $\gamma_{g_{ce}}{}^{+}(G) = 5$.

Most of the following theorems and results are analogy to upper edge geodetic domination number theorems. So we leave detailed proof.

3.3 Theorem: Let G be a connected graph of order p. Then $2 \leq \gamma_{g_{ce}}(G) \leq \gamma_{g_{ce}}{}^{+}(G)$.

3.4 Theorem: For any connected graph G, $\gamma_{g_{ce}}(G) = p$ if and only if $\gamma_{g_{ce}}{}^{+}(G) = p$.

3.5 Theorem: Every extreme-vertex of G belongs to every minimal CEGD set.

3.6 Theorem: Each semi-extreme vertex of G belongs to every minimal CEGD set.

3.7 Theorem: Let G be the complete graph K_p. Then $\gamma_{g_{ce}}{}^{+}(G) = p$.

Proof: For complete graph K_p, every vertex of G is an extreme vertex.

3.8 Theorem: Let G be a connected graph of order p with $\gamma_{g_{ce}}(G) = p-1$. Then $\gamma_{g_{ce}}{}^{+}(G)= p-1$.

3.9 Theorem: For a semi-extreme graph G of order p, $\gamma_{g_e}{}^{+}(G) = n$.

3.10 Theorem: Let G be a connected graph and v be a cut-vertex of G. If S is a minimal CEGD set of G, then every component of $G-v$ contains some vertices of S.

3.11 Theorem: Let G be a connected graph and v be a cut-vertex of G. Then v belongs to every minimal CEGD set.

Proof: Let v be a cut-vertex and $G_1,G_2,\ldots,G_r$, $r \geq 2$ be the components of $G - v$. Let S be a minimal CEGD set of G. Then by Theorem 3.10, it contains at least one element from $G_1,G_2,\ldots,G_r$. Since $<S>$ is connected it must contains v.

3.12 Theorem: For complete bipartite graph $K_{m,n}$,

$$\gamma_{g_c}^+(G) = \begin{cases} 2, & \text{if } m=n=1 \\ n, & \text{if } n\geq 2, m=1 \\ \max\{m,n\}, & \text{if } m,n \geq 2 \end{cases}$$

3.13 Theorem: Let G be a connected graph with m extreme vertices and n cut vertices. Then $\gamma_{g_{ce}}^+(G) \geq m+n$.

Proof: This follows from the Theorem 3.5 and Theorem 3.11.

3.14 Theorem: Let T be a tree with p vertices. Then $\gamma_{g_{ce}}^+(G) = p$.

Proof: Since every vertex of T is either a cut-vertex or an end vertex, the result follows from Theorem 3.5 and Theorem 3.11.

3.15 Theorem: For cycle graph $G = C_p$, $\gamma_{g_{ce}}^+(G) = p - 2$, for $p \geq 5$.

Proof: Consider any consecutive $p-2$ vertices in G. They form a CEGD set of G. Also they are minimal. Therefore $\gamma_{g_{ce}}^+(G) = p - 2$.

4 PATH GRAPHS

For a path graph of p vertices, edge geodetic number is 2 and edge geodetic domination number, $\gamma_{g_e}(G) = \left\lceil \frac{n+4}{3} \right\rceil$. The proof is, $V(P_n) = \{v_1,v_2,\ldots,v_n\}$ then the set $S = \{v_1,v_4,v_7,\ldots,v_n\}$ is the minimum EGD set which contains $\left\lceil \frac{n+4}{3} \right\rceil$ elements.

4.1 Theorem: Let G be the path graph of p vertices. Then

$$\gamma_{g_e}^+(G) = \begin{cases} 2, & \text{if } 2 \leq n \leq 4 \\ 3, & \text{if } n = 5 \\ \left\lceil \frac{n}{3} \right\rceil + 2, & \text{if } n \geq 6 \end{cases}$$

Proof: The first two parts are trivial. For $n \geq 6$, let $V(G) = \{v_1,v_2,\ldots,v_n\}$. Then the set $S = \{v_1,v_2,v_5,v_8,\ldots,v_n\}$ is a minimal EGD set with maximum cardinality. This set contains exactly $\left\lceil \frac{n}{3} \right\rceil + 2$ vertices.

4.2 Theorem: Let G be the path graph of p vertices. Then $\gamma_{g_{ce}}^+(G) = p$.

Proof: Since every path graph is a tree, the result follows by Theorem 3.14.

5 REALISATION RESULTS

5.1 Theorem: For every two positive integers p and q where $2 \leq p \leq q$ there exist some connected graph G with $\gamma_{g_e}(G) = p$ and $\gamma_{g_e}^+(G) = q$.

Proof: Case (i): Let $p = q$. Consider the bipartite graph $G = K_{1,p}$. Then G is a tree with p end vertices. Hence $\gamma_{g_e}(G) = p$. By Theorem 2.13, $\gamma_{g_e}^+(G) = p$.

Case (ii): Let $2 \leq p < q$. Construct the graph G as follows: Take $A=\{x,y\}$ and $S_1 = \{u_1,u_2,\ldots,u_{q-p+1}\}$ as two set of vertices in G such that each vertex u_i in S_1 is adjacent with x and y , and x and y are non-adjacent. Let $S_2 = \{w_1,w_2,..,w_{p-1}\}$ be $p-1$ non adjacent vertices in G all are incident with y only. (Ref: figure 04) Since each vertex in S_2 is an end vertex, it lies in every EGD set of G. Note that S_2 is not itself an EGD set because any edge xu_i does not lie on any of the geodetic path of S_2. But $S_3 = \{x\} \cup S_2$ is a minimum EGD set. Therefore $\gamma_{g_e}(G) = p - 1 + 1 = p$.

Next, take $S_4 = S_1 \cup S_2$. Then S_4 is a minimal edge geodetic set. On the contrary, suppose Q is any proper subset of S_4. Then there exist at least one vertex say $v \in S_4$ such that $v \notin Q$. Assume first $v = w_i$ for some i, $1 \leq i \leq p-1$. Then the edge yw_i does not lie on any geodetic path joining any pair of vertices in Q. Therefore Q is not an EGD set. Next assume $v = u_j$ for some j, $1 \leq j \leq q - p + 1$. Then the edges xu_j or yu_j does not lie on a geodetic path joining any pair of vertices in Q. Hence Q is not an EGD set so that $\gamma_{g_e}^{+}(G) \geq (q-p+1) + p - 1 = q$.

Next we show that there does not exist any minimal EGD set S_5 with $|S_5| > q$. On the contrary, let S_5 be a minimal EGD set with $|S_5| > q$. Now $V(G) = q + 2$ and since S_3 is an EGD set of G, it follows that $|S_5| = q + 1$. Since no cut-vertex belongs to any minimal EGD set, $y \notin S_5$. Therefore $S_5$is not a minimal EGD set and is a contradiction. Therefore $\gamma_{g_e}^{+}(G) = q$

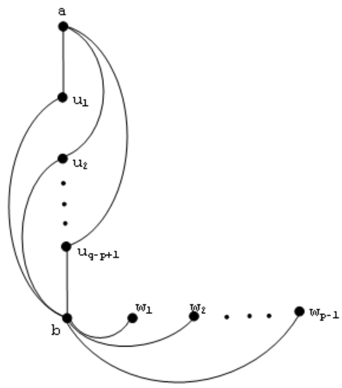


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Figure: 04

5.2 Theorem: For every two positive integers p and q where $2 \leq p \leq q$ there exists some connected graph G with $\gamma_{g_{ce}}(G) = p$ and $\gamma_{g_{ce}}^{+}(G) = q$.

Proof: For $p = q$, consider a complete graph with p vertices. For $p<q$, consider the figure 04 and proof of the Theorem 5.1. Take S_2asa set of $p-2$ end vertices instead of $p-1$. Then $S_2 \cup \{x, y\}$ is a minimum CEGD set with p vertices. Take $S_4 = S_2 \cup S_1 \cup \{y\}$ having $(p-2) + (q-p+1) + 1 = q$ vertices. As in Theorem 5.1, we can prove S_4 is the unique minimal CEGD set. That is $\gamma_{g_{ce}}^{+}(G) = q$.

6 CONCLUSION

We can extent the concept of upper connected edge geodetic domination number to find minimal forcing CEGD number and UCEGD number of join of graphs, UCEGD number of composition of graphs and UCEGD hull number of graphs and so on.

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