Vol.4.Issue.3.2016 (July-Sept.,)

BULLETIN OF MATHEMATICS AND STATISTICS RESEARCH

A Peer Reviewed International Research Journal

http://www.bomsr.com

Email:editorbomsr@gmail.com

ISSN:2348-0580

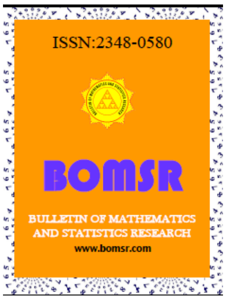
RESEARCH ARTICLE

A SINGLE SERVER NON-MARKOVIAN BERNOULLI VACATION QUEUE WITH TWO TYPE OF SERVICES AND WITH AN OPTIONAL SERVICE

KALYANARAMAN. R

Department of Mathematics, Annamalai University, Annamalainagar-608 002, India.

email: r.kalyan24@rediff.com



ABSTRACT

A single server queue with two type of services and with Bernoulli vacation has been considered. The type 1 service is a phase type service with two service phases. Both the service time distributions are generally distributed. The type 2 service has only one phase of service. In addition the server also provides an optional service. These service time distributions are also general. After completion of service, the server takes vacation based on a Bernoulli process and vacation time distribution is general. For this model the probability generating function for the number of customers in the queue at different server's state are obtained using supplementary variable technique. Some performance measures and particular models are calculated and numerical results are presented.

Keywords: Phases service – Optional service – Bernoulli process – Supplementary variable technique – Vacation – Performance measures.

AMS Subject classification number: 90B22, 60K25 and 60K30.

1. INTRODUCTION

Queueing systems constitute a key tool in modelling and performance analysis of telecommunication systems and computer systems. Poisson arrivals are in many cases a fairly realistic model for the arrival process, but exponential service times are not very common in practice. In many systems the coefficient of variation of the service times will be smaller (or greater) than 1. Therefore, it is essential to analyze the models with generally distributed service times. Due to this reason the $M/G/1$ queue has been studied in various forms by numerous authors including Gaver (1959), Keilson and Kooharian (1960), Bhat (1964), Prabhu (1965) and Cohen (1969), to mention a few.

Recently there have been several contributions considering queueing system of the $M/G/1$ type, in which the server may provide services in phases. The motivation for such type of model comes from some computer and communication networks, where messages are processed in two stages by a single server. The case where both phases of service are exponentially distributed is the so called coxian distribution. Bertsimas and Papaconstantinou (1988) considered such distribution to design a multi server queue with application in a transportation system.

The queueing system with two phase service have studied by Krishna and Lee (1990). Doshi (1991) has extended the two phase queueing system of Krishna and Lee into case of batch. Recently Artalejo and Choudhury (2004) have studied the steady state analysis of an $M/G/1$ queue with repeated attempts and two phase service.

In day to day life, one encounters numerous queueing situations in which all the arriving customers are given the essential service and only some of them may require additional optional service. Such a model was studied by Madan (1994). The other works to be noted here are Madan (2000), Medhi (2002), Al-Jararah and Madan (2003), Jinting Wang (2004), Kalyanaraman et al. (2005) and Jau-Chuan (2008).

In real life, where as soon as the server becomes free, the server shut down the service facility temporarily for a random period of time and thus the server may not be available when the customer arrives to an empty queue, the service starts only after the server returns to the queue. This random period is called vacation period and the queue is called a vacation queue. Miller (1964), first studied a model, where the server is unavailable for service during some random length of time for the M/G/1 queueing system. Bernoulli vacation model at each service completion epoch the decision to take a vacation depends on a Bernoulli distribution. This type of vacation policy was first introduced by Keilson and Servi (1986).

The paper is organized as follows: The corresponding mathematical model is defined in section 2 and the governing differential difference equations, the boundary conditions and the normalizing condition are given in section 3. For this model the probability generating function of the number of customers in queue irrespective of the server state are derived in section 4. Also some performance measures related to this queueing model are derived from these probability generating functions and are given in section 5. In section 6, some particular models are derived. A numerical study is carried out in section 7.

2. The Model

The arrival follows Poisson with rate $\lambda(>0)$ and a single server provides two type of services, respectively called type 1 service and type 2 service. Also the server provides an optional service. The entering customers selects type 1 service with probability p or type 2 service with probability $1-p$. The type 1 service is a phase type service (two phases). After completion of type 1 service, the customer leaves the system, whereas after completion of type 2 service, the customer leaves the system with probability $1-r$ or choose an optional service with probability r. After completion of optional service, the customer leaves the system. The service time distributions are general, the distribution functions are $B_{1,j}(x)$, for type 1 and j^{th} phase of service $(j=1,2)$, $B_{2,1}(x)$, for type 2 service, $B_{2,2}(x)$, for an optional service. The Laplace- Stieltjes transform (LST) for $B_{i,j}(x)$ is $B^*_{i,j}(\theta)$ and finite k^{th} moments are $E(B^k_{i,j}), k\geq 1, i, j=1,2.$

After completion of each service the server may go for a vacation of random length with probability q $(0\leq q\leq 1)$ or may continue to serve the next customer, if any, with probability $(1-q)$. If there are no customers in the queue, at the completion of vacation, the server remains in the system without taking further vacation. Vacation time distribution is also a general distribution with distribution function $V(x)$, Laplace- Stieltjes transform (LST) for $V(x)$ is $V^*(\theta)$ and finite moments are $E(V^k), k \geq 1.$

It may be noted that $B_{i,j}(x), V(x), (B_{i,j}(\infty)=1, B_{i,j}(0)=0, V(\infty)=1, V(0)=0)$ are continuous, so that $\mu_{i,j}(x)dx, (\gamma(x)dx)$ are the first order differential functions (hazard rates) of $B_{i,j}(x), (V(x)).$

For the analysis the supplementary variable (the variable is elapsed time) technique has been used.

Let $\mu_{1,j}(x)dx$ be the conditional probability of completion of the j^{th} phase of type 1 service during the interval $(x, x+dx],$ given that elapsed service time is x so that $\mu_{1,j}(x) = \dfrac{b_{1,j}(x)}{1-B_{1,j}(x)}, (j=1,2)$ and let $\mu_{2,j}(x)dx$ be the conditional probability of completion of the type 2 service and optional service during the interval $(x, x+dx],$ given that elapsed service time is x so that $\mu_{2,j}(x) = \dfrac{b_{2,j}(x)}{1-B_{2,j}(x)}, (j=1,2)$ and let $\gamma(x)dx$ be the conditional probability of completion of the vacation during the interval $(x, x+dx],$ given that elapsed vacation time is x so that $\gamma(x) = \dfrac{v(x)}{1-V(x)}.$

The following notations are introduced to define the model mathematically:

$P_n^{(1,j)}(x,t) = \Pr\{$at time t, there are n customers in the queue excluding one in the type 1 service and is in the j^{th} phase of service and the elapsed service time is $x\}$, $j = 1, 2, n \geq 0$, $P_n^{(2,1)}(x,t) = \Pr\{$at time t, there are n customers in the queue excluding one in the type 2 service and elapsed service time is $x\}, n \geq 0$, $P_n^{(2,2)}(x,t) = \Pr\{$at time t, there are n customers in the queue excluding one in the optional service and elapsed service time is $x\}, n \geq 0$ and

$V_n(x,t) = \Pr\{$at time t, the server is on vacation with elapsed vacation time is x and the number of customers in the queue is $n\}, n \geq 0$. $Q(t) = \Pr\{$at time t, there are no customers in the system and the server is idle$\}$.

Let $P_n^{(i,j)}(x)\ (i, j = 1,2)$, $V_n(x)$ and Q denote the corresponding steady state probabilities.

The probability generating functions for the probabilities $\{P_n^{(i,j)}(x)\}, (i, j = 1,2), \{V_n(x)\}$ are respectively defined as

$$P^{(i,j)}(x,z) = \sum_{n=0}^{\infty} z^n P_n^{(i,j)}(x) \text{ and } V(x,z) = \sum_{n=0}^{\infty} z^n V_n(x).$$

3. The Governing Equations

The differential difference equations related to the model defined in the proceeding section are

$$\frac{d}{dx} P_0^{(1,1)}(x) + (\lambda + \mu_{1,1}(x)) P_0^{(1,1)}(x) = 0 \quad (1)$$

$$\frac{d}{dx} P_n^{(1,1)}(x) + (\lambda + \mu_{1,1}(x)) P_n^{(1,1)}(x) = \lambda P_{n-1}^{(1,1)}(x); n \geq 1 \quad (2)$$

$$\frac{d}{dx}P_0^{(1,2)}(x)+(\lambda+\mu_{1,2}(x))P_0^{(1,2)}(x)=0 \tag{3}$$

$$\frac{d}{dx}P_n^{(1,2)}(x)+(\lambda+\mu_{1,2}(x))P_n^{(1,2)}(x)=\lambda P_{n-1}^{(1,2)}(x);n\geq 1 \tag{4}$$

$$\frac{d}{dx}P_0^{(2,1)}(x)+(\lambda+\mu_{2,1}(x))P_0^{(2,1)}(x)=0 \tag{5}$$

$$\frac{d}{dx}P_n^{(2,1)}(x)+(\lambda+\mu_{2,1}(x))P_n^{(2,1)}(x)=\lambda P_{n-1}^{(2,1)}(x);n\geq 1 \tag{6}$$

$$\frac{d}{dx}P_0^{(2,2)}(x)+(\lambda+\mu_{2,2}(x))P_0^{(2,2)}(x)=0 \tag{7}$$

$$\frac{d}{dx}P_n^{(2,2)}(x)+(\lambda+\mu_{2,2}(x))P_n^{(2,2)}(x)=\lambda P_{n-1}^{(2,2)}(x);n\geq 1 \tag{8}$$

$$\frac{d}{dx}V_0(x)+(\lambda+\gamma(x))V_0(x)=0 \tag{9}$$

$$\frac{d}{dx}V_n(x)+(\lambda+\gamma(x))V_n(x)=\lambda V_{n-1}(x);n\geq 1 \tag{10}$$

$$\lambda Q=(1-q)\left\{\int_0^\infty P_0^{(1,2)}(x)\mu_{1,2}(x)dx+(1-r)\int_0^\infty P_0^{(2,1)}(x)\mu_{2,1}(x)dx+\int_0^\infty P_0^{(2,2)}(x)\mu_{2,2}(x)dx\right\}$$
$$+\int_0^\infty V_0(x)\gamma(x)dx \tag{11}$$

The boundary conditions are

$$P_0^{(1,1)}(0)=\lambda pQ+(1-q)p\left\{\int_0^\infty P_1^{(1,2)}(x)\mu_{1,2}(x)dx+(1-r)\int_0^\infty P_1^{(2,1)}(x)\mu_{2,1}(x)dx+\int_0^\infty P_1^{(2,2)}(x)\mu_{2,2}(x)dx\right\}$$
$$+p\int_0^\infty V_1(x)\gamma(x)dx \tag{12}$$

$$P_n^{(1,1)}(0)=(1-q)p\left\{\int_0^\infty P_{n+1}^{(1,2)}(x)\mu_{1,2}(x)dx+(1-r)\int_0^\infty P_{n+1}^{(2,1)}(x)\mu_{2,1}(x)dx+\int_0^\infty P_{n+1}^{(2,2)}(x)\mu_{2,2}(x)dx\right\}$$
$$+p\int_0^\infty V_{n+1}(x)\gamma(x)dx,n\geq 1 \tag{13}$$

$$P_n^{(1,2)}(0)=\int_0^\infty P_n^{(1,1)}(x)\mu_{1,1}(x)dx,n\geq 0 \tag{14}$$

$$P_0^{(2,1)}(0)=(1-q)(1-p)\left\{\int_0^\infty P_1^{(1,2)}(x)\mu_{1,2}(x)dx+(1-r)\int_0^\infty P_1^{(2,1)}(x)\mu_{2,1}(x)dx+\int_0^\infty P_1^{(2,2)}(x)\mu_{2,2}(x)dx\right\}$$
$$+\lambda(1-p)Q+(1-p)\int_0^\infty V_1(x)\gamma(x)dx \tag{15}$$

$$P_n^{(2,1)}(0)=(1-q)(1-p)\left\{\int_0^\infty P_{n+1}^{(1,2)}(x)\mu_{1,2}(x)dx+(1-r)\int_0^\infty P_{n+1}^{(2,1)}(x)\mu_{2,1}(x)dx+\int_0^\infty P_{n+1}^{(2,2)}(x)\mu_{2,2}(x)dx\right\}$$
$$+(1-p)\int_0^\infty V_{n+1}(x)\gamma(x)dx,n\geq 1 \tag{16}$$

$$P_n^{(2,2)}(0) = r\int_0^\infty P_n^{(2,1)}(x)\mu_{2,1}(x)dx, n \geq 0 \tag{17}$$

$$V_n(0) = q\int_0^\infty P_n^{(1,2)}(x)\mu_{1,2}(x)dx + (1-r)q\int_0^\infty P_n^{(2,1)}(x)\mu_{2,1}(x)dx + q\int_0^\infty P_n^{(2,2)}(x)\mu_{2,2}(x)dx, n \geq 0 \tag{18}$$

and the normalization condition is

$$Q + \sum_{n=0}^{\infty}\int_0^\infty \left[P_n^{(1,1)}(x) + P_n^{(1,2)}(x) + P_n^{(2,1)}(x) + P_n^{(2,2)}(x) + V_n(x)\right]dx = 1 \tag{19}$$

4. The Analysis

Multiplying equations (2), (4), (6), (8) and (10) by z^n, summing from $n=1$ to ∞ and then adding (1), (3), (5), (7) and (9), we get

$$\frac{\frac{d}{dx}P^{(i,j)}(x,z)}{P^{(i,j)}(x,z)} = -s - \mu_{i,j}(x) \tag{20}$$

$$\frac{\frac{d}{dx}V(x,z)}{V(x,z)} = -s - \gamma(x) \tag{22}$$

where $s = \lambda(1-z)$ and $i, j = 1,2.$

Integration of the equations (20) and (22) leads to

$$P^{(i,j)}(x,z) = C_{i,j}(1 - B_{i,j}(x))e^{-sx} \tag{23}$$

$$V(x,z) = C(1 - V(x))e^{-sx} \tag{24}$$

Taking $x = 0$ in equations (23), (24), the constants $C_{i,j}$, $(i, j = 1,2)$, C are obtained as

$$C_{i,j} = P^{(i,j)}(0,z) \tag{25}$$

$$C = V(0,z) \tag{26}$$

Using equations (25), (26) in (23), (24), we get

$$P^{(i,j)}(x,z) = P^{(i,j)}(0,z)(1 - B_{i,j}(x))e^{-sx} \tag{27}$$

$$V(x,z) = V(0,z)(1 - V(x))e^{-sx} \tag{28}$$

Multiplying equations (13), (16) by z^n, summing from $n=1$ to ∞, adding (12), (15), using (11), (27) and (28) with the corresponding equation, we get

$$zP^{(1,1)}(0,z) = \lambda p(z-1)Q + pV^*(s)V(0,z) + p(1-q)\left\{B_{1,2}^*(s)P^{(1,2)}(0,z) + (1-r)B_{2,1}^*(s)P^{(2,1)}(0,z) + B_{2,2}^*(s)P^{(2,2)}(0,z)\right\} \tag{29}$$

$$[z - (1-p)(1-q)(1-r)B_{2,1}^*(s)]P^{(2,1)}(0,z) = \lambda(1-p)(z-1)Q + (1-p)V^*(s)V(0,z) + (1-p)(1-q)\left\{B_{1,2}^*(s)P^{(1,2)}(0,z) + B_{2,2}^*(s)P^{(2,2)}(0,z)\right\} \tag{30}$$

Now multiplying equation (14) by z^n, summing from $n=0$ to ∞ and using equation (27),

$$P^{(1,2)}(0,z) = B_{1,1}^*(s)P^{(1,1)}(0,z) \tag{31}$$

Performing similar operation on equations (17) and (18), we obtain

$$P^{(2,2)}(0,z) = rB_{2,1}^*(s)P^{(2,1)}(0,z) \tag{32}$$

$$V(0,z) = qB_{1,2}^*(s)P^{(1,2)}(0,z) + q(1-r)B_{2,1}^*(s)P^{(2,1)}(0,z) + qB_{2,2}^*(s)P^{(2,2)}(0,z) \tag{33}$$

Using equations (31)-(33) in (29), (30), we get

$$[z - pB_{1,1}^*(s)B_{1,2}^*(s)(1-q+qV^*(s))]P^{(1,1)}(0,z) = \lambda p(z-1)Q + pB_{2,1}^*(s)(1-r+rB_{2,2}^*(s)) \times (1-q+qV^*(s))P^{(2,1)}(0,z) \quad (34)$$

$$[z-(1-p)B_{2,1}^*(s)(1-r+rB_{2,2}^*(s))(1-q+qV^*(s))]P^{(2,1)}(0,z) = \lambda(1-p)(z-1)Q + (1-p)B_{1,1}^*(s) \times B_{1,2}^*(s)(1-q+qV^*(s))P^{(1,1)}(0,z) \quad (35)$$

From equations (34) and (35), we get

$$P^{(1,1)}(0,z) = \frac{\lambda p(z-1)Q}{D(z)} \quad (36)$$

$$P^{(2,1)}(0,z) = \frac{\lambda(1-p)(z-1)Q}{D(z)} \quad (37)$$

where $D(z) = z-(1-q+qV^*(s))[pB_{1,1}^*(s)B_{1,2}^*(s)+(1-p)B_{2,1}^*(s)(1-r+rB_{2,2}^*(s))]$

Using equations (36), (37) in (31), (32), we get

$$P^{(1,2)}(0,z) = \frac{\lambda p(z-1)B_{1,1}^*(s)Q}{D(z)} \quad (38)$$

$$P^{(2,2)}(0,z) = \frac{\lambda r(1-p)(z-1)B_{2,1}^*(s)Q}{D(z)} \quad (39)$$

Using equations (37)-(39) in (33), we get

$$V(0,z) = \frac{\lambda q(z-1)[pB_{1,1}^*(s)B_{1,2}^*(s)+(1-p)B_{2,1}^*(s)(1-r+rB_{2,2}^*(s))]Q}{D(z)} \quad (40)$$

Integration of equations (26) and (27) by parts with respect to x and then using equations (36)-(40), we get

$$P^{(1,1)}(z) = \int_0^\infty P^{(1,1)}(x,z)dx = \frac{p(B_{1,1}^*(s)-1)Q}{D(z)} \quad (41)$$

$$P^{(1,2)}(z) = \int_0^\infty P^{(1,2)}(x,z)dx = \frac{pB_{1,1}^*(s)(B_{1,2}^*(s)-1)Q}{D(z)} \quad (42)$$

$$P^{(2,1)}(z) = \int_0^\infty P^{(2,1)}(x,z)dx = \frac{(1-p)(B_{2,1}^*(s)-1)Q}{D(z)} \quad (43)$$

$$P^{(2,2)}(z) = \int_0^\infty P^{(2,2)}(x,z)dx = \frac{r(1-p)B_{2,1}^*(s)(B_{2,2}^*(s)-1)Q}{D(z)} \quad (44)$$

$$V(z)=\int_0^\infty V(x,z)dx$$

$$=\frac{q(V^*(s)-1)[pB_{1,1}^*(s)B_{1,2}^*(s)+(1-p)B_{2,1}^*(s)(1-r+rB_{2,2}^*(s))]Q}{D(z)} \tag{45}$$

The idle probability Q is obtained using the equation (19) as

$$Q=1-\rho \tag{46}$$

where

$$\rho=\lambda p[E(B_{1,1})+E(B_{1,2})]+\lambda(1-p)[E(B_{2,1})+rE(B_{2,2})]+\lambda qE(V).$$

The utilization factor $(\rho)<1$ is the stability condition under which steady state solution exists.

Equations (41)-(45) together with equation (46) are respectively, the probability generating functions of the number of customers in the queue when the server is, serving phase 1 service, serving phase 2 service, serving type 2 service, serving an optional service and the sever is on vacation.

The probability generating function for the number of customers in the queue irrespective of server state is

$$U(z)=Q+\sum_{i=1}^{2}\sum_{j=1}^{2}P^{(i,j)}(z)+V(z)$$

$$=\frac{(z-1)(1-\rho)}{D(z)} \tag{47}$$

5. The Performance Measures

Using straightforward calculations the following performance measures have been obtained:

(i) The mean number of customers in the queue is

$L_q=\lim_{z\to 1}\frac{d}{dz}U(z)$ where $U(z)$ is given in equation (47).

$$L_q=\lim_{z\to 1}\left[\frac{D(z)-(z-1)D'(z)}{(D(z))^2}\right](1-\rho) \tag{48}$$

Since this limit gives $\frac{0}{0}$ form, so applying L'Hospitals rule twice, we get

$$L_q=\frac{-D''(1)Q}{2(D'(1))^2}$$

$$L_q=\frac{\lambda^2 C_1}{2(1-\rho)} \tag{49}$$

where $C_1=qE(V^2)+2qE(V)[p(E(B_{1,1})+E(B_{1,2}))+(1-p)(E(B_{2,1})+rE(B_{2,2}))]+p[E(B_{1,1}^2)$

$+E(B_{1,2}^2)+2E(B_{1,1})E(B_{1,2})]+(1-p)[E(B_{2,1}^2)+rE(B_{2,2}^2)+2rE(B_{2,1})E(B_{2,2})]$

(ii) The mean waiting time in the queue is

$$W_q=\frac{L_q}{\lambda}$$

where L_q is given in (49).

(iii) The variance of the number of customers in the queue is

$$V_{L_q} = \lim_{z\to 1}\frac{d^2}{dz^2}U(z) + \lim_{z\to 1}\frac{d}{dz}U(z) - \left(\lim_{z\to 1}\frac{d}{dz}U(z)\right)^2 \tag{50}$$

Now equation (48) differentiating with respect to z, we get

$$\lim_{z\to 1}\frac{d^2}{dz^2}U(z) = \lim_{z\to 1}\left[\frac{-(z-1)D(z)D''(z) - 2D'(z)[D(z)-(z-1)D'(z)]}{(D(z))^3}\right](1-\rho)$$

Since this limit gives $\frac{0}{0}$ form, so applying L'Hospitals rule three times, we get

$$\lim_{z\to 1}\frac{d^2}{dz^2}U(z) = \frac{[3(D''(1))^2 - 2D'(1)D'''(1)]Q}{6(D'(1))^3}$$

$$= \frac{\lambda^3[3\lambda C_1^2 + 2(1-\rho)C_2]}{6(1-\rho)^2} \tag{51}$$

where

$$C_2 = p[E(B_{1,1}^3) + 3E(B_{1,1}^2)E(B_{1,2}) + 3E(B_{1,1})E(B_{1,2}^2) + E(B_{1,2}^3)] + (1-p)[E(B_{2,1}^3) + rE(B_{2,2}^3)$$
$$+3rE(B_{2,1}^2)E(B_{2,2}) + 3rE(B_{2,1})E(B_{2,2}^2)] + 3qE(V)\big(p[E(B_{1,1}^2) + 2E(B_{1,1})E(B_{1,2}) + E(B_{1,2}^2)]$$
$$+(1-p)[E(B_{2,1}^2) + 2rE(B_{2,1})E(B_{2,2}) + rE(B_{2,2}^2)]\big) + 3qE(V^2)[p(E(B_{1,1}) + E(B_{1,2}))$$
$$+(1-p)(E(B_{2,1}) + rE(B_{2,2}))] + qE(V^3)$$

Using equations (49) and (51) in (50), we get

$$V_{L_q} = \frac{\lambda^4 C_1^2}{4(1-\rho)^2} + \frac{\lambda^2[2\lambda C_2 + 3C_1]}{6(1-\rho)} \tag{52}$$

(iv) The variance of the waiting time in the queue is

$$V_{W_q} = \frac{V_{L_q}}{\lambda^2}$$

where V_{L_q} is given in (52).

6. Some Particular Models

In this section, two particular models are derived by taking known distributions to service times and vacation time. The service rates are $\mu_{1,1}$ for phase 1 service, $\mu_{1,2}$ for phase 2 service, $\mu_{2,1}$ for type 2 service, $\mu_{2,2}$ for an optional service and the vacation rate is θ.

Model 1: In this model the service times and the vacation time distribution are negative exponential.

$$L_q = \frac{\lambda^2 C_3}{C_4\theta\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}}$$

$$V_{L_q} = \frac{\lambda^3(\lambda C_3^2 + 3C_4C_5)}{C_4^2\theta^2\mu_{1,1}^2\mu_{1,2}^2\mu_{2,1}^2\mu_{2,2}^2} + \frac{\lambda^2 C_3}{C_4\theta\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}}$$

where

$$C_3 = \mu_{2,1}^2\mu_{2,2}^2[p\theta^2(\mu_{1,1}^2 + \mu_{1,2}^2 + \mu_{1,1}\mu_{1,2}) + q\mu_{1,1}^2\mu_{1,2}^2] + (1-p)\theta^2\mu_{1,1}^2\mu_{1,2}^2(\mu_{2,2}^2 + r\mu_{2,1}^2 + r\mu_{2,1}\mu_{2,2})$$
$$+q\theta\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}[p\mu_{2,1}\mu_{2,2}(\mu_{1,1} + \mu_{1,2}) + (1-p)\mu_{1,1}\mu_{1,2}(\mu_{2,2} + r\mu_{2,1})]$$

$$C_4 = \mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}(\theta - \lambda q) - \lambda p\theta\mu_{2,1}\mu_{2,2}(\mu_{1,1} + \mu_{1,2}) - \lambda(1-p)\theta\mu_{1,1}\mu_{1,2}(\mu_{2,2} + r\mu_{2,1})$$

$$C_5 = \mu_{2,1}^3\mu_{2,2}^3[p\theta^3(\mu_{1,1}+\mu_{1,2})(\mu_{1,1}^2+\mu_{1,2}^2)+q\mu_{1,1}^3\mu_{1,2}^3]+(1-p)\mu_{1,1}^3\mu_{1,2}^3\theta^3[\mu_{2,2}^2(\mu_{22}+r\mu_{2,1}) + r\mu_{2,1}^2(\mu_{2,2}+\mu_{2,1})]+q\theta\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}[p\mu_{2,1}^2\mu_{2,2}^2[\mu_{1,1}^2(\theta+\mu_{1,2})+\mu_{1,2}^2(\theta+\mu_{1,1})+\theta\mu_{1,1}\mu_{1,2}] + (1-p)\mu_{1,1}^2\mu_{1,2}^2[\mu_{2,2}^2(\theta+\mu_{2,1})+r\mu_{2,1}^2(\theta+\mu_{2,2})+r\theta\mu_{2,1}\mu_{2,2}]]$$

Model 2: In this model the service times and vacation time distribution are Erlang k.

$$L_q = \frac{\lambda^2 C_6}{2k\theta C_4\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}}$$

$$V_{L_q} = \frac{\lambda^3(3\lambda C_6^2+4C_4C_7)}{12k^2C_4^2\theta^2\mu_{1,1}^2\mu_{1,2}^2\mu_{2,1}^2\mu_{2,2}^2}+\frac{\lambda^2C_6}{2k\theta C_4\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}}$$

where

$$C_6 = p\theta^2\mu_{2,1}^2\mu_{2,2}^2[(k+1)(\mu_{1,1}^2+\mu_{1,2}^2)+2k\mu_{1,1}\mu_{1,2}]+(1-p)\theta^2\mu_{1,1}^2\mu_{1,2}^2[(k+1)(\mu_{2,2}^2+r\mu_{2,1}^2) + 2kr\mu_{2,1}\mu_{2,2}]+q\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}[2k\theta(p\mu_{2,1}\mu_{2,2}(\mu_{1,1}+\mu_{1,2})+(1-p)\mu_{1,1}\mu_{1,2})(r\mu_{2,1}+\mu_{2,2})) + (k+1)\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}]$$

$$C_7 = (k+1)(k+2)[\mu_{2,1}^3\mu_{2,2}^3(p\theta^3(\mu_{1,1}^3+\mu_{1,2}^3)+q\mu_{1,1}^3\mu_{1,2}^3)+(1-p)\theta^3\mu_{1,1}^3\mu_{1,2}^3(r\mu_{2,1}^3+\mu_{2,2}^3)] +3p\theta\mu_{1,1}\mu_{1,2}\mu_{2,1}^3\mu_{2,2}^3[(k+1)(\mu_{1,1}+\mu_{1,2})(\theta^2+kq\mu_{1,1}\mu_{1,2})+kq\theta(k+1)(\mu_{1,1}^2+\mu_{1,2}^2)+2k^2q\theta \times\mu_{1,1}\mu_{1,2}]+3(1-p)\theta\mu_{1,1}^3\mu_{1,2}^3\mu_{2,1}\mu_{2,2}[\theta^2(k+1)(\mu_{2,1}+\mu_{2,2})+kq\theta(k+1)(\mu_{2,2}^2+r\mu_{2,1}^2) + kq\mu_{2,1}\mu_{2,2}(2kr\theta+(k+1)(r\mu_{2,1}+\mu_{2,2}))]$$

7. The Numerical Study

In this section, some numerical results have been calculated. The service rates, the probability are fixed as $\mu_{1,1}=2, \mu_{1,2}=5, \mu_{2,1}=6, \mu_{2,2}=3, q=0.6, r=0.7, p=0.4$ and $k=7$. The arrival rate (λ) and the vacation rate (θ) has been varied from 0.1 to 1.0 and 3.2 (0.2) 4.0. The system performance measures the mean number of customers in the queue (L_q), the mean waiting time in the queue (W_q), the variance of the number of customers in the queue (V_{L_q}) and the variance of the waiting time in the queue (V_{W_q}) have been calculated and are presented in figures and tables.

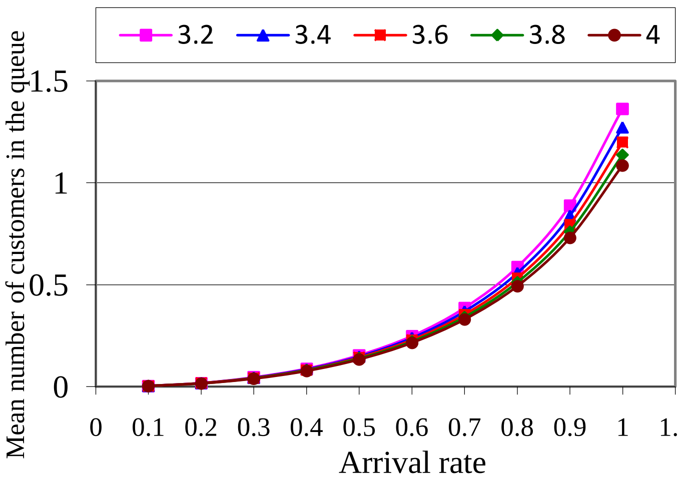


Figure 1: Arrival rate versus mean number of customers in the queue for model 1

From the figures 1 and 2, it is clear that the mean number of customers in the queue for both models are increasing functions of arrival rate. The waiting time also increases for increasing values of arrival rate. In table 1 and 2, the variance of number of customers in the queue and the variance of waiting time in the have been presented. The variances increases for increasing values of arrival rate.

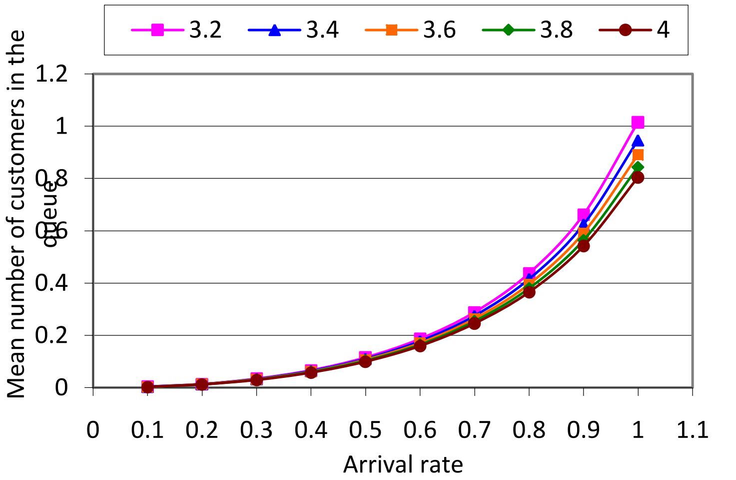


Figure 2: Arrival rate versus mean number of customers in the queue for model 2

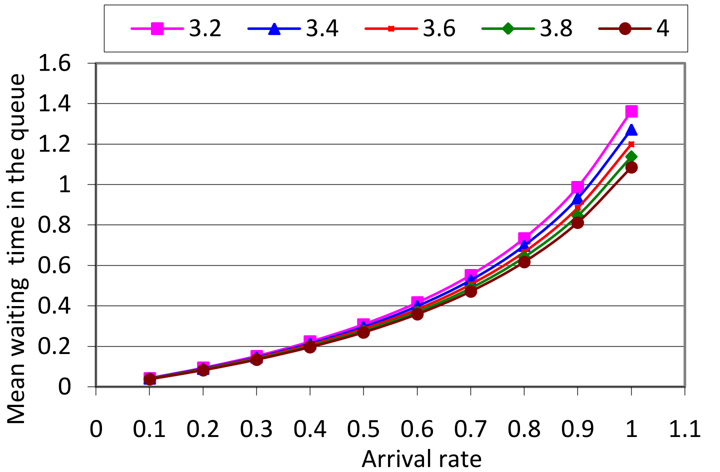


Figure 3: Arrival rate versus mean waiting time in the queue for model 1

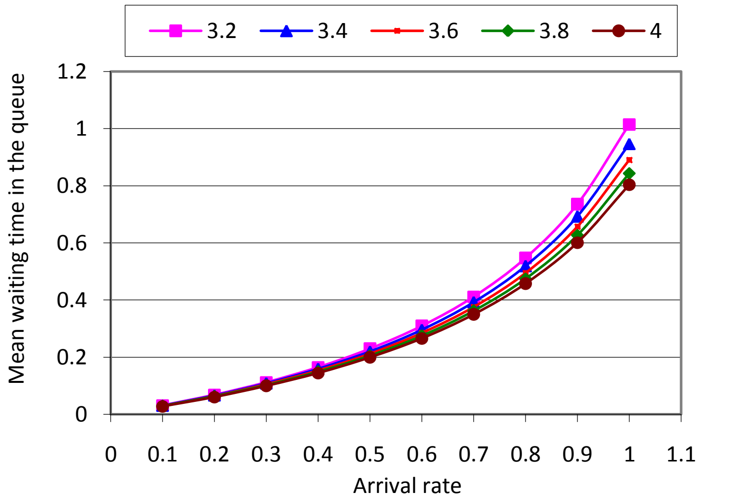


Figure 4: Arrival rate versus mean waiting time in the queue for model 2

Table 1: The variance of number of customers in the queue for model 1

λ	$\mu_{1,1}=7,\mu_{1,2}=5,\mu_{2,1}=6,\mu_{2,2}=3,q=0.6,$ $k=7,r=0.2,p=0.4$				
	$\theta=3.2$	$\theta=3.4$	$\theta=3.6$	$\theta=3.8$	$\theta=4.0$
0.1	0.0048	0.0046	0.0045	0.0043	0.0042
0.2	0.0228	0.0219	0.0212	0.0206	0.0201
0.3	0.0617	0.0593	0.0572	0.0554	0.0539
0.4	0.1336	0.1279	0.1230	0.1189	0.1153
0.5	0.2575	0.2454	0.2353	0.2266	0.2191
0.6	0.4659	0.4417	0.4214	0.4043	0.3896
0.7	0.8162	0.7686	0.7291	0.6960	0.6677
0.8	1.4186	1.3240	1.2465	1.1820	1.1275
0.9	2.5046	2.3086	2.1507	2.0210	1.9129
1.0	4.6268	4.1862	3.8397	3.5611	3.3329

Table 2: The variance of number of customers in the queue for model 2

λ	$\mu_{1,1}=7,\mu_{1,2}=5,\mu_{2,1}=6,\mu_{2,2}=3,q=0.6,$ $r=0.2,p=0.4$				
	$\theta=3.2$	$\theta=3.4$	$\theta=3.6$	$\theta=3.8$	$\theta=4.0$
0.1	0.0034	0.0033	0.0032	0.0031	0.0030
0.2	0.0158	0.0152	0.0147	0.0142	0.0139
0.3	0.0414	0.0398	0.0384	0.0371	0.0361
0.4	0.0872	0.0835	0.0803	0.0775	0.0751
0.5	0.1640	0.1563	0.1497	0.1441	0.1392
0.6	0.2901	0.2749	0.2622	0.2513	0.2419
0.7	0.4978	0.4687	0.4444	0.4237	0.4061
0.8	0.8490	0.7924	0.7455	0.7063	0.6729
0.9	1.4738	1.3583	1.2646	1.1872	1.1223
1.0	2.6813	2.4257	2.2235	2.0600	1.9255

Table 3: The variance of the waiting time in the queue for model 1

λ	$\mu_{1,1}=7,\mu_{1,2}=5,\mu_{2,1}=6,\mu_{2,2}=3,q=0.6,$ $k=7,r=0.2,p=0.4$				
	$\theta=3.2$	$\theta=3.4$	$\theta=3.6$	$\theta=3.8$	$\theta=4.0$
0.1	0.4752	0.4592	0.4454	0.4336	0.4232
0.2	0.5689	0.5481	0.5305	0.5152	0.5019
0.3	0.6858	0.6587	0.6357	0.6160	0.5989
0.4	0.8349	0.7991	0.7688	0.7429	0.7205
0.5	1.0302	0.9817	0.9410	0.9063	0.8765
0.6	1.2942	1.2269	1.1706	1.1230	1.0822
0.7	1.6658	1.5685	1.4880	1.4203	1.3627
0.8	2.2166	2.0688	1.9476	1.8468	1.7618
0.9	3.0921	2.8501	2.6551	2.4951	2.3616
1.0	4.6268	4.1862	3.8397	3.5611	3.3329

Table 4: The variance of the waiting time in the queue for model 2

λ	$\mu_{1,1}=7,\mu_{1,2}=5,\mu_{2,1}=6,\mu_{2,2}=3,q=0.6,$ $r=0.2,p=0.4$				
	$\theta=3.2$	$\theta=3.4$	$\theta=3.6$	$\theta=3.8$	$\theta=4.0$
0.1	0.3405	0.3287	0.3186	0.3097	0.3019
0.2	0.3939	0.3794	0.3668	0.3559	0.3463
0.3	0.4605	0.4422	0.4264	0.4128	0.4008
0.4	0.5453	0.5218	0.5017	0.4843	0.4691
0.5	0.6561	0.6252	0.5989	0.5763	0.5567
0.6	0.8057	0.7637	0.7283	0.6980	0.6719
0.7	1.0158	0.9565	0.9068	0.8648	0.8287
0.8	1.3266	1.2381	1.1649	1.1036	1.0514
0.9	1.8194	1.6770	1.5613	1.4657	1.3856
1.0	2.6813	2.4257	2.2235	2.0600	1.9255

References

[1]. Al-Jararah, J. and Madan, K. C., An $M/G/1$ queue with second optional service with general service time distribution, *International Journal of Information and Management Sciences*, Vol. 14(2), 47-56, 2003.

[2]. Bertsimas, D. and Papaconstamtinou, X., On the steady state solution of the $M/C_2(a,b)/s$ queueing system, *Transport Science*, Vol. 22, 125-138, 1988.

[3]. Bhat, U. N., Imbedded Markov chain analysis of a single server bulk queue, *J. Aust. Math. Soc.*, Vol. 4, 244-263, 1964.

[4]. Chaudhry, M. L. and Templeton, J. G. C., The queueing system $M/G^B/1$ and its ramification, *Euro. J. Oper. Res.*, Vol. 6, 56-60, 1981.

[5]. Cohen, J. W., *The single server queue*, North Holland, Amsterdam, 1969.

[6]. Doshi, B. T., Analysis of a two phase queueing system with general service times, Oper. Res. Lett., Vol. 10, 265-272, 1991.

[7]. Gaver, P., Imbedded Markov chain analysis of a waiting line process in continuous time, *Ann. Math. Statis.*, Vol. 30, 698-720, 1959.

[8]. Jau-Chuan, Ke., An $M^X/G/1$ system with startup server and J additional options for service, *Applied Mathematical Modelling*, Vol. 32(4), 443-458, 2008.

[9]. [9] Jinting Wang, An $M/G/1$ queue with second optional service and server break downs, *Computers and Mathematics with Applications*, Vol. 47(10-11), 1713-1723, 2004.

[10]. Kalyanaraman, R., Thillaigovindan, N., Ayyappan, G. and Manoharan, P., An $M/G/1$ retrial queue with second optional service, *Octogon*, Vol. 13(2), 966-973, 2005.

[11]. Keilson, J. and Kooharian, A., Time dependent queueing processes, *Ann. Math. Statist.*, Vol. 31, 104-112, 1960.

[12]. Keilson, J. and Servi. L., Oscillating random walk models for GI/G/1 vacation systems with Bernoulli schedules, Journal of Applied Probability, Vol. 23, pp. 790-802, 1986.

[13]. Krishna, C. M. and Lee, Y. H., A study of a two phase service, *Oper. Res. Lett.*, Vol. 9, 91-97, 1990.

[14]. Levy, Y. and Yechiali. U., Utilisation of idle time in an M/G/1 queueing system, *Management Science*, Vol. 22 (2), pp. 202-211, 1975.

[15]. Madan, K. C., An $M/G/1$ queueing system with compulsory server vacations, Trabajos De Investigacion Operativa, Vol. 7 (1), 105-115, 1992.

[16]. Madan, K. C., An $M/G/1$ queueing system with additional optional service and no waiting capacity, *Microelectronics and Reliability*, Vol. 34 (3), 521-527, 1994.

[17]. Madan, K. C., An $M/G/1$ queue with second optional service, *Queueing systems*, Vol. 34, 37-46, 2000.

[18]. Medhi, J., A single server Poisson input queue with a second optional channel, *Queueing systems*, Vol. 42, 239-242, 2002.

[19]. Miller, L. W., Alternating priorities in multi-class queue, *Ph.D. Disseration*, Cornell University, Ithaca, N.Y., 1964.

[20]. Prabhu, N. U., *Queues and Inventories*, Wiley, New York, 1965.