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RESEARCH ARTICLE

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ON THE IMPOSSIBILITY OF AN ISOMETRIC IMMERSION OF SOME MANIFOLDS IN EUCLIDEAN SPACE

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ABSTRACT

In this paper, we investigate the possibility of an isometric immersion of manifolds Sol^4 and SL_2R into Euclidean space as a hypersurface.

Keywords: Riemannian manifold, isometric immersion, Riemannian tensor, system of equations of Gauss.

1. INTRODUCTION

Problems of isometric immersions and embeddings of manifolds into Euclidean and the other space, is one of the central problems in differential geometry and in Riemannian geometry, and are studied in these disciplines from many different points of view.

Isometric immersions of Riemannian spaces into Euclidean space or into another Riemannian space is one of methods of construction submanifolds of spaces that have new and interesting geometric properties. The theory of isometric immersions associated with the difficult questions of solvability of nonlinear systems of differential equations in the small and especially difficult during races discretion, as a whole, as well as on topological questions and in obtaining wide arsenal of mathematical tools, used in applied visual geometric ideas.

According to the well-known Hilbert's theorem, Lobachevski plane is not immersed regularly and isometrically into the three-dimensional Euclidean space R^3. In [1] N.V.Efimov strengthened this theorem proving non immersability of the class C^4 into R^3 Lobachevski half-plane, i.e. infinite domain in the hyperbolic plane bounded complete geodesic. Thus, the nonimmersibility properties can be valid not only for complete manifolds, but also for manifolds with boundary, even if the curvature does not change sign.

In [2] Riverts proved the theorem that there is no regular isometric immersion of the three-dimensional Heisenberg group with an arbitrary left-invariant metric into Euclidean space R^4.

2. MAIN PART

Definition 1. Let $f: M \to B$ be a differentiable mapping of maximal rank, where M and B are smooth manifolds of dimension n and m, respectively, $n < m$. Such maps are called immersions.

Here are some examples of an immersion.

Example 1. Let $M = R^1$, $B = R^2$ and $f(t) = (\sin t; \sin 2t)$. Rank of this mapping is the same as the rank of the matrix $(\cos t, 2\cos 2t)$, which is equal to one.

Example 2. Let $M = R^2$, $B = R^3$ and $f(u,v) = (\cos u; \sin u; v)$. Rank of this mapping is equal to two. Indeed, the rank of the this mapping is equal to the rank of the matrix $\begin{pmatrix} -\cos u & \sin u & 0 \\ 0 & 0 & 1 \end{pmatrix}$, it follows that the map $f(u,v) = (\cos u; \sin u; v)$ has the maximal rank.

Definition 2. The immersion of a Riemannian manifold M with metric g_M into a Riemannian manifold N with a metric g_N is said to be isometrical, if $g_M(X,Y) = g_N(f_* X, f_* Y)$ for all $x \in M$ and all $X, Y \in TM$.

Let us consider the question of an immersion of the four-dimensional manifold Sol^4 into the five-dimensional Euclidean space.

On the four-dimensional manifold Sol^4 left-invariant metric is given by the formula

$ds^2 = e^{-2t}(dx^2 + dy^2) + e^{4t} dz^2 + dt^2$.

Theorem 1. The manifold Sol^4 can not be immersed into the five-dimensional Euclidean space.

Proof. Consider Sol^4 with a metric

$ds^2 = e^{-2t}(dx^2 + dy^2) + e^{4t} dz^2 + dt^2$.

The coefficients are in the following form

$g_{11} = e^{-2t}, g_{22} = e^{-2t}$, $g_{33} = e^{4t}$, $g_{44} = 1$, and the rest $g_{ij} = 0.$.

The vector $\vec{r}_{11}$ can be represented as a linear combination of basis vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4, \vec{n}$,

$$\vec{r}_{11} = \Gamma_{11}^1 \vec{r}_1 + \Gamma_{11}^2 \vec{r}_2 + \Gamma_{11}^3 \vec{r}_3 + \Gamma_{11}^4 \vec{r}_4 + \alpha_{11}\vec{n}. \quad (1)$$

To Both parts of the equation (1) we multiply the vector $\vec{r}_i \ (i = 1,2,3,4)$, where $\vec{r}_i \vec{r}_j = g_{ij}$, $\Gamma_{ij,k} = \vec{r}_{ij}\vec{r}_k$.

Hence, we have

$$\vec{r}_{11}\vec{r}_1 = \Gamma_{11}^1 \vec{r}_1\vec{r}_1 + \Gamma_{11}^2 \vec{r}_2\vec{r}_1 + \Gamma_{11}^3 \vec{r}_3\vec{r}_1 + \Gamma_{11}^4 \vec{r}_4\vec{r}_1 + \alpha_{11}\vec{n}\vec{r}_1 = \Gamma_{11}^1 \vec{r}_1\vec{r}_1 + 0 = \Gamma_{11}^1 g_{11}$$

$$\Gamma_{11,1} = \vec{r}_{11}\vec{r}_1 = \frac{1}{2}(g_{11})_1 = \frac{1}{2}(e^{-2t})_x = 0,$$

$$\Gamma_{11}^1 g_{11} = 0, \ \Gamma_{11}^1 = 0, \ \Gamma_{11,1} = 0.$$

Similarly,

$$\Gamma_{ij}^k = 0 \ (i,j,k = 1,2,3), \ \Gamma_{kt}^4 = 0 \ (k,t = 1,2,3,4, k \neq t), \ \Gamma_{44}^i = 0 \ (i = 1,2,3),$$

$$\Gamma_{ij,k} = 0 \ (i,j,k = 1,2,3), \Gamma_{kt,4} = 0 (k,t = 1,2,3,4; k \neq t), \ \Gamma_{44,i} = 0 \ (i = 1,2,3).$$

Next, calculate the coefficients Γ_{ij}^k. To do this, use the equality

$$\vec{r}_{11}\vec{r}_4 = \Gamma_{11}^1 \vec{r}_1\vec{r}_4 + \Gamma_{11}^2 \vec{r}_2\vec{r}_4 + \Gamma_{11}^3 \vec{r}_3\vec{r}_4 + \Gamma_{11}^4 \vec{r}_4\vec{r}_4 + \alpha_{11}\vec{n}\vec{r}_4 = \Gamma_{11}^4 \vec{r}_4\vec{r}_4 = \Gamma_{11}^4 g_{44},$$

and we find that

$$\Gamma_{11,4} = \vec{r}_{11}\vec{r}_4 = \frac{1}{2}(g_{14})_1 - \frac{1}{2}(g_{11})_4 = -\frac{1}{2}(e^{-2t})_t = e^{-2t};$$

$$\Gamma_{11}^4 g_{44} = e^{-2t}, \ \Gamma_{11}^4 = e^{-2t}, \ \Gamma_{11,4} = e^{-2t}.$$

Similarly, using the equality

$\vec r_{22}\vec r_4 = \Gamma^1_{22}\vec r_1\vec r_4 + \Gamma^2_{22}\vec r_2\vec r_4 + \Gamma^3_{22}\vec r_3\vec r_4 + \Gamma^4_{22}\vec r_4\vec r_4 + \alpha_{22}\vec n\vec r_4 = \Gamma^4_{22}\vec r_4\vec r_4 = \Gamma^4_{22}g_{44}$,

and we obtain

$\Gamma_{22,4} = \vec r_{22}\vec r_4 = \frac{1}{2}(g_{24})_2 - \frac{1}{2}(g_{22})_4 = -\frac{1}{2}(e^{-2t})_t = e^{-2t}$;

$\Gamma^4_{22}g_{44} = e^{-2t}$, $\Gamma^4_{22} = e^{-2t}$, $\Gamma_{22,4} = e^{-2t}$.

3) $\vec r_{33}\vec r_4 = \Gamma^1_{33}\vec r_1\vec r_4 + \Gamma^2_{33}\vec r_2\vec r_4 + \Gamma^3_{33}\vec r_3\vec r_4 + \Gamma^4_{33}\vec r_4\vec r_4 + \alpha_{33}\vec n\vec r_4 = \Gamma^4_{33}\vec r_4\vec r_4 = \Gamma^4_{33}g_{44}$,

$\Gamma_{33,4} = \vec r_{33}\vec r_4 = (g_{34})_3 - \frac{1}{2}(g_{33})_4 = -\frac{1}{2}(e^{4t})_t = -2e^{4t}$;

$\Gamma^4_{33}g_{44} = -2e^{4t}$, $\Gamma^4_{33} = -2e^{4t}$, $\Gamma_{33,4} = -2e^{4t}$.

4) $\vec r_{14}\vec r_1 = \Gamma^1_{14}\vec r_1\vec r_1 + \Gamma^2_{14}\vec r_2\vec r_1 + \Gamma^3_{14}\vec r_3\vec r_1 + \Gamma^4_{14}\vec r_4\vec r_1 + \alpha_{14}\vec n\vec r_1 = \Gamma^1_{14}\vec r_1\vec r_1 = \Gamma^1_{14}g_{11}$,

$\Gamma_{14,1} = \vec r_{14}\vec r_1 = (g_{11})_4 = \frac{1}{2}(e^{-2t})_t = -e^{-2t}$;

$\Gamma^1_{14}g_{11} = -e^{-2t}$, $\Gamma^1_{14} = -1$, $\Gamma_{14,1} = -e^{-2t}$.

5) $\vec r_{24}\vec r_2 = \Gamma^1_{24}\vec r_1\vec r_2 + \Gamma^2_{24}\vec r_2\vec r_2 + \Gamma^3_{24}\vec r_3\vec r_2 + \Gamma^4_{24}\vec r_4\vec r_2 + \alpha_{24}\vec n\vec r_2 = \Gamma^2_{24}\vec r_2\vec r_2 = \Gamma^2_{24}g_{22}$,

$\Gamma_{24,2} = \vec r_{24}\vec r_2 = \frac{1}{2}(g_{22})_4 = \frac{1}{2}(e^{-2t})_t = -e^{-2t}$;

$\Gamma^2_{24}g_{22} = -e^{-2t}$, $\Gamma^2_{24} = -1$, $\Gamma_{24,2} = -e^{-2t}$.

6) $\vec r_{34}\vec r_3 = \Gamma^1_{34}\vec r_1\vec r_3 + \Gamma^2_{34}\vec r_2\vec r_3 + \Gamma^3_{34}\vec r_3\vec r_3 + \Gamma^4_{34}\vec r_4\vec r_3 + \alpha_{34}\vec n\vec r_3 = \Gamma^3_{34}\vec r_3\vec r_3 = \Gamma^3_{34}g_{33}$,

$\Gamma_{34,3} = \vec r_{34}\vec r_3 = \frac{1}{2}(g_{33})_4 = \frac{1}{2}(e^{4t})_t = 2e^{4t}$;

$\Gamma^3_{34}g_{33} = 2e^{4t}$, $\Gamma^3_{34} = 2$, $\Gamma_{34,3} = 2e^{4t}$.

To write the Gauss equations calculate the Riemann tensor components using the formula [3]

$$R_{lijk} = \frac{\partial}{\partial u^j}\Gamma_{ik,l} - \frac{\partial}{\partial u^k}\Gamma_{ij,l} + \sum_{\alpha=1}^{4}\Gamma^{\alpha}_{ij}\Gamma_{lk,\alpha} - \sum_{\alpha=1}^{4}\Gamma^{\alpha}_{ik}\Gamma_{lj,\alpha}.$$

This implies the following:

$R_{1212} = -e^{-4t}$, $R_{1313} = 2e^{2t}$, $R_{2323} = 2e^{2t}$,

$R_{1414} = -e^{-2t}$, $R_{2424} = -e^{-2t}$, $R_{3434} = -4e^{4t}$.

The remaining components $R_{l,i,j,k}$ are all zero. Using these values, we obtain Gauss equations

$$R_{1212} = -e^{-4t} = b_{11}b_{22} - b^2_{12} \quad (2)$$

$$R_{2323} = 2e^{2t} = b_{22}b_{33} - b^2_{23} \quad (3)$$

$$R_{1313} = 2e^{2t} = b_{11}b_{33} - b^2_{13} \quad (4)$$

$$R_{3434} = -4e^{4t} = b_{33}b_{44} - b^2_{34} \quad (5)$$

$$R_{1312} = 0 = b_{11}b_{32} - b_{13}b_{12} \quad (6)$$

$$R_{3234} = 0 = b_{33}b_{24} - b_{32}b_{34} \quad (7)$$

$$R_{4243} = 0 = b_{44}b_{23} - b_{42}b_{43} \quad (8)$$

$$R_{1232} = 0 = b_{13}b_{22} - b_{12}b_{32} \quad (9)$$

$$R_{1323} = 0 = b_{12}b_{33} - b_{13}b_{23} \quad (10)$$

$$R_{2324} = 0 = b_{22}b_{34} - b_{23}b_{24} \quad (11)$$

From (2) and (3) it follows that $b_{11} \neq 0, b_{22} \neq 0, b_{33} \neq 0$, because if one of them is zero, we get the following relation

$$2e^{2t} = -b^2_{13} \text{ or } 2e^{2t} = -b^2_{23},$$

which are not true for the real numbers.

If we multiply the equation (7) to b_{23}, the equation (11) to b_{33} and add them by term, we get

$$b_{34}b_{23}^2 - b_{22}b_{34}b_{33} = 0.$$

Taking-out their common factor from bracket we obtain the following

$$b_{34}(b_{22}b_{33} - b_{23}^2) = 0.$$

By (3), the expression in brackets is nonzero. So $b_{34} = 0.$

Further, similarly multiplying the equation (7) to b_{34}, the equation (8) to b_{33}, and adding them by term, we obtain the following relation:

$$b_{23}b_{34}^2 - b_{23}b_{33}b_{44} = 0.$$

Now from the equation

$$b_{23}(b_{33}b_{44} - b_{34}^2) = 0.$$

We get $b_{23} = 0.$

We multiply (6) to b_{12}, the equation (9) to b_{11} and add them term by term, we get

$$b_{13}b_{12}^2 - b_{13}b_{11}b_{22} = 0.$$

Similarly, $b_{13} = 0.$ From (3) and (4) we obtain $b_{11} = b_{22}$.

Then equation (2) has the following form:

$$R_{1,2,1,2} = -e^{-4t} = b_{11}^2 - b_{12}^2.$$

Here $b_{12} \neq 0$, therefore $-e^{-4t} \neq b_{11}^2.$

If we multiply the equation (9) to b_{12}, the equation (10) to b_{11} and add them term by term, we get

$$b_{12}b_{23}^2 - b_{12}b_{22}b_{33} = 0.$$

Thus, $b_{12} = 0$. This contradiction reveals the impossibility of immersion of the four-dimensional manifold Sol^4 into the five-dimensional Euclidean space. Theorem 1 is proved.

Theorem 2. The three-dimensional manifold SL_2R can not be immersed in the four-dimensional Euclidean space.

Proof. Consider SL_2R with the metric

$$ds_{UH^2}^2 = \frac{dx^2 + dy^2 + (dx + ydt)^2}{y^2}$$

The coefficients have the following form

$g_{11} = \frac{2}{y^2}$, $g_{22} = \frac{1}{y^2}$, $g_{33} = 1$, $g_{13} = \frac{1}{y}$, and the remaining $g_{ij} = 0.$

The vector $\vec{r}_{ij}$ can be represented as (1) a linear combination of basis vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{n}$.

Both parts of the equation (1) is multiplied by the vector $\vec{r}_i \, (i = 1,2,3)$, where $\vec{r}_i\vec{r}_j = g_{ij}$, $\Gamma_{ij,k} = \vec{r}_{ij}\vec{r}_k.$

Hence, we have

1) $(\overline{r}_{11} \cdot \overline{r}_1) = \Gamma_{11}^1 g_{11} + \Gamma_{11}^3 g_{31} = \frac{1}{2}(g_{11})_1$;

$(\overline{r}_{11} \cdot \overline{r}_2) = \Gamma_{11}^2 g_{22} = -\frac{1}{2}(g_{11})_2 + (g_{12})_1$; $(\overline{r}_{11} \cdot \overline{r}_3) = \Gamma_{11}^1 g_{13} + \Gamma_{11}^3 g_{33} = -\frac{1}{2}(g_{11})_3 + (g_{13})_1$

$\Gamma_{11,1} = 0,\ \Gamma_{11,2} = \frac{2}{y^3},\ \Gamma_{11,3} = 0,\ \Gamma_{11}^1 = 0,\ \Gamma_{11}^3 = 0,\ \Gamma_{11}^2 = \frac{2}{y}.$

2) $(\overline{r}_{12} \cdot \overline{r}_1) = \Gamma_{12}^1 g_{11} + \Gamma_{12}^3 g_{31} = \frac{1}{2}(g_{11})_2$,

$(\overline{r}_{12}\cdot\overline{r}_2)=\Gamma^2_{12}g_{22}=\frac{1}{2}(g_{22})_1$, $(\overline{r}_{12}\cdot\overline{r}_3)=\Gamma^1_{12}g_{11}+\Gamma^3_{12}g_{33}=\frac{1}{2}((g_{13})_2+(g_{23})_1-(g_{12})_3)$

$\Gamma_{12,2}=0,\ \Gamma_{12,1}=-\frac{2}{y^3},\ \Gamma_{12,3}=-\frac{1}{2y^2},\ \Gamma^1_{12}=-\frac{3}{2y},\ \Gamma^2_{12}=0,\ \Gamma^3_{12}=\frac{1}{y^2}$.

3) $(\overline{r}_{13}\cdot\overline{r}_1)=\Gamma^1_{13}g_{11}+\Gamma^3_{13}g_{31}=\frac{1}{2}(g_{11})_3$;

$(\overline{r}_{13}\cdot\overline{r}_3)=\Gamma^1_{13}g_{13}+\Gamma^3_{13}g_{33}=\frac{1}{2}(g_{33})_1$; $(\overline{r}_{13}\cdot\overline{r}_2)=\Gamma^2_{13}g_{33}=\frac{1}{2}((g_{12})_3+(g_{23})_1-(g_{13})_2)$

$\Gamma_{13,1}=0,\ \Gamma_{13,2}=\frac{1}{2y^2},\ \Gamma_{13,3}=0,\ \Gamma^1_{13}=0,\ \Gamma^3_{13}=0,\ \Gamma^2_{13}=\frac{1}{2}$

4) $(\overline{r}_{22}\cdot\overline{r}_2)=\Gamma^2_{22}g_{22}=\frac{1}{2}(g_{22})_2$;

$(\overline{r}_{22}\cdot\overline{r}_1)=\Gamma^1_{22}g_{11}+\Gamma^3_{33}g_{13}=(g_{12})_2-\frac{1}{2}(g_{22})_1$; $(\overline{r}_{22}\cdot\overline{r}_3)=\Gamma^1_{22}g_{13}+\Gamma^3_{22}g_{33}=(g_{23})_2-\frac{1}{2}(g_{22})_3$;

$\Gamma_{22,1}=0,\ \Gamma_{22,2}=-\frac{1}{y^3},\ \Gamma_{22,3}=0,\ \Gamma^1_{22}=0,\ \Gamma^3_{22}=0,\ \Gamma^2_{22}=-\frac{1}{y}$

5) $(\overline{r}_{23}\cdot\overline{r}_3)=\Gamma^1_{23}g_{13}+\Gamma^3_{23}g_{33}=\frac{1}{2}(g_{33})_2$;

$(\overline{r}_{23}\cdot\overline{r}_2)=\Gamma^2_{23}g_{22}=\frac{1}{2}(g_{22})_3$;

$(\overline{r}_{23}\cdot\overline{r}_1)=\Gamma^1_{23}g_{11}+\Gamma^3_{23}g_{13}=\frac{1}{2}((g_{12})_3+(g_{31})_2-(g_{23})_1)$

$\Gamma_{23,2}=0,\ \Gamma_{23,3}=0,\ \Gamma_{23,1}=-\frac{1}{2y^2},\ \Gamma^1_{23}=-\frac{1}{2},\ \Gamma^2_{23}=0,\ \Gamma^3_{23}=\frac{1}{2y}$.

6) $(\overline{r}_{33}\cdot\overline{r}_1)=\Gamma^1_{33}g_{11}+\Gamma^3_{33}g_{13}=(g_{31})_3-\frac{1}{2}(g_{33})_1$;

$(\overline{r}_{33}\cdot\overline{r}_2)=\Gamma^2_{33}g_{33}=(g_{32})_3-\frac{1}{2}(g_{33})_2$;

$(\overline{r}_{33}\cdot\overline{r}_3)=\Gamma^1_{33}g_{13}+\Gamma^3_{33}g_{33}=\frac{1}{2}(g_{33})_3$; $\Gamma_{33,1}=0,\ \Gamma_{33,2}=0,\ \Gamma_{33,3}=0,\ \Gamma^1_{33}=0,\ \Gamma^2_{33}=0,\ \Gamma^3_{33}=0.$

To write the Gauss equations calculate the Riemann tensor components using the formula [3]

$$R_{lijk}=\frac{\partial}{\partial u^j}\Gamma_{ik,l}-\frac{\partial}{\partial u^k}\Gamma_{ij,l}+\sum_{\alpha=1}^{3}\Gamma^\alpha_{ij}\cdot\Gamma_{lk,\alpha}-\sum_{\alpha=1}^{3}\Gamma^\alpha_{ik}\cdot\Gamma_{lj,\alpha}$$

This implies the following:

$R_{1212}=-\frac{3}{2y^4}$, $R_{1313}=\frac{1}{4y^2}$, $R_{2323}=\frac{1}{4y^2}$, $R_{1223}=-\frac{1}{4y^3}$, $R_{1332}=0$, $R_{2113}=0$.

All other components of Riemannian tensors R_{lijk} are equal zero. Using these values, we obtain Gauss equations

$$R_{1212}=b_{11}b_{22}-b_{12}^2=-\frac{3}{2y^4}\qquad(12)$$

$$R_{1313}=b_{11}b_{33}-b_{13}^2=\frac{1}{4y^2}\qquad(13)$$

$$R_{2323}=b_{22}b_{33}-b_{23}^2=\frac{1}{4y^2}\qquad(14)$$

$$R_{1223}=b_{12}b_{23}-b_{13}b_{22}=-\frac{1}{4y^3}\qquad(15)$$

$$R_{1332}=b_{13}b_{32}-b_{12}b_{33}=0\qquad(16)$$

$$R_{2113}=b_{12}b_{13}-b_{23}b_{11}=0\qquad(17)$$

If we multiply the equation (16) to b_{13}, the equation (17) to b_{33} by term and add them, then we get

$b_{23}b_{13}^2-b_{23}b_{11}b_{33}=0$

Thus, $b_{23} = 0$.

Similarly, from equations (16) and (17) we obtain

$b_{12}b_{13}^2 - b_{11}b_{12}b_{33} = 0$.

This implies $b_{12} = 0$.

Inserting the found values into equations, we get

$$R_{1212} = b_{11}b_{22} = -\frac{3}{2y^4}$$

$$R_{1313} = b_{11}b_{33} - b_{13}^2 = \frac{1}{4y^2}$$

$$R_{2323} = b_{22}b_{33} = \frac{1}{4y^2}$$

$$R_{1223} = b_{13}b_{22} = \frac{1}{4y^3}$$

Together solving these equations leads to a contradiction $b_{33}^2 = -\frac{1}{28}$. This contradiction shows the impossibility of immersion of the three-dimensional manifold SL_2R into the four-dimensional Euclidean space. Theorem 2 is proved.

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